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Berry phase effects in the dynamics of Dirac electrons in doubly special relativity framework

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Abstract

We consider the Doubly Special Relativity (DSR) generalization of Dirac equation in an external potential in the Magueijo–Smolin base. The particles obey a modified energy–momentum dispersion relation. The semiclassical diagonalization of the Dirac Hamiltonian reveals the intrinsic Berry phase effects in the particle dynamics.

Keywords: κ-Minkowski spacetime; DSR; Foldy–Wouthuysen

1. Introduction

Evidence (see [1] for discussion and references) of ultra-high energy cosmic ray particles that violate the Greisen–Zatsepin–Kuzmin bound have compelled theorists to generalize the conventional energy–momentum dispersion law of particles,

$$p^2 = m^2, \tag{1}$$

based on principles of Special theory of Relativity (SR).[1] The extension requires another observer independent dimensional parameter, apart from c, the velocity of light. The second parameter (κ) is expected to be related to Planck energy. Based on this idea Amelino–Camelia [2] has pioneered an extended form of SR, popularly known as Doubly Special theory of Relativity (DSR). The effect of κ appears in the explicit structures of Lorentz transformations in DSR, which are non-linear for momenta and momenta-dependent for coordinates. At the same time κ induces a Non-Commutative (NC) spacetime structure [3], which is referred to as the κ-Minkowski spacetime [4]. Generally the scale of κ is associated with Planck energy, and all κ-induced modifications smoothly disappear in the low energy sector (or equivalently in the limit $\kappa \to \infty$). In the present Letter we take $\lambda \equiv (1/\kappa)$ as the NC parameter and for $\lambda = 0$ one recovers the commutative limit.

A particular form of DSR extension of (1) that we will be concerned with in the present Letter is given by

$$p^2 = M^2[1 - \lambda E]^2, \tag{2}$$

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[1] However, it should be pointed out that very recent data from the Auger cosmic ray observatory does not quite support the observation of ultra-high energy cosmic ray particles although results on the ultra-high energy cosmic photons is still awaited.

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with E being the particle energy. This form is known as the Magueijo–Smolin (MS) construction [5] of DSR. The corresponding κ-Minkowski NC spacetime [4] is

$$\{x^i, x^0\} = i\lambda x^i; \qquad \{x^i, x^j\} = 0. \tag{3}$$

We will comment on the classical nature of NC spacetime algebra, that is being considered here, a little later. In the present article we will focus on the DSR generalization of Dirac equation [6] in MS base [5]. (For alternative constructions of DSR Dirac equation see [7].) We will take up directly the issue of (semi-classical) quantization of the DSR Dirac particle by exploiting the generalized Foldy–Wouthuysen (FW) approach [8]. This extension amounts to a semiclassical $O(\hbar)$ quantization of the Dirac particle in the presence of external interactions and in previous works of some of us [9] this idea has been successfully employed to reveal intrinsic Berry phase [10] effects in the particle dynamics (see [11] for a review). The appearance of Berry phase effects is very natural in FW formalism. The Berry potential is induced by the electron spin coordinates which are treated as "fast" variables as compared to the position coordinates which are considered as "slow" degrees of freedom. The same phenomena was discovered earlier [12] when molecular dynamics was studied in the Born–Oppenheimer approximation. In this context let us emphasize the novelties in the present work: The DSR Dirac equation will be derived in an algebraic formalism which is mathematically very simple and it exploits ideas studied in detail in a previous work of one of the present authors [13]. Interestingly this formalism allows us to introduce interactions in the DSR scenario in a consistent way. Indeed interactions in a DSR framework generally have not appeared in the literature. Lastly the study of DSR Dirac model in FW scheme is entirely new and (not surprisingly) the results obtained by us can indicate new directions in this area.

The Letter is organized as follows: In Section 2 we derive the DSR Dirac equation in Magueijo–Smolin (MS) [5] base. Section 3 is devoted to the FW analysis of the DSR Dirac equation constructed in Section 2. In Section 4 we introduce the external interaction and reveal the intrinsic Berry phase contribution in the present case. The Letter ends with conclusions and areas of future studies in Section 5.

