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Structures on generalized Sasakian-space-forms

Pablo Alegre [1], Alfonso Carriazo [*,1]

Department of Geometry and Topology, Faculty of Mathematics, University of Sevilla, Apdo. Correos 1160, 41080 Sevilla, Spain

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ABSTRACT

In this paper, contact metric and trans-Sasakian generalized Sasakian-space-forms are deeply studied. We present some general results for manifolds with dimension greater than or equal to 5, and we also pay a special attention to the 3-dimensional cases.

1. Introduction

In [1], the authors (jointly with D.E. Blair) introduced the notion of a *generalized Sasakian-space-form* as an almost contact metric manifold (M,ϕ,ξ,η,g) whose curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &\quad + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1.1}$$

where f_1, f_2, f_3 are differential functions on M. From now on we will also use the notation

$$\begin{aligned} R_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y, \\ R_2(X,Y)Z &= g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z, \\ R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \end{aligned}$$

* Corresponding author.
E-mail addresses: psalerue@upo.es (P. Alegre), carriazo@us.es (A. Carriazo).

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so the Riemann curvature tensor of a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is simply given by:

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3.$$

These spaces can be seen as the almost contact versions of generalized complex-space-forms, i.e., almost Hermitian manifolds with curvature tensor given by

$$R(X,Y)Z = F_1\{g(Y,Z)X - g(X,Z)Y\} + F_2\{g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\},$$

F_1, F_2 being differentiable functions. Such a manifold can be denoted by $N(F_1, F_2)$.

Of course, Sasakian-space-forms appear as natural examples of generalized Sasakian-space-forms, with constant functions $f_1 = (c+3)/4$, $f_2 = f_3 = (c-1)/4$, where c denotes the constant ϕ-sectional curvature. One of the main goals of [1] was to look for non-trivial examples, i.e., generalized Sasakian-space-forms with non-constant functions. In this sense, it was proved in [8] that, if $f_2 = f_3$ is not identically zero, M is connected with $\dim(M) \geqslant 5$, and $g(X, \nabla_X \xi) = 0$ for any vector field X orthogonal to ξ, then f_1 and f_2 are constant functions. Nevertheless, we were able to find interesting examples by using a wide variety of geometric constructions, such as warped products and appropriate changes of metric. Actually, we proved in [1] that we can find generalized Sasakian-space-forms with non-constant functions and arbitrary dimensions. Contrary to this fact, it was shown in [17] that, if F_2 is not identically zero, connected generalized complex-space-forms reduce to complex-space-forms in dimensions greater than or equal to 6.

As well as giving the above mentioned examples, we began in [1] the study of the structure of a generalized Sasakian-space-form. In fact, we were able to show that such a space with a K-contact structure must be a Sasakian manifold, and, if the dimension is greater than or equal to 5, it is a Sasakian-space-form. Moreover, we also gave some properties about the possibility of a generalized Sasakian-space-form to admit a contact metric structure, and we studied Bianchi's identities.

In this paper, we continue the study of generalized Sasakian-space-forms by presenting two new sections, preceded by a preliminaries section containing some background on almost contact metric geometry.

The first one is devoted to the study of contact metric generalized Sasakian-space-forms. We mainly prove that, in dimensions greater than or equal to 5, such a space must be a Sasakian manifold, and then the functions f_1, f_2, f_3 must be constant. To do so, we have to recall the notion of a (κ, μ)-space. With respect to 3-dimensional manifolds, we are also able to prove some interesting results, such as an specific writing of the curvature tensor for either a non-Sasakian contact metric generalized Sasakian-space-form, or an almost contact manifold with a particular Ricci operator, similar to that of η-Einstein manifolds.

Finally, in the last section we study trans-Sasakian generalized Sasakian-space-forms. These spaces are important because most of our examples present such a structure. We begin by studying α-Sasakian generalized Sasakian-space-forms, and we prove that their functions must be constant in dimensions greater than or equal to 5. With respect to β-Kenmotsu ones, we offer some interesting equations relating β with functions f_1, f_2, f_3, which allow us to improve some previous results. Moreover, we prove that any three-dimensional (α, β) trans-Sasakian manifold with α, β depending only on the direction of ξ is a generalized Sasakian-space-form. We also offer a result describing locally any β-Kenmotsu generalized Sasakian-space-form as a warped product of an open interval and a Kaehler manifold.

2. Preliminaries

In this section, we recall some general definitions and basic formulas which we will use later. For more background on almost contact metric manifolds, we recommend the reference [4]. Anyway, we will recall some more specific notions and results in the following sections, when needed.

An odd-dimensional Riemannian manifold (M, g) is said to be an *almost contact metric manifold* if there exist on M a $(1,1)$ tensor field ϕ, a vector field ξ (called the *structure vector field*) and a 1-form η such that $\eta(\xi) = 1$, $\phi^2(X) = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$, for any vector fields X, Y on M. In particular, in an almost contact metric manifold we also have $\phi\xi = 0$ and $\eta \circ \phi = 0$.

Such a manifold is said to be a *contact metric* manifold if $d\eta = \Phi$, were $\Phi(X,Y) = g(X, \phi Y)$ is called the *fundamental 2-form* of M. If, in addition, ξ is a Killing vector field, then M is said to be a *K-contact manifold*. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X \xi = -\phi X$, for any vector field X on M. On the other hand, the almost contact metric structure of M is said to be *normal* if $[\phi,\phi](X,Y) = -2\,d\eta(X,Y)\xi$, for any X, Y, where $[\phi,\phi]$ denotes the Nijenhuis torsion of ϕ, given by $[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. A normal contact metric manifold is called a *Sasakian manifold*. It can be proved that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X,$$

for any X, Y.

