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Cyclic Modules Whose Quotients Have All Complement Submodules Direct Summands

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In [20, 21], Osofsky showed that a ring all of whose cyclic modules are injective is semisimple Artinian. Since that time, a cyclic, finitely presented module version of the theorem has been proved in [8] and used to classify certain kinds of rings (see [8, 9]). Also, Ahsan [1] applied the proof in [21] to rings all of whose cyclic modules are quasi-injective, enabling Koehler [17] to show that any ring with all cyclic modules quasi-injective is a product of a semisimple Artinian ring and a noncommutative analog of the injective pre-self-injective rings of Klatt and Levy [16]. Goel and Jain [11] applied the proof of [21] in their study of nonsingular and self-injective rings with every cyclic quasi-continuous. Also, several authors have studied rings for which hypotheses on cyclic modules show that cyclic singular modules are injective (see [9, 10, 24, 25]). In this paper we eliminate extraneous hypotheses used in previous proofs of the theorem in [20, 21] to get a very general result. We prove that a cyclic module M has finite uniform dimension if all quotients of cyclic submodules of M have the property that all complement submodules are direct summands. The modules studied need not have endomorphism rings which are von Neumann regular (at least modulo their Jacobson radicals), a property

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used in previous proofs in the area. Indeed, $\mathbb{Z}$ satisfies the hypotheses of the main theorem. Essentially all previously known theorems in the area follow rather easily. We also show that many of the apparently different classes of rings defined by properties of their cyclic modules are actually the class of rings all of whose singular modules are injective, or very closely related to that class. In addition, we get the beginnings of a structure theorem for rings all of whose cyclics are quasi-continuous.

The proof of the main theorem is categorical even though the statement of the theorem is in terms of cyclic modules, which is not a categorical concept. The argument holds in an $\mathscr{A}\mathscr{B}5$ category $\mathbb{C}$ if cyclic is replaced by a property $\mathscr{P}$ such that any object with $\mathscr{P}$ is finitely generated, a direct summand of an object with $\mathscr{P}$ has $\mathscr{P}$, and if $\mathscr{A}$ and $\mathscr{M}$ are objects with $\mathscr{P}$ such that $\mathscr{A}$ is a direct summand of a quotient object $\mathscr{M}/\mathscr{N}$ then there is a subobject $\mathscr{B}$ of $\mathscr{M}$ such that $\mathscr{B}$ has $\mathscr{P}$ and $\mathscr{B}$ maps onto $\mathscr{A}$ under the natural map $\mathscr{M} \to \mathscr{M}/\mathscr{N}$. This is exploited in our first corollary. However, the most interesting applications in module theory occur when $\mathscr{P}$ is cyclic.

Unless otherwise stated, all modules are objects in the category of unital right modules over a ring R with 1, and all conditions such as Noetherian refer to right modules.

A module M is called CS (for *c*omplement submodules are direct *s*ummands) provided every submodule of M is essential in a direct summand of M, or equivalently, every maximal essential extension of a submodule of M is direct summand of M. This is the terminology of [4], one of the first papers to study this concept. Later other terminology, such as extending module, has been used in place of CS. M is called *quasi-continuous* if it is CS and for any direct summands A and B of M with $A \cap B = 0$, $A + B$ is a direct summand of M. M is called *completely CS* (respectively *completely quasi-continuous*) provided every quotient of M is CS (quasi-continuous).

It is well known that (quasi-)injective modules are quasi-continuous.

THEOREM 1. *Let N be a cyclic module such that every cyclic submodule of N is completely CS. Then N is a finite direct sum of uniform modules.*

Proof. We first prove that N contains no infinite direct sum of nonzero submodules.

Assume that N does contain an infinite direct sum $\bigoplus_{i \in \mathscr{I}} N_i$, where each N_i is nonzero. The proof that this leads to a contradiction proceeds in two steps.

Step 1. There exists a cyclic module M such that M is essential over its countably (but not finitely) generated socle S, every finitely generated submodule of S is a direct summand of M, and every cyclic submodule of M is completely CS.