2. DSR Dirac equation in Magueijo–Smolin base

Let us first put our approach in its proper perspective in the context of quantum NC DSR theories. Our aim is to exploit the canonical framework [13] in constructing the theory and subsequently consider its quantization. For this reason our analysis is completely classical, at least for the time being. Thus we use (classical) Poisson brackets in place of (quantum) commutators. On the other hand, the authors of [14] discuss a quantum DSR theory from the very beginning and consider the spacetime as quantum in nature. However, the classical nature of the field variables are retained in [14] in the sense that products of fields are not replaced by their $*$-product. In this section we will construct the DSR Dirac equation in MS base (see Appendix A). To motivate the present derivation, we have to introduce the full κ-Minkowski Non-Commutative (NC) phase space,

$$\{x^i, x^0\} = \lambda x^i; \qquad \{x^i, x^j\} = 0; \qquad \{x^i, p^j\} = -g^{ij}; \qquad \{p^\mu, p^\nu\} = 0; \tag{4}$$

$$\{x^0, p^i\} = \lambda p^i; \qquad \{x^i, p^0\} = 0; \qquad \{x^0, p^0\} = -1 + \lambda p^0. \tag{5}$$

We are in the classical framework and will interpret the phase space algebra as Poisson brackets. Our metric is diag $g^{00} = -g^{ii} = 1$. For convenience it is expressed in a covariant form,

$$\{x_\mu, x_\nu\} = \lambda(x_\mu\eta_\nu - x_\nu\eta_\mu), \qquad \{x_\mu, p_\nu\} = -g_{\mu\nu} + \lambda\eta_\mu p_\nu, \qquad \{p_\mu, p_\nu\} = 0, \tag{6}$$

where $\eta_0 = 1$, $\eta_i = 0$. For completeness we mention that the Lorentz generator

$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \tag{7}$$

satisfies the *undeformed* Lorentz algebra,

$$\{J^{\mu\nu}, J^{\alpha\beta}\} = g^{\mu\beta}J^{\nu\alpha} + g^{\mu\alpha}J^{\beta\nu} + g^{\nu\beta}J^{\alpha\mu} + g^{\nu\alpha}J^{\mu\beta}, \tag{8}$$

but induces deformations in the transformation laws. The energy–momentum dispersion law, consistent with the form of $J_{\mu\nu}$ given in (7) is the MS relation (2),

$$p^2 = M^2[1 - \lambda E]^2 = M^2[1 - \lambda(\eta p)]^2. \tag{9}$$

Indeed in all the above computations one uses the NC phase space algebra (6).

It is quite obvious that a direct generalization of the physical laws in the NC phase space is complicated, both mathematically as well as conceptually [6,7]. However, there is an easy way out [13] which we now explain. We can introduce a map $(X_\mu, P_\mu) = F^{-1}(x_\mu, p_\mu)$ [13] between (x_μ, p_μ)—the *physical* κ-Minkowski NC phase space and (X_μ, P_μ)—a *completely canonical* phase space. Explicitly the transformation rules are the following:

$$X_\mu \equiv x_\mu\big(1 - \lambda(\eta p)\big) = x_\mu(1 - \lambda E); \qquad P_\mu \equiv \frac{p_\mu}{(1 - \lambda(\eta p))} = \frac{p_\mu}{(1 - \lambda E)}. \tag{10}$$

The (X_μ, P_μ) variables obey canonical Poisson Bracket algebra

$$\{X_\mu, P_\nu\} = -g_{\mu\nu}; \qquad \{X_\mu, X_\nu\} = \{P_\mu, P_\nu\} = 0. \tag{11}$$

We have also shown in [13] that (X_μ, P_ν) have conventional (Special Theoretic) Lorentz transformation properties whereas, as mentioned before, (x_μ, p_ν) obey κ-deformed Lorentz transformation laws [15].

