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NOTE

New Designs with Block Size 7

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An imprimitive permutation group of order 4200 is used for the construction of a 2-(175, 7, 1) design. The design yields also a group divisible design $7-GDD$ and a generalized Bhaskar Rao design $GBRD(25, 100, 28, 7, 7; Z_7)$. © 1998 Academic Press

1. INTRODUCTION

Our notation is in accordance with [4]. A necessary condition for the existence of a 2-$(v, 7, 1)$ design is $v \equiv 1$ or 7 (mod 42). The existence of 2-$(v, 7, 1)$ designs has been resolved for all but 31 values of v (cf. [1], Tables 2.6, 2.7, and [7]). In this note, we give a direct construction of a design with $v = 175 = 42 \cdot 4 + 7$ that settles the undecided case $t = 4$ in Table 2.7 of [1]. The design admits a parallel class fixed by a fixed-point-free automorphism of order 7. This implies the existence of a group divisible design $7-GDD$, and a generalized Bhaskar Rao design $GBRD(25, 100, 28, 7, 7; Z_7)$ which were previously unknown. Julian Abel (private communication) pointed out that the existence of this 2-(175,7,1) design implies also the existence of two other previously undecided 2-$(v, 7, 1)$ designs, namely for $v = 1219 = 42 \cdot 29 + 1$ and $v = 1225 = 42 \cdot 29 + 7$, by applying known recursive constructions.

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2. THE CONSTRUCTION

A significant amount of work appears to have been done on classifying designs with block transitive groups (see, e.g., [2] and the references therein), but not so much seems to be known about designs with point imprimitive and block intransitive groups, apart perhaps from the cyclic 2-designs with non-prime number of points. In this note, we describe a construction of a 2-(175, 7, 1) design with an automorphism group that is point transitive but imprimitive, and has two block orbits. The group structure resembles that of the full automorphism group of the 2-(45, 5, 1) design given in Marshall Hall's book [5]. The Hall 2-(45, 5, 1) design is invariant under a group H of order 360, acting transitively (but imprimitively) on the points, and having two orbits of blocks, one being a parallel class, with the group H acting primitively on the blocks of that parallel class (for a list of small primitive groups see [3]). More precisely, $H = (E_9 \times Z_5) \cdot Q$, where $E_9 \times Z_5$ is a direct product of an elementary abelian group E_9 of order 9 with a cyclic group Z_5 of order 5 and this group is extended by the quaternion group Q of order 8 so that Q acts faithfully on E_9. The group $E_9 \cdot Q$ is a Frobenius group of order 72, and $Z_5 \cdot Q$ is a non-abelian group of order 40. Furthermore, E_9 and Z_5 are both normal in G, and the centralizer of Z_5 in Q is a cyclic group of order 4. This determines the structure of the solvable group H uniquely. The last "centralizer fact" gives us the unique generalization to a group $G = \langle a, b, c, d, e, f, g, h, i, j \rangle$ of order $4200 = 25 \times 24 \times 7$ acting as a transitive permutation group of degree 175:

1. $\langle a, b, c \rangle = E_{25}$ is an elementary abelian group of order 25, and E_{25} is normal in G;

2. $\langle d, e, f \rangle = Q_8$ is a quaternion group of order 8 which acts faithfully and fixed-point-free on E_{25};

3. $\langle g \rangle = Z_3$ is a cyclic group of order 3 which acts fixed-point-free on E_{25} and $Q8 \cdot Z_3$ is isomorphic to $SL(2, 3)$ of order 24;

4. $\langle d, e, g \rangle = SL(2, 3)$ acts fixed-point-free on E_{25} so that $E_{25} \cdot SL(2, 3)$ is a Frobenius group of order 25×24 and $h = fg$ is an element of order 6;

5. $\langle i \rangle = Z_7$ is a subgroup of order 7 which is normal in G and $\langle i \rangle$ commutes with each element in E_{25} and Q_8, but Z_3 acts faithfully on Z_7 so that $Z_7 \cdot Z_3$ is a Frobenius group of order 21;

6. $j = ia$ is an element of order 35;

7. The subgroup $SL(2, 3) \cdot Z_3 \cdot Z_7 = \langle d, e, g, i \rangle$ is of order $168 = 24 \times 7$ acting primitively on the elements of E_{25}.