Proof. Without loss of generality, $\mathscr{I} = \omega$. Let E_0 be a maximal essential extension of N_0 in N. Then $N = E_0 \oplus K_0$. Let π_0 be the projection of N onto K_0 with respect to this decomposition. Then $\text{kernel}(\pi_0) \cap \bigoplus_{i=1}^{\infty} N_i = 0$, so K_0 contains the infinite direct sum of nonzero submodules $\bigoplus_{i=1}^{\infty} \pi_0(N_i)$. K_0 is CS, so for E_1 a maximal essential extension of $\pi_0(N_1)$ in K_0 we have $K_0 = E_1 \oplus K_1$ with projection π_1 to K_1 inducing a monomorphism on $\bigoplus_{i=2}^{\infty} \pi_0(N_i)$. Continuing in this manner, by finite induction we get sequences $\{E_i\}$, $\{K_i\}$ of nonzero submodules of N and $\{\varphi_i = \pi_i \pi_{i=1} \cdots \pi_0\}$ of compositions of projections such that, for each i, $K_i = E_{i+1} \oplus K_{i+1}$, E_{i+1} is a maximal essential extension of $\varphi_i(N_{i+1})$ in K_i, and K_{i+1} contains the infinite direct sum $\bigoplus_{j=i+2}^{\infty} \varphi_{i+1}(N_j)$ of nonzero submodules. Each E_i is cyclic and so has a maximal submodule L_i. Set $M' = N/(\bigoplus_{i=0}^{\infty} L_i)$. Since cyclic submodules of M' are quotients of cyclic submodules of N, every cyclic submodule of M' is completely CS. Moreover, for $S_i = E_i/L_i$, since $N = \bigoplus_{i=0}^{n} E_i \oplus K_n$, $M' = \bigoplus_{i=0}^{n} S_i \oplus (K_n/\bigoplus_{i=n+1}^{\infty} L_i)$. Set $S = \bigoplus_{i=0}^{\infty} S_i \subseteq M'$. Then S has a maximal essential extension M in M' which is a direct summand of M' and hence cyclic. M also has every cyclic submodule completely CS. Let T be a finitely generated submodule of S. Then T is a direct summand of $\bigoplus_{i=0}^{n} S_i$ for some n. Thus T is a direct summand of a direct summand of M'. Hence $T \subseteq M$ is a direct summand of M.

Step 2. Let M and S be as in Step 1. Let ω be a disjoint union of countable sets $\{A_\nu \mid \nu \in \omega\}$. Let X_ν be a maximal essential extension of $\bigoplus_{\mu \in A_\nu} S_\mu$ in M for each $\nu \in \omega$. Since M/S is CS, there is a direct summand $\bar{A}$ of M/S such that $\bar{A}$ is an essential extension of $\sum_{\nu \in \omega} (X_\nu + S)/S$. Since M is cyclic, so is $\bar{A}$. Let A be a cyclic submodule of M such that $\bar{A} = (A+S)/S$. Then for all $\nu \in \omega$, $X_\nu \subseteq A + S$. Since S is semisimple, $S = (S \cap A) + T$ and $A + S = A \oplus T$. Let π project $A + S$ to T with kernel A. Since X_ν is finitely generated, $\pi[X_\nu] \subseteq T$ has finite length. Thus $X_\nu \cap A \cap S$ is of finite colength in $\bigoplus_{\mu \in A_\nu} S_\mu$. In particular, there is a simple module $T_\nu \subseteq (\bigoplus_{\mu \in A_\nu} S_\mu) \cap A$. Let Y be a maximal essential extension of $\bigoplus_{\nu \in \omega} T_\nu$ in A. Then Y is cyclic but has an infinitely generated socle and so cannot be contained in S. Thus $Y/(Y \cap S)$ is a nonzero submodule of $\bar{A}$. But $S \cap Y \cap \bigoplus_{\nu=0}^{n} X_\nu = \bigoplus_{\nu=0}^{n} T_\nu$, which is a direct summand of M, and since M is an essential extension of its socle, $Y \cap \bigoplus_{\nu=0}^{n} X_\nu = \bigoplus_{\nu=0}^{n} T_\nu$. Then $Y \cap \sum_{\nu \in \omega} X_\nu \subseteq S$, contradicting the fact that $\bar{A}$ is essential over $(\sum_{\nu \in \omega} X_\nu + S)/S$.