The above relations in (10) are invertible,

$$x_\mu = X_\mu\big(1 + \lambda(\eta P)\big) = X_\mu(1 + \lambda P_0); \qquad p_\mu = \frac{P_\mu}{(1 + \lambda(\eta P))} = \frac{P_\mu}{(1 + \lambda P_0)}. \tag{12}$$

We note that the above relations are classical and one has to order them appropriately under quantization. However, we will show these mappings survive under some restrictions and in the present work that is sufficient.

Our framework of generalizing physical laws to κ-Minkowski phase space is the following: Start with the known form of a relation in the (auxiliary) canonical phase space (X_μ, P_ν). Now simply map the relation using (10) to the physical NC λ-variables (x_μ, p_ν). This yields the cherished λ-deformed physical law. Indeed it is not obvious that the procedure will work but we have explicitly shown its validity in various instances in [13] in the context of point particle models. We will demonstrate (see Appendix A) that this principle works for the Dirac equation as well.

To this end, we start with the momentum space Dirac equation in canonical phase space:

$$\big(\gamma^\mu P_\mu - M\big)u(\vec{P}) = 0, \tag{13}$$

where γ^μ are the standard "γ matrices". We now provide the one line derivation of the DSR Dirac equation in MS base:

$$\left(\gamma^\mu \frac{p_\mu}{(1 - \frac{(\eta p)}{\kappa})} - M\right)u(\vec{p}) = 0. \tag{14}$$

This is the λ-extended Dirac equation in MS basis. This is a new result. In Appendix A, we substantiate the validity of our claim by comparing (14) with existing results in the literature [6].

For applying the FW transformation on our MS–Dirac equation (14) we express it in Hamiltonian form,

$$\big(E - \vec{\alpha}\,.\,\vec{p} - \beta M(1 - \lambda E)\big)u = 0. \tag{15}$$

After rearranging (15) we obtain,

$$E(1 + \lambda M\beta)u = (\vec{\alpha}\,.\,\vec{p} + \beta M)u, \tag{16}$$

$$Eu = \frac{(1 - \lambda M\beta)}{1 - \lambda^2 M^2}(\vec{\alpha}\,.\,\vec{p} + \beta M)u. \tag{17}$$

Quite surprisingly we find that the Hamiltonian H (or E in (17)) has a *non-Hermitian* structure:

$$H = A + B\beta + C\vec{\alpha}\vec{p}, \tag{18}$$

where

$$A = \frac{-\lambda M^2}{1 - \lambda^2 M^2}, \tag{19}$$

$$B = \frac{M}{1 - \lambda^2 M^2}, \tag{20}$$

$$C = \frac{(1 - \lambda M\beta)}{1 - \lambda^2 M^2}. \tag{21}$$

This is a new result. This feature was not revealed in previous analysis [6,7] since a proper quantum treatment of the DSR Dirac equation was not attempted. This problem is tackled by performing a similarity transformation on H with the matrix D,

$$D = C^{-1/2} = \sqrt{1 + \frac{M}{\kappa}\beta} = a + b\beta, \tag{22}$$

where

$$a = \sqrt{\frac{1}{2}\left(1 + \sqrt{1 - \frac{M^2}{\kappa^2}}\right)}; \qquad b = \sqrt{\frac{M^2}{2\kappa^2(1 + \sqrt{1 - \frac{M^2}{\kappa^2}})}}. \tag{23}$$

This leads to the final Hermitian form of H:

$$H = D(A + B\beta + C\vec{\alpha}\vec{p})D^{-1} = A + B\beta + \left(1 - \frac{M^2}{\kappa^2}\right)^{-\frac{1}{2}} \vec{\alpha}\,.\,\vec{p}. \tag{24}$$

