ELSEVIER Journal of Computational and Applied Mathematics 67 (1996) 327–341

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

The symmetric strong moment problem

A.K. Common[a], J.H. McCabe[b,*]

[a] Institute of Mathematics and Statistics, University of Kent, Canterbury, Kent, CT2 7NF, UK
[b] Department of Mathematical and Computational Sciences, University of St. Andrews, North Haugh, St. Andrews, Fife, KY16 9SS, UK

Received 20 May 1994; revised 18 November 1994

Abstract

Sequences of polynomials that occur as denominators in the two point Padé table for two series expansions are considered in the special case when the series coefficients are solutions of a strong *symmetric* Stieltjes moment problem. The continued fractions whose convergents generate these polynomials as denominators are presented, together with determinant representations for the polynomials and the continued fraction coefficients. The log-normal distribution is used as an example.

Keywords: Continued fractions; Padé approximants; Symmetric distributions

AMS classification: 30B70; 30E05; 41A21

1. Introduction

In this paper we study Padé approximants and continued fractions associated with the two series

$$\mu_0 z^{-1} + \mu_1 z^{-2} + \mu_2 z^{-3} + \cdots + \mu_n z^{-(n+1)} + \cdots \tag{1.1}$$

and

$$-\mu_{-1} - \mu_{-2} z - \mu_{-3} z^2 - \cdots - \mu_{-n} z^{n-1} - \cdots \tag{1.2}$$

in the particular case when

$$\mu_{-n} = \mu_n; \quad n = 1, 2, \ldots \tag{1.3}$$

In Section 2 we consider the denominators $B_{n,m}(z)$ of the two point Padé approximants which correspond to these series. They are generated by a version of the Quotient-Difference algorithm

* Corresponding author.

SSDI 0377-0427(94)00125-1

developed by one of us [7]. It is demonstrated that because of the "symmetry" property (1.3) the two-dimensional array of these denominators is symmetric about a horizontal line between the row $m = 0$ and the row $m = -1$.

In Section 3 the case when the coefficients in (1.1) and (1.2) are related to solutions of a strong Stieltjes moment problem [5]

$$\mu_k = \int_0^\infty t^k \, d\psi(t); \quad k = 0, \pm 1, \pm 2, \ldots$$

with $\psi(t)$ a bounded nondecreasing function of t with infinitely many points of increase is investigated. The property (1.3) of the moments implies that

$$d\psi(t) = -d\psi(1/t), \quad 0 \leqslant t \leqslant \infty \tag{1.4}$$

so long as the moment problem is determinate. This subclass of the strong Stieltjes moment problem arose in a study by one of us [1] of the Discrete Modified K–dV equation. We will construct two sequences of orthogonal polynomials

$$\{Q_n(t);\, n = 0, 1, 2, \ldots\} \quad \text{and} \quad \{q_n(t);\, n = 0, 1, 2, \ldots\}$$

with respect to the weight function $\psi(t)$. The first sequence has been constructed by one of us [8] when considering the strong Hamburger moment problem. The two sequences are interlinked and are shown in Section 4 to arise from a continued fraction of the form

$$S_1(t) \equiv \frac{\mu_0}{t - \mu_1/\mu_2 +} \, \frac{e_1}{1 +} \, \frac{t}{u_3 +} \, \frac{e_2}{1 +} \, \frac{t}{u_5 +} \, \cdots$$

We will demonstrate that the "even part" of $S_1(z)$ is identical to the "even part" of a specified Perron–Carathéodary continued fraction (or PC-fraction)

$$S_2(t) \equiv \frac{\alpha_1}{1 +} \, \frac{1}{\beta_2 t +} \, \frac{\alpha_3 t}{\beta_3 +} \, \frac{1}{\beta_4 t +} \, \frac{\alpha_5 t}{\beta_5 +} \, \cdots$$

with $\alpha_{2n+1} = 1 - \beta_{2n}\beta_{2n+1} \neq 0$, $n = 1, 2, 3, \ldots$ which were introduced by Jones et al. [4]. In general the "odd parts" of $S_1(t)$ and $S_2(t)$ are not identical, but their common "even part" is a T-fraction of the form

$$S_3(t) \equiv \frac{F_1 z}{1 + G_1 z +} \, \frac{F_2 z}{1 + G_2 z +} \, \frac{F_3 z}{1 + F_3 z +} \, \cdots, \quad z = 1/t. \tag{1.5}$$

Thus we have in $S_1(t)$, an interesting even extension of this T-fraction which has another even extension which is the PC-fraction $S_2(t)$.

Finally in Section 5 we will consider the particular example of the log-normal distribution

$$d\psi(t) = \frac{q^{1/2}}{2\kappa\sqrt{\pi t}} \, e^{-(\ln t/2\kappa)^2} \, dt,$$

where κ is a positive constant. This important distribution has been considered by Jones and Thron and co-workers [2, 3] and they have constructed a corresponding T-fraction which is identical to

(1.5) for this distribution. They have shown that it has an "even extension" which is a PC-fraction and in Section 4 we will demonstrate that it has another "even extension" which is also a PC-fraction.

Note: The authors use of the word "Symmetric" in describing the moment problem considered here is different from and more appropriate than that already used by other authors, when looking at moment problems in which the odd order moments are zero.