The last portion of the theorem follows immediately by standard techniques, since any nonuniform module contains a direct sum of two nonzero submodules. If N contains no uniform submodule, one could thus produce an infinite direct sum of submodules. So N contains a uniform submodule, and its maximal essential extension is also uniform and a direct summand.

Now work in a complement. The process stops in a finite number of steps. ∎

COROLLARY 1. *Let R be a ring such that every cyclic (respectively finitely generated, cyclic singular, finitely generated singular, etc.) module is CS. Then every cyclic (respectively finitely generated, cyclic singular, finitely generated singular, etc.) module is a direct sum of uniform modules.*

Proof. These are all properties of type $\mathscr{P}$ discussed before the proof of the theorem. ∎

The ring $\mathbb{Z}$ of integers has every finitely generated module a direct sum of uniform submodules by the basis theorem for finitely generated Abelian groups. However, not every finitely generated Abelian group is CS. Indeed, $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ has a subgroup of order p^2 with no proper essential extension. Thus the converse of Corollary 1 for finitely generated modules is false. However, every cyclic $\mathbb{Z}$-module is CS. It would be interesting to know for which of the classes mentioned, if any, the converse of Corollary 1 is true.

We note that, in the infinitely generated case, every CS module is a direct sum of uniform modules if and only if R is Noetherian (see [19]).

COROLLARY 2. *Let N be a module with every quotient of a cyclic submodule injective. Then N is semisimple.*

Proof. Every cyclic submodule of N is a finite direct sum of indecomposable injective modules, each of which has every cyclic submodule a direct summand and so either 0 or the indecomposable module. Thus N is a sum of simple modules. ∎

COROLLARY 3. (Osofsky [20, 21]). *Let R be a ring for which every cyclic is injective. Then R is semisimple Artinian.*

COROLLARY 4. *Let R be a ring for which every singular cyclic module is injective. Then every singular module is semisimple.*

Proof. Submodules and quotient modules of singular modules are singular. ∎

LEMMA A. (Matlis-Bass). *Let R be a ring such that every direct sum of injective hulls of simples is injective. Then R is Noetherian. (For a proof, see [23, Theorem* 4.1].)

LEMMA B. *Let R be any ring, and T a simple R-module. Then T is singular if and only if $T \cdot \operatorname{socle}(R) = 0$.*

Proof. A maximal ideal is essential in R if and only if it contains the socle of R. Thus a simple module T is singular $\Leftrightarrow$ the annihilator of every nonzero element of T is essential in $R \Leftrightarrow$ the annihilator of every nonzero element of T contains the socle of $R \Leftrightarrow T \cdot \mathrm{socle}(R) = 0$. ▮

COROLLARY 5. *The following are equivalent*:

(a) *Every cyclic singular module is injective.*

(b) *Every singular module is injective.*

Proof. (a) $\Rightarrow$ (b) By Corollary 4, every singular module is semisimple. Since the singular submodule of R cannot contain a direct summand of R, R is nonsingular and the injective hull of a singular module is singular (see [12]). In particular, every singular module is a direct summand of its (singular) injective hull, and is therefore injective.

(b) $\Rightarrow$ (a) is immediate. ▮

COROLLARY 6 (cf Goodearl [13]). *If every cyclic singular R-module is injective, then* $R/\mathrm{socle}(R)$ *is Noetherian and every simple* $R/\mathrm{socle}(R)$*-module is injective.*

Proof. A simple $R/\mathrm{socle}(R)$-module is a simple R-module annihilated by the socle of R and hence singular as an R-module by Lemma B. If E is a direct sum of simple $R/\mathrm{socle}(R)$-modules then E is singular, hence injective as an R-module (Corollary 5), and so as an $R/\mathrm{socle}(R)$-module. Apply Lemma A. ▮

Goodearl [13] has a characterization of rings with every singular module injective.