This H is suitable for the conventional FW procedure with similarity transformation performed by the unitary matrix U,

$$U = \frac{\sqrt{(\frac{1}{1-\lambda^2 M^2})\vec{p}^2 + B^2} + B + \frac{1}{\sqrt{1-\lambda^2 M^2}}\beta\alpha\,.\,\vec{p}}{(2\sqrt{\frac{\vec{p}^2}{1-\lambda^2 M^2} + B^2}(\sqrt{\frac{\vec{p}^2}{1-\lambda^2 M^2} + B^2} + B))^{1/2}}, \tag{25}$$

which can also be written as,

$$U = \frac{(\sqrt{(1-\lambda^2 M^2)\vec{p}^2 + M^2} + M + \sqrt{1-\lambda^2 M^2}\beta\alpha\,.\,\vec{p})}{(2\sqrt{(1-\lambda^2 M^2)\vec{p}^2 + M^2}(\sqrt{(1-\lambda^2 M^2)\vec{p}^2 + M^2} + M))^{1/2}}. \tag{26}$$

In this way we see that U is very similar to the usual FW transformation U_0 for a free particle of momentum $\vec{p}^2$ which is

$$U_0 = \frac{(\sqrt{\vec{p}^2 + M^2} + M + \beta\alpha\,.\,\vec{p})}{(2\sqrt{\vec{p}^2 + M^2}(\sqrt{\vec{p}^2 + M^2} + M))^{1/2}}. \tag{27}$$

Indeed, U in (26) reduces to U_0 for $\lambda = 0$.[2] The energy eigenvalues of H are:

$$\begin{aligned} E_\pm &= A \pm \sqrt{\left(1 - \frac{M^2}{\kappa^2}\right)^{-1}\vec{p}^2 + B^2} = \frac{-\lambda M^2}{1-\lambda^2 M^2} \pm \sqrt{\left(\frac{1}{1-\lambda^2 M^2}\right)\vec{p}^2 + \left(\frac{M}{1-\lambda^2 M^2}\right)^2} \\ &= \frac{1}{1-\lambda^2 M^2}\left(-\lambda M^2 \pm \sqrt{(1-\lambda^2 M^2)\vec{p}^2 + M^2}\right). \end{aligned} \tag{28}$$

Clearly they satisfy the MS dispersion law (2). This can be considered as an alternative way of obtaining the MS dispersion law starting from the MS extension of Dirac equation, in the spirit of [18]. We emphasize the framework adopted by us, i.e. constructing the MS–Dirac equation by exploiting the canonical phase space coordinates and subsequently utilizing the FW formalism to compute the energy eigenvalues that are identical with the MS energy values, is quantum mechanical in nature and is totally new.

We would like to make a comment on the particle and anti-particle spectra corresponding to the MS dispersion law, although this is not directly relevant for the present work. For the normal Dirac particle in commutative spacetime, the particle and anti-particle spectra, $E_\pm = \pm\sqrt{\vec{p}^2 + M^2}$ are symmetrically placed with respect to the zero-energy level. This is obtained by putting $\lambda = 0$ in (28). At the same time it is evident that this feature is absent for the MS spectra (28) with a non-zero λ. However, we can restore this symmetry simply by *shifting* the zero level of the energy value by the *constant* amount $\frac{-\lambda M^2}{1-\lambda^2 M^2}$ in (28). Thus the particle and anti-particle sector energy levels are given by

$$E_\pm = \pm\frac{\sqrt{(1-\lambda^2 M^2)\vec{p}^2 + M^2}}{1-\lambda^2 M^2}. \tag{29}$$

3. Physical position operator and Berry curvature effects on particle dynamics

It is well known that (in the normal commutative spacetime) the position operator $\vec{r}$ in Dirac equation does not represent the physical coordinate of the particle simply because the magnitude of the particle velocity obtained from taking time evolution of $\vec{r}$ is c, even for a massive particle. On the other hand the FW formalism provides a natural prescription for a physical position operator $\vec{r}_+$ that correctly reproduces the particle velocity. Incidentally $\vec{r}_+$ operators are non-commutative in nature and as we shall see below, this feature can be attributed to a Berry phase effect. Phenomena of a similar have been observed before [19] in condensed matter systems. In the present case we find that the Berry curvature effect is further modified by the λ-corrections.