2. Denominators in the two-point Padé table for series with symmetric coefficients

From the two series (1.1), (1.2) one may construct the n–d array [6]

$$\begin{array}{lllllll} \vdots \\ d_1^{-3} & n_2^{-3} & \cdots \\ d_1^{-2} & n_2^{-2} & d_2^{-2} & n_3^{-2} & \cdots \\ d_1^{-1} & n_2^{-1} & d_2^{-1} & n_3^{-1} & d_3^{-1} & n_4^{-1} & \cdots \\ d_1^{0} & n_2^{0} & d_2^{0} & n_3^{0} & d_3^{0} & n_4^{0} & d_4^{0} \quad \cdots \\ d_1^{1} & n_2^{1} & d_2^{1} & n_3^{1} & d_3^{1} & \cdots \\ d_1^{2} & n_2^{2} & d_2^{2} & \cdots \\ d_1^{3} & \cdots \\ \vdots \end{array}$$

from the rhombus rules

$$n_j^k = d_{j-1}^{k+1} + n_{j-1}^{k+1} - d_{j-1}^k, \tag{2.1}$$

$$d_j^k = \frac{n_j^k d_{j-1}^{k-1}}{n_j^{k-1}} \tag{2.2}$$

for $j = 2, 3, \ldots$ and $k = 0, \pm 1, \pm 2, \pm 3, \ldots$ with starting values

$$n_1^k = 0; \quad d_1^k = -\frac{\mu_{-(k+1)}}{\mu_{-k}}; \qquad k = 0, \pm 1, \pm 2, \ldots \tag{2.3}$$

The continued fraction

$$\frac{-\mu_{-1}}{1 + d_1^0 z} + \frac{n_2^0 z}{1 + d_2^0 z} + \frac{n_3^0 z}{1 + d_3^0 z} + \frac{n_4^0 z}{1 + d_4^0 z} + \cdots$$

is an M-fraction for the two series. That is the nth convergent is a ratio of polynomials of degrees $(n-1)$ and n respectively and agrees with n terms of each of the two series when expanded accordingly. Similarly the continued fraction

$$\frac{-\mu_{-(k+1)}}{1 + d_1^k z} + \frac{n_2^k z}{1 + d_2^k z} + \frac{n_3^k z}{1 + d_3^k z} + \frac{n_4^k z}{1 + d_4^k z} + \cdots$$

corresponds to the two series

$$\frac{\mu_{-k}}{z} + \frac{\mu_{-k+1}}{z^2} + \frac{\mu_{-k+2}}{z^3} + \frac{\mu_{-k+3}}{z^4} + \frac{\mu_{-k+4}}{z^5} + \cdots$$

and

$$-\mu_{-(k+1)} - \mu_{-(k+2)}z - \mu_{-(k+3)}z^2 - \mu_{-(k+4)}z^3 - \cdots$$

for $k = 0, \pm 1, \pm 2, \ldots$

By multiplying the above continued fraction by z^k and adding the first k terms of the series

$$-\mu_{-1} - \mu_{-2}z - \mu_{-3}z^2 - \mu_{-4}z^3 - \cdots$$

if $k > 0$, but the first $|k|$ terms of the series

$$\frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \frac{\mu_2}{z^3} + \frac{\mu_3}{z^4} + \frac{\mu_4}{z^5} + \cdots$$

if k is negative, we obtain the whole of the two point Padé table for these two series [6]. However, the denominators of these continued fractions are unaltered by these additions and multiplications. Hence the denominators of the convergents of these continued fractions as k takes the values $0, +1, +2, \ldots$ are simply the denominators of the rational functions that appear in the two point Padé table, except for those in the first column. These latter ones however are either monomials or constants.

Denote the rth denominator of the continued fraction

$$\frac{-\mu_{-(k+1)}}{1 + d_1^k z} \; \frac{n_2^k z}{+\, 1 + d_2^k z} \; \frac{n_3^k z}{+\, 1 + d_3^k z} \; \frac{n_4^k z}{+\, 1 + d_4^k z +} \cdots \tag{2.4}$$

by $B_{r,k}(z)$. Then $B_{r,k}(z)$ is a polynomial of degree r in z and we can form the table

$$\begin{array}{lllll} \vdots & & & & \\ B_{1,-1}(z) & B_{2,-1}(z) & B_{3,-1}(z) & B_{4,-1}(z) & \cdots \\ B_{1,0}(z) & B_{2,0}(z) & B_{3,0}(z) & B_{4,0}(z) & \cdots \\ B_{1,1}(z) & B_{2,1}(z) & B_{3,1}(z) & B_{4,1}(z) & \cdots \\ \vdots & & & & \end{array} \tag{2.5}$$

We now obtain the symmetry property of this table stated in Section 1 by starting from the following result:

Theorem 2.1. *Let the coefficients of the series* (1.1), (1.2) *satisfy* (1.3).
Then

$$d_1^k = \frac{1}{d_1^{-(k+1)}}, \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{2.6}$$

and

$$d_r^k = \frac{1}{d_r^{-(k+1)}}, \tag{2.7}$$

$$n_r^k = \frac{+\, n_r^{-(k+1)}}{d_r^{-(k+1)} d_{r-1}^{-(k+1)}} \tag{2.8}$$

for $r = 2, 3, 4, \ldots$ *and* $k = 0, \pm 1, \pm 2, \pm 3, \ldots$

The relations (2.6) follow immediately from (2.3), whilst (2.7) and (2.8) may be proved by mathematical induction on r using the rhombus rules (2.1) and (2.2).