In [15], J.A. Oubiña introduced the notion of a trans-Sasakian manifold. An almost contact metric manifold M is a *trans-Sasakian manifold* if there exist two functions α and β on M such that

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.1}$$

for any X, Y on M. If $\beta = 0$, M is said to be an *α-Sasakian manifold*. Sasakian manifolds appear as examples of α-Sasakian manifolds, with $\alpha = 1$. If $\alpha = 0$, M is said to be a *β-Kenmotsu manifold*. Kenmotsu manifolds are particular examples with $\beta = 1$. If both α and β vanish, then M is a *cosymplectic manifold*.

In particular, from (2.1) it is easy to see that the following equations hold for a trans-Sasakian manifold:

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.2}$$

$$\mathrm{d}\eta = \alpha\Phi.$$

Finally, let us point out that all the functions we will refer to during this paper will be differentiable functions on the corresponding manifolds.

3. Contact metric generalized Sasakian-space-forms

In [1] we began the study of the structure of a generalized Sasakian-space-form. In fact, we were able to prove that, in dimensions greater than or equal to 5, any connected generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with a K-contact (in particular, Sasakian) structure must be a Sasakian-space-form. Moreover, this is also true if M is a contact metric manifold such that $f_3 = f_1 - 1$. Actually, we also proved in [1, Theorem 3.10] that in any contact metric generalized Sasakian-space-form, $f_1 - f_3$ is a constant. But, the general question remained still open: is it possible to find interesting examples (i.e., with non-constant functions) of contact metric generalized Sasakian-space-forms non-satisfying the above conditions?

To look for an answer to this question, we first have to recall that, given $\kappa, \mu \in \mathbf{R}$, a contact metric manifold is said to be a *(κ, μ)-space* ([5]) if

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{3.1}$$

for any vector fields X, Y on M, where $hX = 1/2(L_\xi\phi)X$, L being the usual Lie derivative. In fact, the name (κ, μ)-space was introduced by E. Boeckx in [7]. This notion was later generalized by R. Sharma in [16], by studying contact metric manifolds satisfying (3.1) with κ, μ being differentiable functions. Then, these manifolds were named *generalized (κ, μ)-spaces*. Later, T. Koufogiorgos and C. Tsichlias showed in [13, Theorem 3.5] that, in dimensions greater than or equal to 5, the functions κ, μ must be constant. We also need to recall Theorem 1 of [5], which we slightly adapt to our current notation:

Theorem 3.1. *(See [5].) Let (M, ϕ, ξ, η, g) be a (κ, μ)-space. Then, $\kappa \leqslant 1$. If $\kappa = 1$, then $h = 0$ and M is a Sasakian manifold. If $\kappa < 1$, M admits three mutually orthogonal and integrable distributions $\mathcal{D}(0)$, $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ determined by the eigenspaces of h, where $\lambda = \sqrt{1-\kappa}$. Moreover,*

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = (\kappa - \mu)\{g(\phi Y_\lambda, Z_{-\lambda})\phi X_\lambda - g(\phi X_\lambda, Z_{-\lambda})\phi Y_\lambda\}, \tag{3.2}$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = (\kappa - \mu)\{g(\phi Y_{-\lambda}, Z_\lambda)\phi X_{-\lambda} - g(\phi X_{-\lambda}, Z_\lambda)\phi Y_{-\lambda}\}, \tag{3.3}$$

$$R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = \kappa g(\phi X_\lambda, Z_{-\lambda})\phi Y_{-\lambda} + \mu g(\phi X_\lambda, Y_{-\lambda})\phi Z_{-\lambda}, \tag{3.4}$$

$$R(X_\lambda, Y_{-\lambda})Z_\lambda = -\kappa g(\phi Y_{-\lambda}, Z_\lambda)\phi X_\lambda - \mu g(\phi Y_{-\lambda}, X_\lambda)\phi Z_\lambda, \tag{3.5}$$

$$R(X_\lambda, Y_\lambda)Z_\lambda = (2(1+\lambda) - \mu)\{g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda\}, \tag{3.6}$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (2(1-\lambda) - \mu)\{g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}\}, \tag{3.7}$$

where $X_\lambda, Y_\lambda, Z_\lambda \in \mathcal{D}(\lambda)$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in \mathcal{D}(-\lambda)$.

The relationship between (κ, μ)-spaces and generalized Sasakian-space-forms is established by the following result, which can be proved by computing $R(X, Y)\xi$ from (1.1):

Proposition 3.2. *Any contact metric generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is a generalized (κ, μ)-space with $\kappa = f_1 - f_3$ and $\mu = 0$.*

Moreover, by virtue of Theorem 3.1 and either Theorem 3.10 from [1] or the above mentioned Theorem 3.5 from [13], we can state this proposition:

Proposition 3.3. *If $M(f_1, f_2, f_3)$ is a non-Sasakian contact metric generalized Sasakian-space-form, with dimension greater than or equal to 5, then $\kappa = f_1 - f_3 < 1$ is constant on M.*

Therefore, by virtue of Propositions 3.2 and 3.3, we already know that any non-Sasakian contact metric generalized Sasakian-space-form, with dimension greater than or equal to 5, is a $(\kappa, 0)$-space with $\kappa < 1$. We can now use Theorem 3.1 to obtain the following results:

Lemma 3.4. *If $M(f_1, f_2, f_3)$ is a non-Sasakian contact metric generalized Sasakian-space-form, with dimension greater than or equal to 5, then $f_2 = \kappa = 0$.*

Proof. From (3.2) we have

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = \kappa\{g(\phi Y_\lambda, Z_{-\lambda})\phi X_\lambda - g(\phi X_\lambda, Z_{-\lambda})\phi Y_\lambda\}, \tag{3.8}$$

for any $X_\lambda, Y_\lambda \in \mathcal{D}(\lambda)$ and $Z_{-\lambda} \in \mathcal{D}(-\lambda)$. But, from (1.1) we get