COROLLARY 7 (Damiano [8]). *Let R be a ring such that every cyclic module not isomorphic to R is injective. Then R is Noetherian.*

Proof. By Corollary 6, $R/\mathrm{socle}(R)$ is Noetherian. If $\mathrm{socle}(R) \neq 0$, then $R/\mathrm{socle}(R)$ is semisimple Artinian by Corollary 3 since every quotient of it is annihilated by $\mathrm{socle}(R)$, and R is not. Suppose $R \neq \mathrm{socle}(R)$. Let $y \in R$, $yR/\mathrm{socle}(yR)$ simple. If $\mathrm{socle}(yR)$ is not of finite length, then $\mathrm{socle}(yR) = S \oplus T \oplus U$, where each of S, T, U has infinite length. Since S is not a direct summand of yR, yR/S is not projective. Hence yR/S is injective. Then $T \oplus U$ embeds in yR/S, and $yR/S = E(T) \oplus E(U) \oplus K$ for some injective hulls of T and U. Then $yR/(S \oplus T \oplus U) \approx E(T)/T \oplus E(U)/U \oplus K$ is not simple, a contradiction. Thus the socle of yR is of finite length. Since each simple submodule of R is injective, the socle of yR is a direct summand of yR and $y \in \mathrm{socle}(R)$. Thus $R = \mathrm{socle}(R)$ is semisimple Artinian. ▮

COROLLARY 8 (See Smith [25]). *The following are equivalent*:

(a) *R is a ring with the property that every cyclic module is an extension of a projective module by an injective module.*

(b) *Every singular module is injective.*

Proof. (a) ⇒ (b) Let R/I be a cyclic singular module. Then I is essential in R so R/I cannot contain a nonzero projective submodule. Hence R/I must be injective. Apply Corollary 5.

(b) ⇒ (a) is a result of Smith [25, Corollary 3.7], which follows from the Goodearl classification [13] of rings with every singular module injective. ∎

There is another property equivalent to those in Corollary 5 that has been studied by P. Dan and D. van Huynh [9], namely rings for which the singular submodule of every cyclic is injective. This is clearly implied by Corollary 5(b) and implies Corollary 5(a).

We observe some known facts and an open question on rings all of whose singular modules are injective. In general, every singular R-module being injective does not imply that R is Noetherian (see [13, Example 3.2]). Nor can we conclude that if R is a Noetherian V-ring, that is, every simple R-module is injective, R must have all singular R-modules injective. Cozzens and Johnson [7] have examples of Noetherian V-domains of arbitrary global dimension. Any proper quotient of a Noetherian domain must be singular, but if every proper quotient of a Noetherian domain is injective, that domain must be hereditary [6]. R may have every singular right module injective without having the property on the left [13, Example 3.8]. However, if R is a Noetherian domain with every singular module injective, it is unknown if R has that property on the left.

We next come to a property that is actually somewhat stronger than every singular module is injective. These are the "CDPI-rings" of [24, 25].

PROPOSITION 1. *Let R be a ring for which every cyclic module is a direct sum of a projective module and an injective module. Then R is Noetherian and hereditary.*

Proof. By Corollary 8, every singular R-module is injective, and by Corollary 6, $R/\text{socle}(R)$ is Noetherian. By a result of Chatters [3, Theorem 3.1], it is enough to show that every cyclic module is a direct sum of a projective module and a Noetherian module. This will follow if every cyclic injective module is Noetherian. Let xR be a cyclic injective R-module, and let $S = \text{socle}(R)$. Then xR/xS is a cyclic R/S-module and hence Noetherian. If xS is not of finite length, then it decomposes into a direct sum $\bigoplus_{i=0}^{\infty} X_i$, where each X_i has infinite length. Let E_i be an injective hull of X_i in xR. Then each E_i is cyclic and so not semisimple of infinite length. Then xR/xS contains the infinite direct sum $\bigoplus_{i=0}^{\infty} E_i/X_i$, contradicting the property that xR/xS is Noetherian.

That R is hereditary follows as in [13]. Let E be an injective module and $K \subseteq E$. Then $E = E(K) \oplus L$, where $E(K)$ is an injective hull of K in E, so $E(K)/K$ is singular and hence injective. Thus $E/K \approx E(K)/K \oplus L$ is injective. ▮

We observe that the ring of 2×2 upper triangular matrices over a field and the examples of Cozzens [5] are examples of rings over which every cyclic is either injective or projective. Such rings are Noetherian and hereditary with every singular module injective. On the other hand, the ring of 3×3 upper triangular matrices over a field has a simple module which is neither injective nor projective. This shows that hereditary plus Noetherian is not sufficient to prove that every singular is injective.