If z is replaced by $1/z$ in the continued fraction (2.4) we obtain, after a sequence of similarity transformations, the continued fraction

$$\frac{-\mu_{-(k+1)}z}{z + d_1^k} + \frac{n_2^k z}{z + d_2^k} + \frac{n_3^k z}{z + d_3^k} + \frac{n_4^k z}{z + d_4^k} + \cdots$$

and writing this in the same form as (2.4) we obtain

$$\frac{-\mu_{-(k+1)}z/d_1^k}{1 + z/d_1^k} + \frac{n_2^k z/d_1^k d_2^k}{1 + z/d_2^k} + \frac{n_3^k z/d_2^k d_3^k}{1 + z/d_3^k} + \cdots$$

But from the symmetry relations (2.6) to (2.8), this continued fraction can be written as

$$\frac{-\mu_{-(k+1)}d_1^{-(k+1)}z}{1 + d_1^{-(k+1)}z} + \frac{n_2^{-(k+1)}z}{1 + d_2^{-(k+1)}z} + \frac{n_3^{-(k+1)}z}{1 + d_3^{-(k+1)}z} + \cdots$$

which of course has as its coefficients the elements that form another row of the $n - d$ array. Again the denominators are not effected by the first partial numerator so that the following result is obtained:

Theorem 2.2. *The table of denominator polynomials in* (2.5) *is symmetric about an horizontal line between* $B_{1,0}(z)$ *and* $B_{1,-1}(z)$ *in that any polynomial* $B_{n,m}(z)$ *can be obtained from its image polynomial* $B_{n,-(m+1)}(z)$ *by reversing the coefficients and normalising so that* $B_{n,m}(0) = 1$.

Of particular interest are the elements on the two rows

$$\begin{array}{lllll} B_{1,-1}(z) & B_{2,-1}(z) & B_{3,-1}(z) & B_{4,-1}(z) & \cdots \\ B_{1,0}(z) & B_{2,0}(z) & B_{3,0}(z) & B_{4,0}(z) & \cdots \end{array}$$

A continued fraction which has as denominators of its convergents the "sawtooth" sequence

$$B_{1,0}(z), B_{2,-1}(z), B_{3,0}(z), B_{4,-1}(z), B_{5,0}(z), \ldots$$

has been used in the solution of the strong Hamburger moment problem and in extending some classical distributions [8]. In an earlier work [7] it was shown how to construct the particular extension of this fraction that has, as its denominators, the "battlement" sequence

$$B_{1,0}(z),\, B_{1,-1}(z),\, B_{2,-1}(z),\, B_{2,0}(z),\, B_{3,0}(z),\, B_{3,-1}(z),\, B_{4,-1}(z),\, \dots$$

and is related to continued fraction of the form

$$\frac{-\mu_{-1}}{1}\,\frac{}{+}\,\frac{s_1 z}{1}\,\frac{}{+}\,\frac{t_2}{1}\,\frac{}{+}\,\frac{s_2}{z}\,\frac{}{+}\,\frac{t_3}{1}\,\frac{}{+}\,\frac{s_3 z}{1}\,\frac{}{+}\,\frac{t_4}{1}\,\frac{}{+}\cdots$$

The sequence is formed by taking the denominators of the second and higher convergents of this fraction.

3. The strong Stieltjes moment problem

Here we will construct two interlinked sets of orthogonal polynomials corresponding to the strong Stieltjes moment problem (1.4), namely

$$\mu_k = \int_0^\infty t^k \, d\psi(t), \quad k = 0, \pm 1, \pm 2, \dots$$

with the symmetry property (1.3).

For any strong Stieltjes distribution

$$H_n^{(m)} > 0; \quad n = 0, 1, \dots, \quad m = 0, \pm 1, \pm 2, \dots, \tag{3.1}$$

where the Hankel determinants are defined in the usual way by

$$H_n^{(m)} = \begin{vmatrix} \mu_m & \mu_{m+1} & \cdots & \mu_{m+n-1} \\ \mu_{m+1} & \mu_{m+2} & \cdots & \mu_{m+n} \\ \vdots & & & \\ \mu_{m+n-1} & \cdots & & \mu_{m+2n-2} \end{vmatrix}. \tag{3.2}$$

The case $m = 0$ is the standard result for series of Stieltjes when $\psi(t)$ has infinitely many points of increase. The case $m \neq 0$ is obtained by taking the weight function $\phi(t) = \int_0^t u^m \, d\psi(u)$ in this standard result.

It is straightforward to prove that when (1.3) holds,

$$H_n^{(-m)} = H_n^{(m-2n+2)} \tag{3.3}$$

and, from Jacobi's identity, that

$$(H_{n-1}^{(-n+2)})^2 - (H_{n-1}^{(-n+3)})^2 + H_{n-2}^{(-n+3)} H_n^{(-n+1)} = 0; \quad n = 2, 3, \dots \tag{3.4}$$

This identity will be used later in this work.