We now limit ourself to the dynamical evolution of a positive energy particle whose energy is formally the projection (we denote the formal projection operation by $\mathcal{P}$) of the Hamiltonian on the positive energy subspace. Physically, we thus consider the adiabatic approximation which allows to neglect the interband transition, by identifying the momentum degree of freedom as slow and the spin degree of freedom as fast, similarly to the nuclear configuration in adiabatic treatment of molecular problems. To be coherent with this projection on the positive energy subspace, the same projection has to be done for all dynamical operators. Therefore the

[2] Once again, in an algebraic way, U can be computed directly from U_0 by the change of variable $\vec{p} \to \sqrt{1-\lambda^2 M^2}\,\vec{p}$. This is simply because in our equation (24) the factor $(1-\lambda^2 M^2)^{-\frac{1}{2}}$ appears in front of $\vec{\alpha}\,.\,\vec{p}$.

physical position operator $\vec{r}_+$ is obtained as

$$\vec{r}_+ = \mathcal{P}\big(UD\vec{r}D^{-1}U^+\big) = \mathcal{P}\big(U\vec{r}U^+\big) \tag{30}$$

with the MS position operator $\vec{r} = i\hbar\frac{\partial}{\partial\vec{p}}$. Notice that we are using the canonical representation of the position operator (valid for commutative spacetime) for the λ-variables as well. This is allowed if one recalls from (4), (5), (6) that the spatial sector remains unaffected in this NC spacetime. This can also be justified in a more rigorous way by going to the canonical phase space framework (10). This leads to the following explicit form of $\vec{r}_+$:

$$\vec{r}_+ = i\hbar\mathcal{P}\left(U\frac{\partial}{\partial\vec{p}}U^+\right) = i\hbar\frac{\partial}{\partial\vec{p}} + \mathcal{P}\left(Ui\hbar\frac{\partial U^+}{\partial\vec{p}}\right) \equiv \vec{r} + \mathcal{A}_{(r)} \tag{31}$$

where we have introduced $\mathcal{A}_{(r)}$, the so-called Berry connection. The (r) in $\mathcal{A}_{(r)}$ indicates that this Berry connection accompanies the position $\vec{r}$. (In general, Berry connections can appear both in coordinate and momentum.) Indeed the state space of the Dirac electron is spanned by the basis of plane waves of the form $|p,\varphi_\alpha\rangle = |p\rangle|\varphi_\alpha\rangle$, where $|\varphi_\alpha\rangle$ is a zero-momentum spinor, such that $U(p)|\varphi_\alpha\rangle = |u_\alpha(p)\rangle$ which is a spinor solution of the Dirac equation. For $|\varphi_\alpha\rangle = (1,0,0,0)$ and $|\varphi_\beta\rangle = (1,0,0,0)$ the matrix elements of $\mathcal{A}_{(r)}$ are then

$$(\mathcal{A}_{(r)})_{\alpha\beta} = i\hbar\langle\varphi_\alpha|U(p)\frac{\partial U^+(p)}{\partial\vec{p}}|\varphi_\beta\rangle = i\hbar\big\langle u_\alpha(p)\big|\frac{\partial}{\partial\vec{p}}\big|u_\beta(p)\big\rangle.$$

This is the definition of the Berry connection. The explicit structure of $\mathcal{A}_{(r)}$ in the present context is

$$\mathcal{A}_{(r)} = \mathcal{P}\big(U\vec{r}U^{-1}\big) = \hbar\frac{\sqrt{(1-\lambda^2M^2)}(\vec{p}\times\vec{\sigma})}{2\sqrt{(1-\lambda^2M^2)\vec{p}^2+M^2}(\sqrt{(1-\lambda^2M^2)\vec{p}^2+M^2}+M)}, \tag{32}$$

with $\vec{\sigma}$ being the Pauli matrices. Note that for a free particle the momentum is FW invariant, i.e. $\vec{p}_+ = \mathcal{P}(U\vec{p}U^{-1}) = \vec{p}$.