The generating permutations $a, b, c, ..., j$ are listed in Table 2.4. Base blocks for our 2-(175,7,1) design are $B_1 = \{1, 60, 81, 161, 75, 168, 83\}$ and $B_2 = \{1, 20, 21, 22, 23, 24, 25\}$. The block B_1 is fixed by the cyclic subgroup of order 6 generated by h and the orbit of B_1 under G consists of 700 blocks. The block B_2 is fixed by the cyclic subgroup of order 7 generated by i and its orbit under G consists of 25 pairwise disjoint blocks (a parallel class), being the cycles of h.

Remark 2.1. Deleting the 25 mutually disjoint blocks fixed by the cyclic subgroup $\langle h \rangle = Z_7$ yields a (previously unknown) group divisible design $7-GDD$ [8].

TABLE 2.4

The Generating Permutations

$a =$ (1 2 3 4 5)(6 10 11 12 13)(7 14 54 18 57)(8 15 17 56 66)
(9 16 55 65 19)(20 26 27 28 29)(21 42 34 46 38)(22 50 43 35 47)
(23 39 51 44 36)(24 48 40 52 45)(25 37 49 41 53)(30 59 68 72 71)
(31 60 69 63 74)(32 61 64 67 75)(33 62 70 73 58)(76 89 81 93 85)
(77 97 90 82 94)(78 86 98 91 83)(79 95 87 99 92)(80 84 96 88 100)
(101 114 106 118 110)(102 122 115 107 119)(103 111 123 116 108)
(104 120 112 124 117)(105 109 121 113 125)(126 139 131 143 135)
(127 147 140 132 144)(128 136 148 141 133)(129 145 137 149 142)
(130 134 146 138 150)(151 164 156 168 160)(152 172 165 157 169)
(153 161 173 166 158)(154 170 162 174 167)(155 159 171 163 175);

$b =$ (1 6 7 8 9)(2 10 14 15 16)(3 11 54 17 55)(4 12 18 56 65)(5 13 57 66 19)
(20 30 31 32 33)(21 76 101 126 151)(22 77 102 127 152)(23 78 103 128 153)
(24 79 104 129 154)(25 80 105 130 155)(26 59 60 61 62)(27 68 69 64 70)
(28 72 63 67 73)(29 71 74 75 58)(34 81 106 131 156)(35 82 107 132 157)
(36 83 108 133 158)(37 84 109 134 159)(38 85 110 135 160)(39 86 111 136 161)
(40 87 112 137 162)(41 88 113 138 163)(42 89 114 139 164)(43 90 115 140 165)
(44 91 116 141 166)(45 92 117 142 167)(46 93 118 143 168)(47 94 119 144 169)
(48 95 120 145 170)(49 96 121 146 171)(50 97 122 147 172)(51 98 123 148 173)
(52 99 124 149 174)(53 100 125 150 175);

$c =$ (1 16 17 18 13)(2 55 56 57 6)(3 65 66 7 10)(4 19 8 14 11)
(5 9 15 54 12)(20 62 64 63 71)(21 164 131 118 85)(22 172 140 107 94)
(23 161 148 116 83)(24 170 137 124 92)(25 159 146 113 100)(26 70 67 74 30)
(27 73 75 31 59)(28 58 32 60 68)(29 33 61 69 72)(34 168 135 101 89)
(35 169 127 122 90)(36 153 136 123 91)(37 171 138 125 80)(38 151 139 106 93)
(39 173 141 108 78)(40 174 142 104 95)(41 175 130 109 96)(42 156 143 110 76)
(43 157 144 102 97)(44 158 128 111 98)(45 154 145 112 99)(46 160 126 114 81)
(47 152 147 115 82)(48 162 149 117 79)(49 163 150 105 84)(50 165 132 119 77)
(51 166 133 103 86)(52 167 129 120 87)(53 155 134 121 88);