A set of orthogonal polynomials $\{Q_n(z): n = 0, 1, \dots\}$ may be defined [8] by the requirement that

$$\int_0^\infty Q_n(t)t^{-2[n/2]+s}\,\mathrm{d}\psi(t) = \begin{cases} 0, & 0 \leqslant s \leqslant n-1, \\ \gamma_n, & s = n \geqslant 1. \end{cases} \tag{3.5}$$

The polynomials have the monic form

$$Q_{2n}(z) = \frac{1}{H_{2n}^{(-2n)}} \begin{vmatrix} \mu_{-2n} & \cdots & \mu_0 \\ \vdots & & \vdots \\ \mu_{-1} & & \mu_{2n-1} \\ 1 & z\cdots & z^{2n} \end{vmatrix}$$

and

$$Q_{2n+1}(z) = \frac{1}{H_{2n+1}^{(-2n)}} \begin{vmatrix} \mu_{-2n} & \cdots & \mu_1 \\ \vdots & & \vdots \\ \mu_0 & & \mu_{2n+1} \\ 1 & z\cdots & z^{2n+1} \end{vmatrix}$$

when the normalisation constants are defined by

$$\gamma_{2n} = H_{2n+1}^{(-2n)}/H_{2n}^{(-2n)}; \qquad \gamma_{2n+1} = H_{2n+2}^{(-2n)}/H_{2n+1}^{(-2n)}.$$

The existence of these polynomials is guaranteed by the positivity property (3.1) of the Hankel determinants.

In the symmetric case we can define another set of orthogonal polynomials by making use of the symmetry property (1.4) of the corresponding weight function. With this property and making the substitution $t \to 1/t$ in (3.5) it follows that

$$\int_0^\infty Q_n(1/t)\,t^{+2[n/2]-s}\,\mathrm{d}\psi(t) = 0, \quad 0 \leqslant s \leqslant n-1.$$

It is then straightforward to prove the following results.

Theorem 3.1. *The set of polynomials*

$$q_n(t) = (-)^n \frac{H_n^{(-2[n/2])}}{H_n^{(-2[n/2]+1)}} Q_n(1/t)t^n; \quad n = 0, 1, \dots$$

are monic and satisfy the orthogonality conditions

$$\int_0^\infty q_{2n}(t)t^{-s}\,\mathrm{d}\psi(t) = 0, \quad 0 \leqslant s \leqslant 2n-1, \tag{3.6}$$

$$\int_0^\infty q_{2n+1}(t)t^{-s-1}\,d\psi(t) = 0, \quad 0 \leqslant s \leqslant 2n. \tag{3.7}$$

These may be compared with (3.5) which can be written in the form

$$\int_0^\infty Q_{2n}(t)t^{-s-1}\,d\psi(t) = 0, \quad 0 \leqslant s \leqslant 2n-1, \tag{3.8}$$

$$\int_0^\infty Q_{2n+1}(t)t^{-s}\,d\psi(t) = 0, \quad 0 \leqslant s \leqslant 2n. \tag{3.9}$$

The two sequences of orthogonal polynomials are interlinked by mixed recurrence relations given by the following result:

Theorem 3.2.

$$Q_{2n}(t) = q_{2n}(t) + e_{2n}q_{2n-1}(t), \tag{3.10}$$

$$Q_{2n+1}(t) = tq_{2n}(t) - [a_{2n+1}/e_{2n} + e_{2n+1}]Q_{2n}(t) \tag{3.11}$$

for $n = 1, 2, \ldots$

$$q_{2n+1}(t) = Q_{2n+1}(t) + e_{2n+1}Q_{2n}(t), \tag{3.12}$$

$$q_{2n+2}(t) = tQ_{2n+1}(t) - \left[\frac{\alpha_{2n+2}}{e_{2n+1}} + e_{2n+2}\right]q_{2n+1}(t) \tag{3.13}$$

for $n = 0, 1, 2, \ldots$, *where*

$$e_{2n} = \frac{H_{2n-1}^{(-2n+3)}H_{2n+1}^{(-2n)}}{H_{2n}^{(-2n+1)}H_{2n}^{(-2n)}}, \qquad e_{2n+1} = \frac{H_{2n}^{(-2n)}H_{2n+2}^{(-2n-1)}}{H_{2n+1}^{(-2n)}H_{2n+1}^{(-2n+1)}}, \tag{3.14}$$

$$\alpha_{2n} = \left\{\frac{H_{2n}^{(-2n+1)}}{H_{2n-1}^{(-2n+2)}}\right\}^2 \frac{H_{2n-2}^{(-2n+1)}}{H_{2n}^{(-2n)}}, \qquad \alpha_{2n+1} = \frac{H_{2n}^{(-2n+1)}H_{2n-1}^{(-2n+2)}}{\{H_{2n}^{(-2n+1)}\}^2}. \tag{3.15}$$

Proof. To prove (3.10) we consider the polynomial of degree $2n-1$ given by

$$P_{2n-1}(t) \equiv Q_{2n}(t) - q_{2n}(t).$$

Then from (3.6) to (3.9)

$$\int_0^\infty P_{2n-1}(t)t^{-s-1}\,d\psi(t) = 0, \quad 0 \leqslant s \leqslant 2n-2,$$

so that $P_{2n-1}(t)$ is a constant multiple of $q_{2n-1}(t)$. We write

$$Q_{2n}(t) - q_{2n}(t) = e_{2n}q_{2n-1}(t)$$

and determine e_{2n} by setting $t = 0$. Then

$$\frac{H_{2n}^{(-2n+1)}}{H_{2n}^{(-2n)}} - \frac{H_{2n}^{(-2n)}}{H_{2n}^{(-2n+1)}} = -\frac{H_{2n-1}^{(-2n+2)}}{H_{2n-1}^{(-2n+3)}} e_{2n}. \tag{3.16}$$

Using (3.4) with $n \to 2n+1$ and $H_{2n}^{(-2n+2)} = H_{2n}^{(-2n)}$ from (3.3) in (3.16) it may be demonstrated that e_{2n} is given by (3.14), α_{2n+2} by (3.15) and so (3.10) follows. The relation (3.12) with e_{2n+1} given by (3.14) and α_{2n+1} by (3.15) may be proved in a similar way by considering $Q_{2n+1}(t) - q_{2n+1}(t)$ which is a polynomial of degree $2n$.