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = f_2\{g(\phi Y_\lambda, Z_{-\lambda})\phi X_\lambda - g(\phi X_\lambda, Z_{-\lambda})\phi Y_\lambda\}, \tag{3.9}$$

since X_λ and Y_λ are orthogonal to $Z_{-\lambda}$, and all of them are orthogonal to ξ. We have also taken into account that $g(X, \phi Y_\lambda) = 0$, given that, if $Y_\lambda \in \mathcal{D}(\lambda)$, then $\phi Y_\lambda \in \mathcal{D}(-\lambda)$. As we are working on a manifold with dimension greater than or equal to 5, we can choose X_λ such that it is orthogonal to Y_λ. Hence, if we put $Z_{-\lambda} = \phi X_\lambda$, from (3.8) and (3.9) we deduce $f_2 = \kappa$.

Let us now choose two unit and mutually orthogonal vector fields $X_\lambda, Y_\lambda \in \mathcal{D}(\lambda)$. Then, $\phi X_\lambda \in \mathcal{D}(-\lambda)$ and, from (3.5), we have

$$R(X_\lambda, \phi X_\lambda)Y_\lambda = -\kappa g(\phi^2 X_\lambda, Y_\lambda)\phi X_\lambda = \kappa g(X_\lambda, Y_\lambda)\phi X_\lambda = 0,$$

and so:

$$R(X_\lambda, \phi X_\lambda, Y_\lambda, \phi Y_\lambda) = 0. \tag{3.10}$$

On the other hand, from (1.1) we get

$$R(X_\lambda, \phi X_\lambda, Y_\lambda, \phi Y_\lambda) = -2f_2. \tag{3.11}$$

Thus, from (3.10) and (3.11) we deduce $f_2 = 0$. □

Theorem 3.5. *Any contact metric generalized Sasakian-space-form $M(f_1, f_2, f_3)$, with dimension greater than or equal to 5, is a Sasakian manifold. Therefore, if M is connected, then f_1, f_2 and f_3 must be constant functions.*

Proof. If M is a non-Sasakian contact metric manifold, then, by virtue of Proposition 3.3 and Lemma 3.4, we obtain $f_2 = 0$ and $f_1 = f_3$. Another consequence is that M must be a $(0, 0)$-space, and so Eq. (3.4) looks like

$$R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = 0, \tag{3.12}$$

for any $X_\lambda \in \mathcal{D}(\lambda)$, $Y_{-\lambda}, Z_{-\lambda} \in \mathcal{D}(-\lambda)$. But, from (1.1) and Lemma 3.4, we have

$$R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = f_1 g(Y_{-\lambda}, Z_{-\lambda})X_\lambda. \tag{3.13}$$

If we now take $Y_{-\lambda} = Z_{-\lambda}$ and we compare (3.12) and (3.13), we deduce $f_1 = 0$. As all three functions vanish, it follows from (1.1) that a non-Sasakian contact metric generalized Sasakian-space-form, with dimension greater than or equal to 5, should be a flat manifold. But, as a contact metric manifold with such a dimension cannot be flat, we deduce that M is a Sasakian manifold, and the proof concludes. □

Hence, the answer to our question in dimensions greater than or equal to 5 is negative. Now, what can we say about 3-dimensional contact metric generalized Sasakian-space-forms? First, let us mention that the writing of the curvature tensor of a 3-dimensional generalized Sasakian-space-form is not unique. Actually, if M^3 is an almost contact metric manifold such that its curvature tensor can be simultaneously written as

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3 \tag{3.14}$$

and

$$R = f_1^* R_1 + f_2^* R_2 + f_3^* R_3, \tag{3.15}$$

then the functions f_i and f_i^* are related as follows,

$$f_1^* = f_1 + f, \qquad f_2^* = f_2 - f/3, \qquad f_3^* = f_3 + f, \tag{3.16}$$

where f is a function on M. To prove this fact it is enough to consider a ϕ-basis $\{X, \phi X, \xi\}$ and to calculate $R(X, \xi; \xi, X)$ and $R(X, \phi X; X, \phi X)$ by using both (3.14) and (3.15). Therefore, we obtain the system

$$\begin{cases} (f_1^* - f_1) - (f_3^* - f_3) = 0, \\ (f_1^* - f_1) + 3(f_2^* - f_2) = 0, \end{cases}$$

which general solution is given by (3.16).

Conversely, if $M^3(f_1, f_2, f_3)$ is a generalized Sasakian-space-form and we define the functions f_i^* as above, for any function f on M, then it is also a generalized Sasakian-space-form $M^3(f_1^*, f_2^*, f_3^*)$. This can be proved by computing from (3.14) and (3.16)

$$R = \sum_{i=1}^{3} f_i R_i = \sum_{i=1}^{3} f_i^* R_i + (-f R_1 + f/3 R_2 - f R_3).$$

But it is easy to check that the last term vanishes and so we obtain (3.15).

Therefore, in order to consider an unique writing of the curvature tensor of a three-dimensional generalized Sasakian-space-form, we will choose that satisfying $f_2^* = 0$.