As we have mentioned before, the physical position operators $\vec{r}_+$ are no longer commutative:

$$\begin{aligned} &\big[r_+^i, r_+^j\big] = i\hbar\varepsilon^{ijk}\Theta_k = i\hbar\big(\big(\nabla_{(p)}^i\mathcal{A}_{(r)}^j - \nabla_{(p)}^j\mathcal{A}_{(r)}^i\big) + \big[\mathcal{A}_{(r)}^i, \mathcal{A}_{(r)}^i\big]\big), \\ &\big[p^i, p^j\big] = 0, \qquad \big[p^i, r_+^j\big] = -i\hbar\delta^{ij} \end{aligned} \tag{33}$$

with the *non-Abelian* Berry curvature defined by

$$\vec{\Theta} = -\frac{\hbar}{((1-\lambda^2M^2)\vec{p}^2+M^2)^{3/2}}\left(M\vec{\sigma} + \frac{(1-\lambda^2M^2)(\vec{p}\,.\,\vec{\sigma})\vec{p}}{\sqrt{(1-\lambda^2M^2)\vec{p}^2+M^2}+M}\right). \tag{34}$$

The dynamics is generated by the Hamiltonian equations of motion,

$$\dot{O} = i[H, O], \tag{35}$$

where O is a generic operator and H is the Hamiltonian. In the present instance with E_+ representing H, we find,

$$\dot{\vec{r}}_+ = \frac{\vec{p}}{\sqrt{(1-\lambda^2M^2)\vec{p}^2+M^2}}; \qquad \dot{\vec{p}} = 0. \tag{36}$$

Also one finds that the non-relativistic expression for velocity is obtained for $M = 1/\lambda$ which in fact corresponds to the upper bound of mass (or energy) available for the particle. In the next section we will study the dynamics of the MS Dirac particle in presence of external interaction.

4. DSR dynamics in presence of scalar interaction and Berry curvature effects

In general it is not possible to perform the FW transformation exactly when interactions are present but one can always resort to a perturbative study for a "small" potential term. But here instead of a perturbative expansion we consider a semiclassical approximation ($O(\hbar)$) and in addition limit ourself to $O(\lambda)$ effect only.

Once again we start with the free MS–Dirac equation (14) in canonical phase space and introduce an interaction potential $V(\vec{R})$,

$$\big\{\gamma^\mu P_\mu - M - \gamma^0V(\vec{R})\big\}u(\vec{P}) = 0. \tag{37}$$

This interaction can be thought of as scalar component of $U(1)$ interaction, in a particular frame where the vector potential vanishes.

Translated to λ-variables this equation reads

$$\left\{\gamma^\mu\frac{p_\mu}{(1-\lambda(\eta p))} - M - \gamma^0V\big(\vec{r}(1-\lambda E)\big)\right\}u(\vec{p}) = 0. \tag{38}$$

Let us consider the λ in its lowest non-trivial order. Hence to $O(\lambda)$ the Dirac equation is

$$\big(1+\lambda V(\vec{r})+\lambda\vec{r}\,.\,\nabla V(\vec{r})\big)Eu(\vec{p})=\big(A+B'\beta+C(\vec{\alpha}\,.\,\vec{p})+V(\vec{r})\big)u(\vec{p}) \tag{39}$$

with $B'=B-\lambda MV(\vec{r})$ and A, B, C given in Eqs. (19)–(21).

To fit in the FW scheme (39) is rewritten as

$$Eu(\vec{p})=\big(1-\lambda V(\vec{r})-\lambda\vec{r}\,.\,\nabla V(\vec{r})\big)\big(A+B'\beta+C(\vec{\alpha}\,.\,\vec{p})+V(\vec{r})\big)u(\vec{p}). \tag{40}$$

The same procedure as in the free case is needed to render this H Hermitian and we obtain a Hamiltonian that is Hermitian in $O(\lambda)$ and $\hbar$:

$$\begin{aligned} H &= D\big(1-\lambda V(\vec{r})-\lambda\vec{r}\,.\,\nabla V(\vec{r})\big)\big(A+B'\beta+C\vec{\alpha}\,.\,\vec{p}+V(\vec{r})\big)D^{-1} \\ &= \big(1-\lambda V(\vec{r})-\lambda\vec{r}\,.\,\nabla V(\vec{r})\big)D\big(A+B'\beta+C\vec{\alpha}\vec{p}+V(\vec{r})\big)D^{-1} \\ &= \big(1-\lambda V(\vec{r})-\lambda\vec{r}\,.\,\nabla V(\vec{r})\big)\left(A+B'\beta+\left(1-\frac{M^2}{\kappa^2}\right)^{-\frac{1}{2}}\vec{\alpha}\,.\,\vec{p}+V(\vec{r})\right). \end{aligned} \tag{41}$$