Now (3.10), (3.12) may be used to rewrite (3.11) in the form

$$e_{2n+1}Q_{2n}(t) + Q_{2n+1}(t) = \left[t + \left(\frac{a_{2n+1}}{e_{2n}} - b_{2n+1} \right) \right] q_{2n}(t) - \frac{a_{2n+1}}{e_{2n}} Q_{2n}(t).$$

But

$$\frac{a_{2n+1}}{e_{2n}} - b_{2n+1} = \frac{H_{2n+1}^{(-2n)}}{H_{2n+1}^{(-2n+1)} H_{2n}^{(-2n+1)}} [H_{2n}^{(-2n)} - H_{2n}^{(-2n+2)}]$$

and the right-hand side is zero from (3.5) so that (3.13) follows. Finally (3.13) is proved in a similar manner. □

4. Continued fractions related to the symmetric strong Stieltjes moment problem

The mixed recurrence relations given by Theorem (3.2) may be used to generate the sequence of polynomials $\{B_n(t) : n = 0, 1, \ldots\}$ with

$$B_{4r}(t) = Q_{2r}(t), \quad B_{4r+1}(t) = Q_{2r+1}(t), \quad B_{4r+2}(t) = q_{2r+1}(t), \quad B_{4r+3}(t) = q_{2r+2}(t)$$

for $r = 0, 1, 2, 3, \ldots$

These recurrence relations may be written in the form

$$B_n(t) = u_n B_{n-1}(t) + v_n B_{n-2}(t): \quad n = 1, 2, \ldots \tag{4.1}$$

with

$$u_{4r} = 1, \quad v_{4r} = e_{2r}; \qquad r = 1, 2, \ldots$$

$$u_{4r+1} = -\frac{H_{2r}^{(-2r)} H_{2r+1}^{(-2r+1)}}{H_{2r}^{(-2r+1)} H_{2r+1}^{(-2r)}}, \quad v_{4r+1} = t; \qquad r = 1, 2, \ldots \tag{4.2}$$

$$u_{4r+2} = 1, \quad v_{4r+2} = e_{2r+1}; \qquad r = 0, 1, \ldots \tag{4.3}$$

$$u_{4r+3} = -\frac{H_{2r+1}^{(-2r+1)} H_{2r+2}^{(-2r)}}{H_{2r+1}^{(-2r)} H_{2r+2}^{(-2r-1)}}, \quad v_{4r+3} = t; \qquad r = 0, 1, 2, \ldots \tag{4.4}$$

and we have defined $B_{-1}(t) \equiv 0$, $u_1 = t - \mu_1/\mu_0$ and $v_1 = \mu_0$. The expressions for u_{4r+1}, u_{4r+3} in terms of Hankel determinants may be obtained by using (3.3), (3.4) to simplify the expressions for

$a_{2r+1}/e_{2r} + e_{2r+1}$ and $\alpha_{2r+2}/e_{2r+1} + e_{2r+2}$ in terms of these determinants. A common form for both odd and even e_k may be derived from (3.14) using (3.3) and takes the form

$$e_k = \frac{H_{k-1}^{(-k+3)} H_{k+1}^{(-k)}}{H_k^{(-k+1)} H_k^{(-k)}}; \quad k = 1, 2, \ldots \tag{4.5}$$

A corresponding sequence of polyomials $\{A_n(t); n = 0, 1, 2, \ldots\}$ exists with $A_{-1} \equiv 1$, $A_0(t) \equiv 0$ and

$$A_n(t) = \int_0^\infty \frac{[B_n(t) - B_n(x)]\,\mathrm{d}\psi(x)}{(l - x)}; \quad n = 1, 2, 3, \ldots$$

and it is straightforward to demonstrate that they satisfy the same recurrence relations as the $B_n(t)$'s, i.e.

$$A_n(t) = u_n A_{n-1}(t) + v_n A_{n-2}(t); \quad n = 1, 2, \ldots \tag{4.6}$$

It may be shown that sequence of polynomials $Q_0(t), q_1(t), Q_2(t), q_3(t), Q_4(t), \ldots$ is the sequence of denominators that would appear in the standard (nonsymmetric) strong Stieltjes moment problem.

A consequence of the recurrence relations (4.1), (4.6) is that $\{A_n, B_n\}$ are the nth numerator and denominator, respectively, of the continued fraction

$$\mathop{K}_{n=1}^{\infty}\left(\frac{v_n}{u_n}\right) \equiv \frac{v_1}{u_1 +} \frac{v_2}{u_2 +} \frac{v_3}{u_3 +} \cdots$$

$$= \frac{\mu_0}{t - \mu_1/\mu_0 +} \frac{e_1}{1 +} \frac{t}{u_3 +} \frac{e_2}{1 +} \frac{t}{u_5 +} \frac{e_3}{1 +} \cdots \tag{4.7}$$

The even part of this fraction is the continued fraction

$$\mathop{K}_{n=1}^{\infty}\left(\frac{v_n^*}{u_n^*}\right) \equiv \frac{v_1^*}{u_1^* +} \frac{v_2^*}{u_2^* +} \frac{v_3^*}{u_3^* +} \cdots, \tag{4.8}$$

where

$$v_1^* = v_1 u_2 = \mu_0, \quad u_1^* = v_2 + u_1 u_2 = t - \mu_0/\mu_1,$$

$$v_n^* = -v_{2n-2} v_{2n-1} u_{2n-4} u_{2n} = -t e_{n-1}; \quad n = 2, 3, \ldots$$

$$u_n^* = v_{2n-1} u_{2n} + u_{2n-2}(v_{2n} + u_{2n-1} u_{2n}) = t + e_n + u_{2n-1}: \quad n = 2, 3, \ldots$$