TABLE 2.4—*Continued*

The Generating Permutations

$d =$ (1)(2 3 5 4)(6 66 9 14)(7 12 8 55)(10 15 19 57)(11 56 65 54)(13 17 16 18)
(20)(21)(22)(23)(24)(25)(26 27 29 28)(30 75 33 60)(31 72 32 70)(34 38 46 42)
(35 50 43 47)(36 44 39 51)(37 49 53 41)(40 45 52 48)(58 74 59 61)
(62 63 71 64)(67 73 69 68)(76 135 151 114)(77 144 152 122)(78 133 153 111)
(79 142 154 120)(80 150 155 109)(81 143 168 106)(82 127 165 102)
(83 148 161 116)(84 134 175 125)(85 131 164 118)(86 136 158 108)
(87 149 174 112)(88 130 171 105)(89 139 160 110)(90 132 157 115)
(91 128 173 103)(92 137 170 124)(93 126 156 101)(94 140 172 107)
(95 145 167 117)(96 138 163 121)(97 147 169 119)(98 141 166 123)
(99 129 162 104)(100 146 159 113);

$e =$ (1)(2 18 5 17)(3 16 4 13)(6 7 9 8)(10 65 19 11)
(12 66 55 14)(15 56 57 54)(20)(21)(22)(23)(24)(25)(26 63 29 64)
(27 62 28 71)(30 31 33 32)(34 164 46 85)(35 94 43 172)(36 148 39 116)
(37 113 53 146)(38 131 42 118)(40 170 52 92)(41 100 49 159)
(44 83 51 161)(45 137 48 124)(47 140 50 107)(58 68 59 73)
(60 72 75 70)(61 67 74 69)(76 101 151 126)(77 102 152 127)(78 103 153 128)
(79 104 154 129)(80 105 155 130)(81 89 168 160)(82 144 165 122)(84 163 175 96)
(86 166 158 98)(87 95 174 167)(88 150 171 109)(90 97 157 169)(91 133 173 111)
(93 135 156 114)(99 142 162 120)(106 139 143 110)(108 123 136 141)
(112 145 149 117)(115 147 132 119)(121 134 138 125);

$f =$ (1)(2 5)(3 4)(6 9)(7 8)(10 19)(11 65)(12 55)(13 16)
(14 66)(15 57)(17 18)(20)(21)(22)(23)(24)(25)(26 29)(27 28)(30 33)
(31 32)(34 46)(35 43)(36 39)(37 53)(38 42)(40 52)(41 49)(44 51)(45 48)
(47 50)(54 56)(58 59)(60 75)(61 74)(62 71)(63 64)(67 69)(68 73)(70 72)
(76 151)(77 152)(78 153)(79 154)(80 155)(81 168)(82 165)(83 161)(84 175)
(85 164)(86 158)(87 174)(88 171)(89 160)(90 157)(91 173)(92 170)(93 156)
(94 172)(95 167)(96 163)(97 169)(98 166)(99 162)(100 159)(101 126)
(102 127)(103 128)(104 129)(105 130)(106 143)(107 140)(108 136)(109 150)
(110 139)(111 133)(112 149)(113 146)(114 135)(115 132)(116 148)(117 145)
(118 131)(119 147)(120 142)(121 138)(122 144)(123 141)(124 137)(125 134);