With respect to three-dimensional contact metric manifolds, D.E. Blair, T. Koufogiorgos and S. Sharma proved in [6] that, for such a manifold (M, ϕ, ξ, η, g), the following conditions are equivalent: i) M is η-Einstein, ii) $Q\phi = \phi Q$, and iii) M is a $(\kappa, 0)$-space, for a certain constant κ. Let us recall that Q denotes the Ricci operator and that an η-Einstein manifold is a contact metric manifold such that

$$Q = aI + b\eta \otimes \xi,$$

where a, b are differentiable functions on M. All these conditions hold on a three-dimensional contact metric generalized Sasakian-space-form $M(f_1, f_2, f_3)$. Actually, condition iii) was really proved in [1, Theorem 3.10], no matter what the dimension of M was, with $\kappa = f_1 - f_3$. On the other hand, by taking into account that the Riemann curvature tensor of a three-dimensional manifold is given by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y - 3\tau\big(g(Y,Z)X - g(X,Z)Y\big), \tag{3.17}$$

where τ denotes the scalar curvature, the computation of $R(X,Y)\xi$ from (1.1) and (3.17) proves that

$$QX = (3\tau - f_1 + f_3)X - 3(\tau - f_1 + f_3)\eta(X)\xi, \tag{3.18}$$

for any vector field X on M, and so M is an η-Einstein manifold with $a = (3\tau - f_1 + f_3)$ and $b = -3(\tau - f_1 + f_3)$. Finally, condition ii) holds for a generalized Sasakian-space-form of any dimension:

Proposition 3.6. *Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-form. Then, $Q\phi = \phi Q$, where Q denotes the Ricci operator on M.*

Proof. An straightforward computation with respect to a ϕ-basis gives that

$$QX = 2mf_1 X + 3f_2(X - \eta(X)\xi) - f_3(X + (2m-1)\eta(X)\xi),$$

for any vector field X on M, where $\dim M = 2m + 1$. Therefore, it is easy to check that

$$Q\phi X = 2mf_1\phi X + 3f_2\phi X - f_3\phi X = \phi QX,$$

and the proof concludes. □

Moreover, it was also proved in [6] that, if (M, ϕ, ξ, η, g) is a three-dimensional contact metric manifold such that $Q\phi = \phi Q$, then M is a Sasakian manifold, a flat manifold, or a space of constant ξ-sectional curvature $\kappa < 1$ and constant ϕ-sectional curvature $-\kappa$. With this in mind, we obtain the following result, which establishes a relationship between the functions of a three-dimensional contact metric generalized Sasakian-space-form:

Proposition 3.7. *Let $M(f_1, f_2, f_3)$ be a three-dimensional, non-Sasakian, contact metric generalized Sasakian-space-form. Then, the following equation holds*:

$$2f_1 + 3f_2 - f_3 = 0. \tag{3.19}$$

Proof. First of all, let us point out that the ξ-sectional curvature of a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is given by $f_1 - f_3$, while its ϕ-sectional curvature is equal to $f_1 + 3f_2$ (see [1, Proposition 3.11]). Now, by virtue of Proposition 3.6 and the above recalled result from [6], it follows that one of the following cases must hold:

Case 1. M is a flat manifold. In such a case, both the ξ-sectional and the ϕ-sectional curvatures of M vanish, and so

$$f_1 + 3f_2 = 0 = -(f_1 - f_3). \tag{3.20}$$

Case 2. M is a space of constant ξ-sectional curvature $\kappa < 1$ and constant ϕ-sectional curvature $-\kappa$. In such a case, $f_1 + 3f_2 = -\kappa$ and $f_1 - f_3 = \kappa$, and so

$$f_1 + 3f_2 = -(f_1 - f_3). \tag{3.21}$$

Finally, both (3.20) and (3.21) imply (3.19). □

Of course, Eq. (3.19) is consistent with (3.16), i.e., if f_i, $i = 1, 2, 3$, satisfy (3.19), then f_i^* also do, as can be trivially checked. On the other hand, we can now improve (3.18) in the non-Sasakian case:

Proposition 3.8. *Let $M(f_1, f_2, f_3)$ be a three-dimensional, non-Sasakian, contact metric generalized Sasakian-space-form. Then, the Ricci operator of M is given by*

$$Q = 2\kappa\eta\otimes\xi, \tag{3.22}$$

and its scalar curvature satisfies

$$\tau = \frac{\kappa}{3}, \tag{3.23}$$

where $\kappa = f_1 - f_3 < 1$ is a constant.

Proof. To compute the scalar curvature of M, it is enough to work with respect to a ϕ-basis and to take into account the values of its ξ-sectional and ϕ-sectional curvatures. Hence, we have

$$6\tau = 4(f_1 - f_3) + 2(f_1 + 3f_2),$$

which, by virtue of (3.19), implies

$$3\tau = f_1 - f_3. \tag{3.24}$$

Therefore, from (3.18) and (3.24), it follows

$$QX = 2(f_1 - f_3)\eta(X)\xi.$$

But we already know that, in the conditions of the statement, $f_1 - f_3$ is a certain constant $\kappa < 1$. □

As a consequence, we also obtain for a manifold in the above conditions the writing of its curvature tensor with $f_2^* = 0$:

Theorem 3.9. *Let $M(f_1, f_2, f_3)$ be a three-dimensional, non-Sasakian, contact metric generalized Sasakian-space-form. Then, we can write*

$$R = -\kappa R_1 - 2\kappa R_3, \tag{3.25}$$

where $\kappa = f_1 - f_3 < 1$ is a constant.

Proof. Eq. (3.25) follows from (3.17), (3.22) and (3.23) by a straightforward computation. □

Therefore, according to the above theorem, any three-dimensional, non-Sasakian, contact metric generalized Sasakian-space-form admits a writing of its curvature tensor with constant functions. An explicit example of such a space can be provided just by considering $\mathbf{R}^3$ or the torus T^3 with $\eta = 1/2(\cos z\,dx + \sin z\,dy)$ and $g_{ij} = (1/4)\delta_{ij}$, which is an η-Einstein (non-Sasakian) contact metric manifold [6]. As g is a flat metric, this manifold is a generalized Sasakian-space-form, with vanishing functions.