Substituting these expressions in (4.8),

$$\mathop{K}_{n=1}^{\infty}\left(\frac{v_n^*}{u_n^*}\right) = \frac{F_1}{t + G_1 +} \frac{F_2 t}{t + G_2 +} \frac{F_3 t}{t + G_3 +} \frac{F_4 t}{t + G_4 +} \cdots \tag{4.9}$$

$$= \frac{F_1 \lambda}{1 + \lambda G_1 +} \frac{F_2 \lambda}{1 + \lambda G_2 +} \frac{F_3 \lambda}{1 + \lambda G_3 +} \frac{F_4 \lambda}{1 + \lambda G_4 +} \cdots \tag{4.10}$$

with

$$\lambda = 1/t, \quad F_1 = \mu_0, \qquad G_1 = -\mu_0/\mu_1, \tag{4.11}$$

$$F_n = -e_{n-1} = \frac{-H_{n-2}^{(-n+4)} H_n^{(-n+1)}}{H_{n-1}^{(-n+2)} H_{n-1}^{(-n+1)}}, \quad n = 2, 3, \ldots, \tag{4.12}$$

$$G_n = u_{2n-1} + e_n = \frac{-H_{n-1}^{(-n+1)} H_n^{(-n)}}{H_{n-1}^{(-n+2)} H_n^{(-n+1)}} + \frac{H_{n-1}^{(-n+3)} H_{n+1}^{(-n)}}{H_n^{(-n+1)} H_n^{(-n)}} \tag{4.13}$$

$$= \frac{-H_{n-1}^{(-n+1)} H_n^{(-n+1)}}{H_n^{(-n)} H_{n-1}^{(-n+2)}}, \quad n = 2, 3, \ldots \tag{4.14}$$

on using (3.3) with $n = m = k$ and Jacobi's identity. The right-hand side of (4.9) and (4.10) are called respectively M-fractions and T-fractions.

The T-fraction in (4.10) then has our continued fraction (4.7) as an "even extension". We will show another "even extension" is a Perron–Carathéodary fraction (PC-fraction) introduced by Jones et al. [4]. These fractions are of the form

$$\mathop{K}_{n=1}^{\infty} \left(\frac{\alpha_n}{\beta_n}\right) = \frac{\alpha_1}{1} + \frac{\lambda}{\beta_2} + \frac{\alpha_3}{\beta_3} + \frac{\lambda}{\beta_4} + \frac{\alpha_5}{\beta_5} + \cdots \tag{4.15}$$

with

$$\alpha_1 = \mu_0, \quad \alpha_{2n+1} = (1 - \beta_{2n}\beta_{2n+1}) \neq 0; \quad n = 0, 1, \ldots \tag{4.16}$$

We now have the following result:

Theorem 4.1. *The T-fraction* (4.10) *with* $\{F_n, G_n\}$ *given by* (4.11)–(4.14) *is equivalent to the even part of the PC-fraction* (4.21) *when* $\lambda \to -\lambda$ *and*

$$\beta_{2n} = \beta_{2n+1} = \frac{H_n^{(-n)}}{H_n^{(-n+1)}}; \quad n = 1, 2, \ldots \tag{4.17}$$

Proof. The even part of the fraction (4.15) with $\{\alpha_n, \beta_n\}$ given by (4.16), (4.17) is the fraction $K_{n=1}^{\infty} (a_n/b_n)$, where

$$a_1 = \mu_0\beta_2, \quad b_1 = \lambda + \beta_2, \quad a_2 = -\lambda(1 - \beta_2\beta_3)/\beta_4, \tag{4.18}$$

$$a_n = -\lambda(1 - \beta_{2n-2}\beta_{2n-1})\beta_{2n-4}\beta_{2n}, \quad n = 3, 4, \ldots, \tag{4.19}$$

$$b_n = \beta_{2n} + \beta_{2n-2}\lambda, \quad n = 2, 3, 4, \ldots \tag{4.20}$$

This fraction is equivalent to the fraction $K_{n=1}^{\infty} (a_n^*/b_n^*)$, where

$$a_n^* = \frac{a_n}{\beta_{2n}\beta_{2n-2}}, \quad b_n^* = \frac{b_n}{\beta_{2n}}; \qquad n = 1, 2, \ldots, \qquad \beta_0 \equiv 1. \tag{4.21}$$

Using the expressions (4.17) for the β_n's in (4.18)–(4.21),

$$a_1^* = \mu_0, \qquad b_1^* = 1 + \lambda \frac{H_1^{(0)}}{H_1^{(-1)}} = 1 + \lambda \frac{\mu_0}{\mu_1}, \tag{4.22}$$

$$a_n^* = -\lambda \left[\frac{\beta_{2n-4}}{\beta_{2n-2}} - \beta_{2n-1}\beta_{2n-4} \right] = \lambda \frac{H_{n-2}^{(-n+4)} H_n^{(-n+1)}}{H_{n-1}^{(-n+1)} H_{n-1}^{(-n+2)}}, \quad n = 2, 3, \ldots, \tag{4.23}$$

$$b_n^* = 1 + \frac{H_{n-1}^{(-n+1)} H_n^{(-n+1)}}{H_{n-1}^{(-n+2)} H_n^{(-n)}} \lambda; \quad n = 2, 3, \ldots, \tag{4.24}$$

where the identity (3.4) has been used to simplify the expression for a_n^*. Comparing the expressions (4.22)–(4.24) for the $\{a_n^*, b_n^*\}$ with expressions (4.11)–(4.14) for the $\{F_n, G_n\}$ we see that the even part of the PC-fraction (4.15) is equivalent to the T-fraction (4.10) when $\lambda \to -\lambda$. □