$g =$ (1)(2 6 19)(3 7 56)(4 8 54)(5 9 10)(11 13 66)(12 57 18)(14 65 16)
(15 17 55)(20 21 23)(22 25 24)(26 76 158)(27 101 141)(28 126 123)
(29 151 86)(30 160 39)(31 143 51)(32 106 44)(33 89 36)(34 103 67)(35 130 112)
(37 79 169)(38 153 59)(40 102 138)(41 129 115)(42 78 58)(43 105 149)
(45 152 84)(46 128 69)(47 155 95)(48 77 175)(49 104 132)(50 80 167)
(52 127 121)(53 154 97)(60 168 161)(61 131 173)(62 114 166)(63 93 108)
(64 156 136)(68 85 133)(70 139 148)(71 135 98)(72 110 116)(73 164 111)
(74 118 91)(75 81 83)(82 125 124)(87 94 150)(88 117 107)(90 100 142)
(92 144 96)(99 119 113)(109 174 172)(120 157 159)(122 163 170)(134 137 165)
(140 171 145)(146 162 147);

TABLE 2.4—*Continued*

The Generating Permutations

$h=$ (1)(2 9 19 5 6 10)(3 8 56 4 7 54)(11 16 66 65 13 14)
(12 15 18 55 57 17)(20 21 23)(22 25 24)(26 151 158 29 76 86)
(27 126 141 28 101 123)(30 89 39 33 160 36)(31 106 51 32 143 44)
(34 128 67 46 103 69)(35 105 112 43 130 149)(37 154 169 53 79 97)
(38 78 59 42 153 58)(40 127 138 52 102 121)(41 104 115 49 129 132)
(45 77 84 48 152 175)(47 80 95 50 155 167)(60 81 161 75 168 83)
(61 118 173 74 131 91)(62 135 166 71 114 98)(63 156 108 64 93 136)
(68 164 133 73 85 111)(70 110 148 72 139 116)(82 134 124 165 125 137)
(87 172 150 174 94 109)(88 145 107 171 117 140)(90 159 142 157 100 120)
(92 122 96 170 144 163)(99 147 113 162 119 146);

$i=$ (1 20 21 22 23 24 25)(2 26 42 50 39 48 37)(3 27 34 43 51 40 49)
(4 28 46 35 44 52 41)(5 29 38 47 36 45 53)(6 30 76 77 78 79 80)
(7 31 101 102 103 104 105)(8 32 126 127 128 129 130)
(9 33 151 152 153 154 155)(10 59 89 97 86 95 84)(11 68 81 90 98 87 96)
(12 72 93 82 91 99 88)(13 71 85 94 83 92 100)(14 60 114 122 111 120 109)
(15 61 139 147 136 145 134)(16 62 164 172 161 170 159)
(17 64 131 140 148 137 146)(18 63 118 107 116 124 113)
(19 58 160 169 158 167 175)(54 69 106 115 123 112 121)
(55 70 156 165 173 162 171)(56 67 143 132 141 149 138)
(57 74 110 119 108 117 125)(65 73 168 157 166 174 163)
(66 75 135 144 133 142 150);

$j=$ (1 26 34 35 36 24 37 3 28 38 22 39 40 41 5 20 42 43 44 45 25 2 27 46
47 23 48 49 4 29 21 50 51 52 53)
(6 59 81 82 83 79 84 11 72 85 77 86 87 88 13 30 89 90 91 92 80 10 68
93 94 78 95 96 12 71 76 97 98 99 100)
(7 60 106 107 108 104 109 54 63 110 102 111 112 113 57 31 114 115
116 117 105 14 69 118 119 103 120 121 18 74 101 122 123 124 125)
(8 61 131 132 133 129 134 17 67 135 127 136 137 138 66 32 139 140
141 142 130 15 64 143 144 128 145 146 56 75 126 147 148 149 150)
(9 62 156 157 158 154 159 55 73 160 152 161 162 163 19 33 164 165
166 167 155 16 70 168 169 153 170 171 65 58 151 172 173 174 175).

Remark 2.2. The orbit matrix under $\langle h \rangle = Z_7$ of the 700 blocks that are not fixed by h yields a (new) generalized Bhaskar Rao design $GBRD(25, 100, 28, 7, 7; Z_7)$ [6].

Remark 2.3. The 2-(175,7,1) design contains a parallel class fixed by $\langle h \rangle = Z_7$, as well as some other parallel classes. We do not know whether the design is resolvable or not.

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