We next look at rings all of whose cyclic modules are quasi-continuous. If every finitely generated module is quasi-continuous, we use a lemma of Bouhy and Mohamed (2, Corollary 3.5], our Lemma C) to show that every finitely generated module is injective, and thus the ring is semisimple. However, if only cyclics are quasi-continuous, the situation is quite different. Here we get a partial classification of rings with all cyclics quasi-continuous. The existing characterization of rings with all cyclics quasi-injective which currently is spread out over four papers [21, 1, 17, 16] is presented here since except for the proof of maximality in [16] it follows quickly from the quasi-continuous classification.

The following lemma is in [18], which quotes [2]. Since Ref. [2] is not readily available a short proof is included here.

LEMMA C. *Let M be a quasi-continuous module, $M = A \oplus B$. Then B is A-injective; that is, any homomorphism from a submodule of A to B extends to a map from A to B.*

Proof. Let $f: C \to B$, where $C \subseteq A$. Set $D = \{x - f(x) \mid x \in C\}$. Clearly $D \cap B = 0$. Let E be a complement of B in M such that $E \supseteq D$. Then $M = E \oplus B$ has projection π onto B with kernel E. Since for all $x \in C$, $0 = \pi(x - f(x)) = \pi(x) - f(x)$, $\pi \mid A$ extends f to A. ▮

We remark that this lemma is essentially the difference between a quasi-continuous direct sum and a CS direct sum.

If every finitely generated module is quasi-continuous, then $R \oplus M$ is quasi-continuous for all finitely generated M, so every finitely generated M is (R-)injective.

LEMMA D. *Let M be a completely quasi-continuous module such that $M = \bigoplus_{i \in \mathscr{I}} M_i$, where each M_i is uniform. Then $\forall \varphi \in \hom_R(M_i, M_j)$ with $i \neq j$, $\varphi \neq 0 \Rightarrow \varphi$ is onto. If in addition each M_i is projective, then φ is an isomorphism.*

Proof. Assume $\exists \varphi \in \hom_R(M_i, M_j)$, $\varphi \neq 0$, for some $i \neq j$. Set $K = \text{kernel}(\varphi)$. Then $M_j \oplus M_i/K$ is a quotient of M and so is quasi-continuous. There is an isomorphism ψ: image$(\varphi) \to M_i/K$ defined on a submodule of the uniform module M_j, so by Lemma C, ψ extends to a homomorphism $\bar{\psi}$ from M_j to M_i/K with kernel$(\bar{\psi}) \cap \text{image}(\varphi) = 0$. Thus $\bar{\psi}$ is monic. Since ψ is already onto, we conclude that $M_j = \text{domain}(\bar{\psi}) = \text{image}(\varphi)$. If M_j is projective, φ splits, and since M_i is uniform, kernel $\varphi = 0$. ∎

PROPOSITION 2. *Let R be a ring such that every cyclic module is quasi-continuous. Then R is a ring direct product $\prod_{i=1}^{n} R_i$, where for each i, either R_i is simple Artinian or any two nonzero right ideals of R_i have nonzero intersection. If any R_i is local, then it has linearly ordered right ideals.*

Proof. By Theorem 1, $R_R = \bigoplus_{i=1}^{n} M_i$, where each M_i is uniform. Nonzero homomorphisms between distinct M_i are onto by Lemma D, and since the M_i are projective, homomorphisms between distinct M_i are isomorphisms. Thus if R_i is the direct sum of all the M_j which are isomorphic to M_i, R is the ring direct product of the distinct R_i, and each R_i is a full ring of matrices over a ring $S_i = \hom_{R_i}(M_i, M_i)$. Assume R_i is not uniform. Then there is a $j \neq i$ with $M_i \approx M_j$. Then by Lemma C, M_i is quasi-injective, so S_i is local. Since R_i is Morita equivalent to S_i, R_i has a unique simple module and every quotient of an indecomposable projective is indecomposable. If M_i is not simple, let xR be a proper submodule of M_i and let K be a maximal submodule of xR. Then $M_i/K \oplus M_j$ satisfies the hypotheses of Lemma D, but has a homomorphism between distinct indecomposable direct summands that is not onto. This contradiction shows that M_i must be simple if there is another of the M_j isomorphic to it. Thus each R_i is either simple Artinian or uniform. Now assume a uniform R_i is local. If $x, y \in R_i$, then R_i/I is indecomposable for any right ideal $I \subseteq R_i$, so R_i/I must be uniform. Thus if $x, y \in R_i$, one of $xR_i/xR_i \cap yR_i$, $yR_i/xR_i \cap yR_i$ must be zero; that is, the right ideals of R_i must be linearly ordered. ∎