5. The log-normal distribution

An important example of a weight function where the moments $\{\mu_k, k = 0, 1, \ldots\}$ have the symmetry property (1.3) is the log-normal distribution $\mathrm{d}\psi(t)$ where

$$\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = \frac{q^{1/2}}{2\kappa\sqrt{\pi}} \mathrm{e}^{-(\ln t/2\kappa)^2}, \quad q = \mathrm{e}^{-2\kappa^2}$$

and κ is a positive constant. Related orthogonal polynomials and corresponding continued fractions have been studied recently for this distribution by Cooper et al. [2] in terms of the moments

$$c_k = (-)^k \int_0^\infty t^{k+1}\, \mathrm{d}\psi(t), \quad k = 0, \pm 1, \pm 2, \ldots$$

$$= (-)^k \mu_{k+1}.$$

It is straightforward to show that

$$\mu_k = q^{1/2 - k^2/2}; \quad k = 0, \pm 1, \pm 2, \ldots \tag{5.1}$$

so that they have the symmetry property (1.3). The Hankel determinants $\mathscr{H}_k^{(n)}$ of the c_k's are related to those defined by (3.2) in terms of the μ_k's in the following way:

$$H_k^{(m+1)} = (-)^{mk} \mathscr{H}_k^{(m)}, \quad k = 0, 1, 2, \qquad m = 0, \pm 1, \pm 2, \ldots$$

These determinants are of Vandermonde type and using the expression for $\mathscr{H}_k^{(m)}$ given in [2],

$$H_k^{(m)} = [q^{-m+k-2)^2/2-(m+k-2)}]^k q^{-k(k^2-1)/6} \times \prod_{j=1}^{k-1} (1 - q^j)^{k-j}. \tag{5.2}$$

Substituting these Hankel determinants in (4.2)–(4.5),

$$u_{2n+1} = -q^{-n-1/2}, \quad e_n = q^{-n+1/2}(1-q^n), \quad n = 1, 2, \ldots$$

Then our fraction given in (4.7) has the form

$$\mathop{K}_{n=1}^{\infty}\left(\frac{v_n}{u_n}\right) = \frac{\mu_0}{t - \mu_1/\mu_0 +} \; \frac{q^{-1/2}(1-q)}{1} \; \frac{t}{+(-)q^{-3/2} +} \; \frac{q^{-3/2}(1-q^2)}{1} \; \frac{t}{+(-)q^{-3/2} +} \cdots$$

and it is the even part of the T-fraction

$$\mathop{K}_{n=1}^{\infty}\left(\frac{v_n^*}{u_n^*}\right) = \frac{q^{1/2}\lambda}{1 - \lambda q^{1/2} +} \; \frac{(-)q^{-1/2}(1-q)\lambda}{1 - \lambda q^{1/2} \quad +} \; \frac{(-)q^{-3/2}(1-q^2)\lambda}{1 - \lambda q^{1/2} \quad +} \cdots \tag{5.3}$$

This is exactly the T-fraction constructed in [2] from the Laurent series

$$L_\infty = \sum_{k=0}^{\infty} c_k \lambda^{-k}, \qquad L_0 = -\sum_{k=1}^{\infty} c_{-k}\lambda^k,$$

where $\lambda \to -\lambda$. We see that this fraction has an even extension which takes the form of the PC-fraction given by (4.15). In this case of the log-normal distribution it has the form

$$\mathop{K}_{n=1}^{\infty}\left(\frac{\alpha_n}{\beta_n}\right) = \frac{\lambda q^{1/2}}{1} \; \frac{\lambda}{+q^{-1/2} +} \; \frac{(1-q^{-1})}{q^{-1/2}} \; \frac{\lambda}{+q^{-1} +} \; \frac{(1-q^{-2})}{q^{-1} \quad +} \cdots$$

and generally from (4.16), (4.17) and (5.2),

$$\alpha_{2n+1} = (1 - q^{-n}), \quad \beta_{2n} = \beta_{2n+1} = q^{-n/2}.$$

We finally investigate the $n-d$ array and corresponding array of Padé denominators in this log-normal case. To do this we set $\lambda = 1/z$ in (5.3) and transform it to the equivalent M-fraction

$$S_4(z) \equiv \frac{-1}{1 - q^{-1/2}z -} \; \frac{q^{-3/2}(1-q)z}{1 - q^{-1/2}z \quad -} \; \frac{q^{-5/2}(1-q^2)z}{1 - q^{-1/2}z \quad -} \; \frac{q^{-7/2}(1-q^3)z}{1 - q^{-1/2}z \quad -} \cdots \tag{5.4}$$

which corresponds to the series (1.1), (1.2) with μ_k given by (5.1). The denominator polynomials satisfy the recurrence relations

$$B_{n+1,0}(z) = (1 - q^{-1/2}z)B_{n,0}(z) - q^{-n+1/2}(1 - q^{n-1})zB_{n-1,0}(z)$$

for $n = 1, 2, 3, \ldots$ with $B_{0,0}(z) = 1$ and $B_{1,0} = 1 - q^{-1/2}z$.