To finish this section, let us show a kind of converse of the fact of a three-dimensional contact metric generalized Sasakian-space-form being an η-Einstein manifold:

Theorem 3.10. *Let M be a three-dimensional almost contact metric manifold with*

$$Q = aI + b\eta\otimes\xi, \tag{3.26}$$

where a, b are differentiable functions on M. Then, it is a generalized Sasakian-space-form $M(f_1^*, f_2^*, f_3^*)$ with $f_1^* = 2a - 3\tau$, $f_2^* = 0$ and $f_3^* = -b$.

Proof. This result can be obtained from (3.17) and (3.26) by a direct calculation. □

Let us notice that, if M is already a non-Sasakian, contact metric generalized Sasakian-space-form $M(f_1, f_2, f_3)$, it follows from Proposition 3.8 and Theorem 3.10 that it should also be a generalized Sasakian-space-form $M(f_1^*, f_2^*, f_3^*)$ with

$$f_1^* = -(f_1 - f_3), \qquad f_2^* = 0, \qquad f_3^* = -2(f_1 - f_3).$$

But, by virtue of (3.19), we have

$$f_1^* = f_1 + 3f_2, \qquad f_2^* = 0, \qquad f_3^* = f_3 + 3f_2.$$

Hence, the change from f_i to f_i^* operated by Theorem 3.10 in this case is just that of (3.16) with $f = 3f_2$ in order to obtain $f_2^* = 0$.

4. Trans-Sasakian generalized Sasakian-space-forms

As the main examples obtained in [1] and in [2] are trans-Sasakian manifolds, it is interesting to study generalized Sasakian-space-forms with such a structure. On the other hand, Marrero showed in [14] that a trans-Sasakian manifold of dimension greater than or equal to 5 is either α-Sasakian, β-Kenmotsu or cosymplectic. Actually, cosymplectic manifolds can be seen as α-Sasakian (resp. β-Kenmotsu) ones with $\alpha = 0$ (resp. $\beta = 0$). Therefore, in this section we will separately study either α-Sasakian or β-Kenmotsu generalized Sasakian-space-forms.

With respect to the first ones, we have:

Proposition 4.1. *Let $M(f_1, f_2, f_3)$ be an α-Sasakian generalized Sasakian-space-form. Then α does not depend on the direction of ξ and the following equation holds*:

$$f_1 - f_3 = \alpha^2. \tag{4.1}$$

Moreover, if M is connected, then α is a constant.

Proof. From (2.1) and (2.3), it follows

$$R(X,\xi)\xi = \xi(\alpha)\phi X + \alpha^2 X, \tag{4.2}$$

$$R(X,\phi X)\xi = X(\alpha)X + \phi X(\alpha)\phi X, \tag{4.3}$$

for any unit vector field X on M, orthogonal to ξ. But, from (1.1) we obtain:

$$R(X,\xi)\xi = (f_1 - f_3)X, \tag{4.4}$$

$$R(X,\phi X)\xi = 0. \tag{4.5}$$

Hence, from (4.2) and (4.4), we deduce (4.1) and $\xi(\alpha) = 0$. Similarly, from (4.3) and (4.5) we have $X(\alpha) = 0$, for any unit vector field X, orthogonal to ξ. Therefore, if M is connect, then α must be constant. □

To prove the following theorem, we need to recall some formulae from [1]. There, by working with Bianchi's second identity for a generalized Sasakian-space-form, we obtained the following equations, where X and Y are unit vector fields, orthogonal to ξ, and such that $g(X,Y) = g(X,\phi Y) = 0$:

$$Y(f_1) - 3f_2 g\big(\phi Y, (\nabla_X \phi)X\big) = 0, \tag{4.6}$$

$$-2X(f_2) + 3f_2\big\{g\big(\phi Y, (\nabla_Y \phi)X\big) + g\big(X, (\nabla_{\phi Y}\phi)Y\big)\big\} = 0, \tag{4.7}$$

$$\xi(f_1) + f_3\big\{g(X, \nabla_X \xi) + g(\nabla_Y \xi, Y)\big\} = 0, \tag{4.8}$$

$$f_2 g\big(\xi, (\nabla_X \phi)X\big) + f_3 g(\nabla_Y \xi, \phi Y) = 0. \tag{4.9}$$

By a similar way as we did in [1], we can also obtain:

$$\xi(f_2) + f_2\big\{g\big(\xi, (\nabla_X \phi)\phi X\big) + g\big(\xi, (\nabla_Y \phi)\phi Y\big)\big\} = 0. \tag{4.10}$$

Now, we can give some important information about functions f_1, f_2, f_3:

Theorem 4.2. *Let $M(f_1, f_2, f_3)$ be a connected α-Sasakian generalized Sasakian-space-form, with dimension greater than or equal to 5. Then, f_1, f_2, f_3 are constant functions, related as follows*:

i) *If $\alpha = 0$, then $f_1 = f_2 = f_3$ and M is a cosymplectic manifold of constant ϕ-sectional curvature.*
ii) *If $\alpha \neq 0$, then $f_1 - \alpha^2 = f_2 = f_3$.*