There is a property of modules between quasi-continuity and quasi-injectivity. A module M is called continuous if it is quasi-continuous and every submodule isomorphic to a direct summand is a direct summand. A continuous uniform module must have local endomorphism ring. We thus can eliminate the semiperfect hypothesis from a result in Jain and Mohamed [14].

COROLLARY 9. *Let R be a ring. Then every cyclic R-module is continuous if and only if R is a finite ring direct product of simple Artinian rings and rings with right ideals linearly ordered and nil Jacobson radical.*

Proof. If every cyclic module is continuous, then by Proposition 2, we need only show that $J(R)$ is nil. Let x be a nonunit of R and set $I=\bigcap_{n=1}^{\infty} x^nR$. Clearly $xI\subseteq I$, and left multiplication by x induces a homomorphism from R/I to R/I. If the kernel of this homomorphism is 0, then since R/I is indecomposable continuous, it is onto. Then there is a $y\in R$ with $1-xy\in I\subseteq J(R)$, contradicting $x\in J(R)$. We conclude that the kernel is nonzero; that is, there is an $r\in R-I$ with $xr\in I$. By linear ordering of right ideals, there is an $n\in\mathbb{N}$ with $x^n\in rR$. Thus for some $s\in R$, $x^n=rs$. Then $x^{n+1}=xrs\in I$, so $x^{n+1}\in I\subseteq x^{n+2}R$. If $x^{n+1}=x^{n+2}z$, then $x^{n+1}(1-xz)=0$, and since $x\in J(R)$, $x^{n+1}=0$.

For the converse, observe that a ring direct product $R=\prod_{i=1}^{n} R_i$ is continuous if and only if each R_i is continuous. Let R_i be a factor which is not simple Artinian. The linear ordering of right ideals ensures that every quotient of R_i is indecomposable and CS. Moreover, if φ: $R_i/I\to R_i/I$ is not onto, then $\varphi(1)$ is nilpotent, so kernel(φ) contains a nonzero power of $\varphi(1)$. Thus φ is not a monomorphism, and the only submodule of R_i/I isomorphic to a direct summand of R_i/I is R_i/I itself. Thus each R_i is continuous. ∎

Remark. If the right (respectively left) ideals of a ring are linearly ordered and the Jacobson radical is nil, then every right (left) ideal is two sided. Indeed, let $r, x\in R$. If $rxR\supset xR$ then there is some $s\in J(R)$ with $0\neq x=rxs=r^nxs^n$ for all n, but $s^n=0$ for some n. We conclude $rxR\subseteq xR$. (Symmetrically, $Rxr\subseteq Rr$.) Thus there is no need to include anything about one-sided ideals being two sided in the hypotheses of Corollaries 9 and 10.

COROLLARY 10 (Ahsan-Koehler). *A self-injective ring with every cyclic quasi-injective is a finite product of rings R_i, each of which is either simple Artinian or a ring with linearly ordered right and left ideals, nil Jacobson radical, and R_i is linearly compact; that is, any family $\{x_\alpha+I_\alpha\}$ of cosets of ideals of R_i which has the intersection of every finite subset nonempty must have nonempty intersection.*

Proof. By Corollary 9, we need only consider the case where R has linearly ordered right ideals and nil Jacobson radical. Since finitely generated left ideals of a self-injective ring must be annihilators, the (finitely generated) left ideals of R must also be linearly ordered. The proof of linear compactness and the converse to Corollary 10 are essentially in [16] except one has to be careful about the sides of annihilators. ∎

We observe that the property in Corollary 9 is not left-right symmetric, whereas that in Corollary 10 is. For example, if F is a field and σ an endomorphism of F which is not onto, the twisted power series ring $F[\![X;\sigma]\!]$ with coefficients written on the right has linearly ordered right ideals which are two sided, namely the ideals generated by the powers of X, so every cyclic right module is quasi-continuous and every proper cyclic right module is even continuous, but cyclic left modules need not be CS (the ring itself and many proper cyclics are not). Any proper quotient ring of $F[\![X;\sigma]\!]$ is completely continuous on the right, but not on the left.