The $n-d$ array for the two series is

$$
\begin{array}{llllll}
\vdots & & & & & \\
(k=-3) & -q^{5/2} & -q^{3/2}(1-q) & -q^{5/2} & -q^{1/2}(1-q^2) & \cdots \\
(k=-2) & -q^{3/2} & -q^{1/2}(1-q) & -q^{3/2} & -q^{-1/2}(1-q^2) & \cdots \\
(k=-1) & -q^{1/2} & -q^{-1/2}(1-q) & -q^{1/2} & -q^{-3/2}(1-q^2) & \cdots \\
(k=0) & -q^{-1/2} & -q^{-3/2}(1-q) & -q^{-1/2} & -q^{-5/2}(1-q^2) & \cdots \\
(k=1) & -q^{-3/2} & -q^{-5/2}(1-q) & -q^{-3/2} & -q^{-7/2}(1-q^2) & \cdots \\
(k=2) & -q^{-5/2} & -q^{-7/2}(1-q) & -q^{-5/2} & -q^{-9/2}(1-q^2) & \cdots \\
(k=3) & -q^{-7/2} & -q^{-9/2}(1-q) & -q^{-7/2} & -q^{-11/2}(1-q^2) & \\
\vdots & & & & &
\end{array}
$$

It is easily established that

$$n_j^k = -q^{-(j+k)+1/2}(1-q^{j-1}) = -[n_j^{k+1}\, n_j^{k-1}]^{1/2},$$
$$d_j^k = -q^{-k-1/2} = -[d_j^{k+1}\, d_j^{k-1}]^{1/2}.$$

The sequence of polynomials $\{B_{j,0}(z);\ j=0,1,\dots\}$ are the denominators of the convergents of the continued fraction $S_4(z)$ given in (5.14) whilst the sequence $\{B_{j,-1}(z); j=0,1,\dots\}$ are the denominators of the convergents of the continued fraction

$$S_5(z) \equiv \frac{-1}{1-q^{1/2}z-}\ \frac{q^{-1/2}(1-q)z}{1-q^{1/2}z}\ -\ \frac{q^{-3/2}(1-q^2)z}{1-q^{1/2}z}\ -\ \frac{q^{-5/2}(1-q^3)z}{1-q^{1/2}z}\ -\ \cdots$$

Since $S_5(z)$ is obtained from $S_4(z)$ by replacing z by qz, it follows that

$$B_{n,-1}(z) = B_{n,0}(qz); \quad n=0,1,2,\dots$$

Similarly the denominators of the continued fraction

$$S_6(z) \equiv \frac{-1}{1-q^{-3/2}z-}\ \frac{q^{-5/2}(1-q)z}{1-q^{-3/2}z}\ -\ \frac{q^{-7/2}(1-q^2)z}{1-q^{-3/2}z}\ -\ \frac{q^{-9/2}(1-q^3)z}{1-q^{-3/2}z}\ -\ \cdots$$

are given by $B_{n,0}(q^{-1}z);\ n=0,1,2,\dots$.

In general the denominators of the continued fraction

$$S_7(z) \equiv \frac{-1}{1+d_1^k z+}\ \frac{n_2^k z}{1+d_2^k z+}\ \frac{n_3^k z}{1+d_3^k z+}\ \frac{n_4^k z}{1+d_4^k z+}\ \cdots$$

are the polynomials $B_{n,0}(q^{-k}z)$ for $k=0,\ \pm 1,\ \pm 2,\dots$. They may also be written in terms of the entries in the mth row of the denominators table (2.15) as the sequence $\{B_{n,m}(q^{m-k}z);\ r=0,1,2,\dots\}$. It follows that any sequence of denominators in this table may be written in terms of those along any row. In particular the sequence of denominators of any continued fraction that corresponds to the power series (1.1), (1.2) may be expressed in terms of $B_{n,0}(z);\ n=0,1,2,\dots$

References

[1] A.K. Common, A solution of the initial value problem for half-infinite integrable lattice systems, *Inverse Problems* **8** (1992) 393–408.

[2] S.C. Cooper, W.B. Jones and W.J. Thron, Orthogonal Laurent-polynomials and continued fractions associated with log-normal distributions, *J. Comput. Appl. Math.* **32** (1990) 39–46.

[3] W.B. Jones, A. Magnus and W.J. Thron, PC-fractions and orthogonal Laurent polynomials for log-normal distributions, *J. Math. Anal. Appl.* **170** (1992) 225–244.

[4] W.B. Jones, O. Njåstad and W.J. Thron, Continued fractions associated with trigonometric and other strong moment problems, *Constructive Approx.* **2** (1986) 197–211.

[5] W.B. Jones, W.J. Thron and H. Waadeland, A strong Stieltjes moment problem, *Trans. Amer. Math. Soc.* **261** (1980) 503–528.

[6] J.H. McCabe, A formal extension of the Padé table to include two-point Padé quotients. *J. Inst. Math. Appl.* **15** (1975) 363–372.

[7] J.H. McCabe, The quotient-difference algorithm and the Padé table: An alternative form and a general continued fraction, *Math. Comput.* **41** (1983) 183–197.

[8] J.H. McCabe and A. Sri Ranga, On the extensions of some classical distributions, *Proc. Edinburgh Math. Soc.* **34** (1991) 19–29.