Proof. By putting (2.1) for an α-Sasakian manifold in both (4.6) and (4.7), we get $Y(f_1) = X(f_2) = 0$, for any unit vector fields X, Y, orthogonal to ξ. But, from (2.3) with $\beta = 0$, (4.8) and (4.10), we deduce $\xi(f_1) = \xi(f_2) = 0$. Therefore, f_1 and f_2 must be constant functions. On the other hand, from (2.1), (2.3) and (4.9), we find

$$(f_2 - f_3)\alpha = 0. \tag{4.11}$$

Now, by virtue of Proposition 4.1, we know that α is a constant. If $\alpha = 0$, (4.1) implies $f_1 = f_3$, and so f_3 is also a constant. But, as we have already pointed out, the ϕ-sectional curvature of M is given by $f_1 + 3f_2$, which is constant. Hence, M must be a cosymplectic manifold of constant ϕ-sectional curvature c. In such a case, it is well-known that $f_1 = f_2 = f_3 = c/4$, and the statement i) holds. To prove statement ii), we just have to take into account that, if $\alpha \neq 0$, then (4.11) implies $f_3 = f_2$, which is a constant function. The rest of the equality follows directly from (4.1). □

Let us notice that statement i) of Theorem 4.2 improves Corollary 4.11 from [1]. On the other hand, an example of statement ii) above is that of Sasakian-space-forms ($\alpha = 1$). In such a manifold, $f_1 = (c+3)/4$ and $f_2 = f_3 = (c-1)/4 = f_1 - 1$, where c denotes the constant ϕ-sectional curvature. Moreover, we can give some nice examples of α-Sasakian generalized Sasakian-space-forms, with $\alpha \neq 0, 1$, from the theory of slant submanifolds in Sasakian manifolds (for some background on that theory, see [9,10]). Actually, for any $\theta \in (0, \pi/2)$, the immersion

$$x(u, v, w, s, t) = 2(u, 0, w, 0, v\cos\theta, v\sin\theta, s\cos\theta, s\sin\theta, t)$$

defines a 5-dimensional minimal θ-slant submanifold M in the Sasakian-space-form $(\mathbf{R}^9(-3), \phi_0, \xi, \eta, g)$. Now, we consider on M the induced almost contact metric structure (ϕ, ξ, η, g), where $\phi = (\sec\theta)T$, T being the tangential component of ϕ_0. It can be checked that $(\nabla_X\phi)Y = \cos\theta(g(X,Y)\xi - \eta(Y)X)$, for any vector fields X, Y tangent to M, which means that M is an α-Sasakian manifold with $\alpha = \cos\theta \in (0, 1)$. Finally, by working with Gauss' equation and by taking into account that M is a totally contact geodesic submanifold, it can be proved that M is a generalized Sasakian-space-form with constant functions $f_1 = 0$ and $f_2 = f_3 = -\cos^2\theta$. For more details and other examples, we refer to [3].

With respect to β-Kenmotsu generalized Sasakian-space-forms, we can first prove the following result, which can be seen as the β-Kenmotsu version of Proposition 4.1:

Proposition 4.3. *Let $M(f_1, f_2, f_3)$ be a β-Kenmotsu generalized Sasakian-space-form. Then, β depends only on the direction of ξ and the functions f_1, f_3 and β satisfy the equation*:

$$f_1 - f_3 + \xi(\beta) + \beta^2 = 0. \tag{4.12}$$

Proof. From (2.1) and (2.3), it follows

$$R(X,\xi)\xi = \left(-\xi(\beta) - \beta^2\right)X, \tag{4.13}$$

$$R(X,\phi X)\xi = X(\beta)\phi X - \phi X(\beta)X, \tag{4.14}$$

for any unit vector field X on M, orthogonal to ξ. But, from (1.1) we obtained (4.4) and (4.5) above. Hence, from (4.4) and (4.13), we deduce (4.12). Similarly, from (4.5) and (4.14) we have $X(\beta) = 0$, for any unit vector field X, orthogonal to ξ. Therefore, β depends only on the direction of ξ. □

But, if we look for a result (more or less) similar to Theorem 4.2, we can recall Theorem 4.10 from [1], which we adapt to our current notation:

Theorem 4.4. *(See [1].) Let $M(f_1, f_2, f_3)$ be a β-Kenmotsu generalized Sasakian-space-form, with dimension greater than or equal to* 5. *Then, f_1, f_2, f_3 depend only on the direction of ξ and the following equations hold*:

$$\xi(f_1) + 2\beta f_3 = 0, \tag{4.15}$$

$$\xi(f_2) + 2\beta f_2 = 0. \tag{4.16}$$

We also pointed out in [1] how, by integrating with respect to t in the above equations, we deduce that, locally,

$$f_1 = \tilde{F}_1 - 2\int \beta f_3\,dt, \qquad f_2 = \tilde{F}_2 e^{-2\int \beta\,dt},$$

where $\partial\tilde{F}_1/\partial t = \partial\tilde{F}_2/\partial t = 0$.

By virtue of Proposition 4.3 and Theorem 4.4, we can deduce:

Corollary 4.5. *Let $M(f_1, f_2, f_3)$ be a β-Kenmotsu generalized Sasakian-space-form, with dimension greater than or equal to* 5, *and such that $f_1 + \beta^2 > 0$. Then, the following equation holds*:

$$2\beta = -\xi\left(\log(f_1 + \beta^2)\right). \tag{4.17}$$

Proof. Eq. (4.17) can be directly obtained from (4.12) and (4.15). □

On the other hand, let us notice that Eq. (4.12) is the same equation we imposed as a condition in [2, Theorem 3.1]. Moreover, the other condition on that theorem about β depending only on the direction of ξ is always satisfied, by virtue of Proposition 4.3. Therefore, this proposition allows as to improve [2, Theorem 3.1] as follows:

Theorem 4.6. *Let $M(f_1, f_2, f_3)$ be a β-Kenmotsu generalized Sasakian-space-form. If we consider a generalized D-conformal deformation with a, b depending only on the direction of ξ, such that $b > 0$ and $a \neq 0$ at any point of M, then we obtain a generalized Sasakian-space-form $M(f_1^*, f_2^*, f_3^*)$, with functions*:

$$f_1^* = \frac{1}{b}\left(f_1 + \frac{a^2-b}{a^2}\beta^2 - \frac{\beta}{a^2}\xi(b) - \frac{\xi(b)^2}{4a^2b}\right), \qquad f_2^* = \frac{1}{b}f_2,$$
$$f_3^* = \frac{1}{b}\left(f_3 - \frac{a^2-b}{a^2}\xi(\beta) - \frac{b}{a^3}\beta\xi(a) - \frac{\xi(a)\xi(b)}{2a^3} + \frac{\xi^2(b)}{2a^2} - \frac{\xi(b)^2}{2a^2b}\right).$$

If we now work on dimension 3, we can consider an (α,β) trans-Sasakian manifold M, with α,β depending only on the direction of ξ. Given any vector field W on M, for any other unit vector field Z_i, orthogonal to ξ, a direct calculation from (2.3) shows:

$$R(\xi,Z_i,Z_i,W) = R(Z_i,W,\xi,Z_i) = \left(\alpha^2 - \xi(\beta) - \beta^2\right)\eta(W).$$

Thus, if we choose a local orthonormal frame $\{Z_1,Z_2,\xi\}$, we can directly compute

$$Q\xi = 2\left(\alpha^2 - \xi(\beta) - \beta^2\right)\xi,$$

and

$$\eta(QX) = g(Q\xi,X) = 2\left(\alpha^2 - \xi(\beta) - \beta^2\right)\eta(X),$$

for any vector field X. Therefore, if we now compute $R(X,Y)\xi$ from (3.17), we get:

$$R(X,Y)\xi = \eta(Y)QX - \eta(X)QY + \left(2\left(\alpha^2 - \xi(\beta) - \beta^2\right) - 3\tau\right)\left\{\eta(Y)X - \eta(X)Y\right\}. \tag{4.18}$$

On the other hand, a direct calculation from (2.1) and (2.3) gives:

$$R(X,Y)\xi = \left(\alpha^2 - \xi(\beta) - \beta^2\right)\left\{\eta(Y)X - \eta(X)Y\right\} + \left(\xi(\alpha) + 2\alpha\beta\right)\left\{\eta(Y)\phi X - \eta(X)\phi Y\right\}. \tag{4.19}$$

Therefore, if we compare (4.18) and (4.19), we can deduce

$$QX = \left(3\tau - \alpha^2 + \xi(\beta) + \beta^2\right)X + \left(\xi(\alpha) + 2\alpha\beta\right)\phi X - 3\left(\tau - \alpha^2 + \xi(\beta) + \beta^2\right)\eta(X)\xi, \tag{4.20}$$

for any vector field X. Hence, if we put (4.20) in (3.17), we obtain

$$\begin{aligned} R(X,Y)Z = {} & (3\tau - 2A)\left\{g(Y,Z)X - g(X,Z)Y\right\} \\ & + B\left\{g(X,\phi Z)Y - g(Y,\phi Z)X + g(Y,Z)\phi X - g(X,Z)\phi Y\right\} \\ & + 3(\tau - A)\left\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\right\}, \end{aligned} \tag{4.21}$$

where $A = \alpha^2 - \xi(\beta) - \beta^2$ and $B = \xi(\alpha) + 2\alpha\beta$. But, by virtue of [11, Theorem 3.2], $B = 0$ in any trans-Sasakian manifold. Therefore, we have:

Theorem 4.7. *Let M be a three-dimensional (α,β) trans-Sasakian manifold such that α,β depend only on the direction of ξ. Then M is a generalized Sasakian-space-form $M(f_1,f_2,f_3)$ with functions*:

$$f_1 = 3\tau - 2\left(\alpha^2 - \xi(\beta) - \beta^2\right), \qquad f_2 = 0, \qquad f_3 = 3\tau - 3\left(\alpha^2 - \xi(\beta) - \beta^2\right). \tag{4.22}$$

In particular:

i) *If $\alpha = 0$, then M is a generalized Sasakian-space-form $M(f_1,f_2,f_3)$ with functions*:

$$f_1 = 3\tau + 2\left(\xi(\beta) + \beta^2\right), \qquad f_2 = 0, \qquad f_3 = 3\tau + 3\left(\xi(\beta) + \beta^2\right). \tag{4.23}$$

ii) *If $\alpha \neq 0$ at any point of M and $\beta = -\xi(\alpha)/(2\alpha)$, then M is a generalized Sasakian-space-form $M(f_1,f_2,f_3)$ with functions*:

$$f_1 = 3\tau - 2\left(\alpha^2 + \frac{\xi(\xi(\alpha))}{2\alpha} - \frac{3\xi(\alpha)^2}{4\alpha^2}\right), \qquad f_2 = 0, \qquad f_3 = 3\tau - 3\left(\alpha^2 + \frac{\xi(\xi(\alpha))}{2\alpha} - \frac{3\xi(\alpha)^2}{4\alpha^2}\right). \tag{4.24}$$

Proof. Since $B = 0$ in (4.21), we directly get $R = f_1R_1 + f_3R_3$, where f_1 and f_3 are given by (4.22). Of course, (4.23) and (4.24) follow directly from (4.22). $\square$

Let us notice how the writings we have obtained for the curvature tensor in the previous theorem are those with $f_2 = 0$.

Statement i) of Theorem 4.7 implies that any three-dimensional β-Kenmotsu manifold with β depending only on the direction of ξ is a generalized Sasakian-space-form. Actually, this fact can be seen as a three-dimensional converse of Proposition 4.3, which stated that, given a β-Kenmotsu manifold, if it is a generalized Sasakian-space-form, then β depends only on the direction of ξ.