We observe that $\mathbb{Z}_{\mathbb{Z}}$ is a uniform completely quasi-continuous module that does not have a local endomorphism ring. Let p be a prime in $\mathbb{Z}$ and S a multiplicatively closed subset of $\mathbb{Z}$ containing no element of $\{p^i, 0\}$. Let R be the ring $\mathbb{Z}_S \oplus \mathbb{Z}_{p^\infty}$ with $[\mathbb{Z}_{p^\infty}]^2 = 0$. Then for appropriate S, R is not local. It may be semilocal and have its Jacobson radical properly containing its nonzero nil radical. Thus the uniform ring factors in Proposition 2 need not have linearly ordered right ideals even if they are semilocal and nonprime. Exactly what properties they have is an open question.

If some family of cyclics not containing R such as the singular cyclics or the proper cyclics are completely CS or completely quasi-continuous, Proposition 2 fails although Lemma D applies. Theorem 1 may prove useful in these cases, as it has in the completely injective case. A study of rings with proper cyclics quasi-injective is contained in [15], where complete determination of their structure is reduced to the prime, nonlocal case. Rings with all singular modules CS or quasi-continuous remain an open area of study. Even if quasi-injective is replaced by injective, it is not clear how to characterize those simple hereditary Noetherian domains that have every proper cyclic injective.

We conclude by looking at an interesting example. Let F be a field and σ an endomorphism of F. Let $\mathscr{R} = F[X;\sigma]$, that is,

$$\mathscr{R} = \left\{ \sum_{i=0}^{n} X^i \alpha_i \mid n \in \mathbb{N}, \alpha_i \in F, \alpha X = X\sigma(\alpha) \right\}.$$

We look at some effects of properties of F and σ on the CS properties for cyclic $\mathscr{R}$-modules.

If F is algebraically closed of characteristic $p > 0$ and σ is the Frobenius map $\alpha \mapsto \alpha^p$, then $\mathscr{R}$ has precisely two nonisomorphic simples, $\mathscr{R}/X\mathscr{R}$ and $\mathscr{R}/(X-1)\mathscr{R}$, and the latter is injective. It is not difficult to see that every cyclic $\mathscr{R}$-module is either torsion-free or a direct sum of an injective semisimple module and a module $\mathscr{R}/X^i\mathscr{R}$ for some $i \in \mathbb{N}$. Thus every proper cyclic is CS and indeed quasi-injective, although $\mathscr{R}_{\mathscr{R}}$ is not continuous. Localizing with respect to the Öre set of powers of X gives a ring for which every proper cyclic is injective.

If we assume that F is separably closed but not perfect and σ the Frobenius map, then localizing with respect to the Öre set of powers of X gives the same localization as in the previous case. However, the ring $\mathscr{R}$ is not completely CS whenever σ is not onto. Indeed, assume there is a $p(X) = \alpha + X\beta + X^2 \in \mathscr{R}$ with $\alpha \neq 0$ such that $p(X)\mathscr{R} = (X-\alpha)\mathscr{R} \cap (X-b)\mathscr{R}$ and if $(X-c)\mathscr{R} \supset p(X)\mathscr{R}$ then $c \notin \sigma(F)$. Then if $Xp(X) \in (X-d)\mathscr{R}$, where $d \neq 0$, we have $Xp(X) = (X-d)\, X(X-e) = X(X-\sigma(d))(X-e)$, so $p(X) = (X-\sigma(d))(X-e)$, a contradiction. Moreover $Xp(X)$ cannot be equal to $q(X)X$, which has all coefficients in $\sigma(F)$. Thus $\mathscr{R}/Xp(X)\mathscr{R}$ has a unique maximal submodule $X\mathscr{R}/Xp(X)\mathscr{R}$ which is a direct sum of two simple modules, so $\mathscr{R}/Xp(X)\mathscr{R}$ is not CS. To find such a $p(X)$ take a monic generator of $(X-t)\mathscr{R} \cap (X - t\sigma(t))\mathscr{R}$ for any $t \in F$, $t \notin \sigma(F)$. One computes that such a $p(X)$ is of the form $t\sigma(a) + X\sigma(b) + X^2$, where $a = t^2(\sigma(t-1))/(t-1)$ and $b = a + t$. Then if $p(X) = (X-c)(X-d)$, $d + \sigma(c) \in \sigma(F)$, so $d \in \sigma(F)$ and $c \notin \sigma(F)$.

Lest one assume that the bad behavior on the left is crucial in preventing $\mathscr{R}$ from having all cyclic modules CS, we now assume that σ is onto. We may also replace $\mathscr{R}$ by its localization with respect to any multiplicatively closed set $\mathscr{S}$ such that $s\mathscr{R} = \mathscr{R}s$ for all $s \in \mathscr{S}$. We observe that $\mathscr{R}$ is a principal right and left ideal domain so every nonzero element of $\mathscr{R}$ is a product of irreducible polynomials which generate maximal right ideals. Let $U = u\mathscr{R}$ be a uniserial cyclic module with unique composition series $0 = U_0 \subset U_1 \subset U_2 \subset U_3 = U$. For $0 \neq v \in U_2/U_1$, assume that the annihilator of v is not a two-sided ideal. Since U is uniserial, there is a unique maximal right ideal containing the annihilator $(0:u)$ of u. Without loss of generality, that is not the annihilator $(0:v)$ of v. Then

$$\mathscr{R}/(0:u) \cap (0:v) \approx \mathscr{R}/(0:u) \oplus \mathscr{R}/(0:v)$$

has a uniserial cyclic submodule of length 2 not contained in a uniserial of length 3. This submodule can have no proper essential extensions and is not a direct summand, so $\mathscr{R}$ is not completely CS.

Thus if $\mathscr{R}$ is completely CS and $S = s\mathscr{R}$ is any simple $\mathscr{R}$-module with $(0:s) = tR$ not a two-sided ideal, $E(S)/S$ must be (injective) semisimple. Assume that $\mathscr{R}$ is completely CS and S is not divisible by some nonzero irreducible $r \in \mathscr{R}$. $r\mathscr{R}$ cannot be a two-sided ideal since $t\mathscr{R} + \mathscr{R}r \neq \mathscr{R}$. If $s' \notin Sr$, the map $ra \mapsto s'a$ from $r\mathscr{R}$ to S extends to a map φ from $\mathscr{R}$ to $E(S)$ with $\varphi(1) = y$. Then $y\mathscr{R}/S$ is injective (so, for example, $t \neq r$, so $t\mathscr{R} + \mathscr{R}t = \mathscr{R}$). This condition is actually equivalent to every simple other than $\mathscr{R}/X\mathscr{R}$ being injective (see [22]), a condition which implies $\mathscr{R}$ completely CS.

As an illustration of the last discussion, if F is the field with four elements, ϑ a primitive cube root of 1, $\sigma(\alpha) = \alpha^2$, let $\mathscr{M} = \mathscr{R}/(X^4 + X^3 +$

$X^2\vartheta + X + \vartheta^2)\mathscr{R}$ and $p(X) = (X^2 + X\vartheta + \vartheta)$. Then $p(X)\mathscr{R}$ is uniserial, and the only submodule of $\mathscr{M}$ of length 3 containing $p(X)$ is generated by $(X - \vartheta)$ and contains a second simple submodule $\mathscr{T}$ generated by $(X^3 + X^2\vartheta + X + \vartheta)$. If $\mathscr{U}$ is the submodule generated by $(X + 1)$, then $\mathscr{U}$ is uniserial and $\mathscr{M} = \mathscr{U} \oplus \mathscr{T}$, so $p(X)\mathscr{R}$ is not a direct summand by Krull–Schmidt. Thus $\mathscr{M}$ is not CS.

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