Moreover, it is easy to check that the functions f_1, f_3 in (4.23) and β satisfy Eq. (4.12). By virtue of that equation, we can also notice that, for a β-Kenmotsu manifold, (4.20) looks like (3.18), which we obtained for a contact metric three-dimensional generalized Sasakian-space-form.

With respect to statement ii) of Theorem 4.7, let us point out that, if $\alpha \neq 0$ at any point of M, then we can write

$$\alpha = \frac{1}{\sigma}, \quad \beta = \frac{\xi(\sigma)}{2\sigma},$$

where σ is a function on M. In fact, we do have examples of three-dimensional trans-Sasakian manifolds with these non-zero α, β. In [14], J.C. Marrero considered a (M, ϕ, ξ, η, g) three-dimensional Sasakian manifold and the generalized D-conformal deformation

$$g^* = \sigma g + (1-\sigma)\eta \otimes \eta,$$

σ being a positive function on M such that $\xi(\sigma) \neq 0$, to obtain an (α,β) trans-Sasakian manifold $(M, \phi, \xi, \eta, g^*)$ with the previous values for α, β (actually, an explicit example of this construction is given in [14] by using the Heisenberg group). Therefore, if we choose σ such that it only depends on the direction of ξ, then, by virtue of Theorem 4.7, we obtain that $(M, \phi, \xi, \eta, g^*)$ is a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with functions:

$$f_1 = 3\tau - \frac{2}{\sigma^2}\left(1 - \frac{\xi(\xi(\sigma))\sigma}{2} + \frac{\xi(\sigma)^2}{4}\right), \qquad f_2 = 0, \qquad f_3 = 3\tau - \frac{3}{\sigma^2}\left(1 - \frac{\xi(\xi(\sigma))\sigma}{2} + \frac{\xi(\sigma)^2}{4}\right). \tag{4.25}$$

In [1], we were able to find interesting examples of β-Kenmotsu generalized Sasakian-space-forms by considering warped products of the real line **R** and complex-space-forms. To finish this section, we can offer a kind of local converse of this fact, by proving that any β-Kenmotsu generalized Sasakian-space-form is, locally, a warped product of an interval and a Kaehler manifold. First, we need to recall a theorem from [12]:

Theorem 4.8. *(See [12].) Let M be a Kenmotsu manifold. Then, for any point $p \in M$, there exists an open neighbourhood $U(p)$ such that $U = (-\varepsilon, \varepsilon) \times_F V$, where $(-\varepsilon, \varepsilon)$ is an open interval, V is a Kaehler manifold and $F(t) = ce^t$.*

The following theorem was already proved by J.C. Marrero in [14]. Here, we offer a slightly different proof, and we are able to determine the warping function f:

Theorem 4.9. *Let M be a β-Kenmotsu manifold such that β depends only on the direction of ξ. Then, for any point $p \in M$, there exists an open neighbourhood $U(p)$ such that $U = (-\varepsilon, \varepsilon) \times_f V$, where $(-\varepsilon, \varepsilon)$ is an open interval, V is a Kaehler manifold and $f = ce^{\int \beta\, dt}$.*

Proof. We first look for a generalized D-conformal deformation on M in order to obtain a Kenmotsu manifold. Hence, if we consider

$$g^* = bg + (1-b)\eta \otimes \eta, \tag{4.26}$$

b being a function depending only on the direction of ξ, by applying [2, Proposition 3.3] we know that the resulting manifold M^* is β^*-Kenmotsu, where $\beta^* = \beta + \xi(b)/(2b)$. Then, if we want it to be a Kenmotsu manifold, it is enough to take b satisfying:

$$\beta + \frac{\xi(b)}{2b} = 1. \tag{4.27}$$

Actually, from (4.27), b can be locally determined just by working on a local parametrization such that $\xi = \partial/\partial t$. Thus, (4.27) can be seen as

$$\frac{\partial}{\partial t}(\log b) = 2(1-\beta),$$

and so

$$b = c_1 e^{2\int (1-\beta)\, dt} = c_1 e^{2t - 2\int \beta\, dt}. \tag{4.28}$$

If we now apply Theorem 4.8 to the obtained Kenmotsu manifold M^*, we know that, locally, it is a warped product $(-\varepsilon, \varepsilon) \times_F V$ where V is a Kaehler manifold and $F(t) = c_2 e^t$. If we denote by G the metric on V, it is related to g^* as

$$g^* = \eta \otimes \eta + (F \circ \pi)^2 \sigma^* G, \tag{4.29}$$

where π and σ denote the projections on $(-\varepsilon, \varepsilon)$ and V, respectively. But, from (4.26) and (4.29) we deduce:

$$g = \eta \otimes \eta + \left(\frac{F}{\sqrt{b}} \circ \pi\right)^2 \sigma^* G.$$

Therefore, locally, M is a warped product $(-\varepsilon, \varepsilon) \times_f V$, where V is a Kaehler manifold and

$$f = \frac{F}{\sqrt{b}} = \frac{c_2 e^t}{\sqrt{b}}, \tag{4.30}$$

with $c_2 > 0$. Finally, from (4.28) and (4.30), we get $f = ce^{\int \beta\, dt}$. □

As a consequence, we obtain the local converse we refereed to above:

Corollary 4.10. *Let $M(f_1, f_2, f_3)$ be a β-Kenmotsu generalized Sasakian-space-form. Then, for any point $p \in M$, there exists an open neighbourhood $U(p)$ such that $U = (-\varepsilon, \varepsilon) \times_f V$, where $(-\varepsilon, \varepsilon)$ is an open interval, V is a Kaehler manifold and $f = ce^{\int \beta\, dt}$.*

Proof. It follows directly from Theorem 4.9, since, by virtue of Proposition 4.3, we already know that in a β-Kenmotsu generalized Sasakian-space-form, β depends only on the direction of ξ. □

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