In Ref. [20] we have shown how to diagonalize Dirac like Hamiltonians in the presence of external fields at the semiclassical order (to get corrections beyond the semiclassical order see Ref. [21]). During this process of diagonalization, we have shown that in general (in the presence of electromagnetic or gravitational fields), both position and momentum operators acquire a Berry-phase contribution making the coordinate and momentum algebra non-commutative. Note that in the presence of interactions, the projected (on the positive energy subspace) dynamical operators emerge naturally during the diagonalization, without resorting to the adiabatic approximation.

Therefore applying this diagonalization procedure in our case leads for the positive energy sector to the following semiclassical diagonal Hamiltonian

$$H_+=\mathcal{P}\big(UHU^{-1}\big) \tag{42}$$

where U is the same matrix as in the free case with M replaced by an effective mass $M(1-\lambda V(\vec{r}))$. This leads to

$$H_+=\big(1-\lambda V(\vec{r}_+)-\lambda\vec{r}_+\nabla V(\vec{r}_+)\big)\Big(\sqrt{\vec{p}^2+\big(1-\lambda V(\vec{r}_+)\big)^2M^2}-\lambda M^2+V(\vec{r}_+)\Big)+o\Big(\sqrt{\lambda^2+\hbar^2}\Big) \tag{43}$$

where $\vec{r}_+=\vec{r}+\mathcal{A}_{(\lambda,r)}$ with $\mathcal{A}_{(\lambda,r)}$ the same as in the free case with the effective mass $M(1-\lambda V(\vec{r}_+))$. We observe a renormalization of the energy through the term

$$\big(1-\lambda V(\vec{r}_+)-\lambda\vec{r}_+\,.\,\nabla V(\vec{r}_+)\big).$$

Note that the momentum get also a Berry potential correction which at the order considered is negligible $\vec{p}_+=\vec{p}+O(\lambda\hbar)$. The commutation relations are also unchanged except for the mass renormalization. Hence, the dynamics in presence of the potential is derived to be,

$$\dot{\vec{r}}_+=\nabla_{(p)}H_+-\dot{\vec{p}}\times\vec{\Theta};\qquad \dot{\vec{p}}=-\nabla_{(r_+)}H_+, \tag{44}$$

where $\vec{\Theta}$ is given in (34) (with M replaced by an effective mass $M(1-\lambda V(\vec{r}))$). Note that the effect of Berry curvature is manifested very directly once an interaction is present. Indeed, the equation for the velocity contains an anomalous velocity term $\dot{\vec{p}}\times\vec{\Theta}$ of order $\hbar$ which causes an additional displacement of the electrons orthogonally to the momentum $\vec{p}$. This phenomenon depends on the particle spin through the Berry curvature. Typically, in canonical spacetime, with V being an electrostatic potential, this is the well studied spin Hall effect. In fact it should be mentioned that the topics of spin Hall effect of electrons or photons, gravitational Hall effect for photons, that have become areas of intense research, all owe there existence to this type of Berry curvature effect. However, as we point out below, modelling the analogue of spin Hall effect for MS particle is probably more complicated.

5. Concluding remarks

In this Letter we have considered particle dynamics in κ-Minkowski spacetime which is a particular form of non-commutative spacetime. Indeed quantization of the particle model is tricky because of the operatorial form of non-commutativity involved in the phase space commutation relations. The novelty of our work lies in the fact that we have been able to bypass this problem by exploiting a semiclassical approach of the Foldy–Wouthuysen formalism. Indeed this makes our analysis semi-classical in nature with quantum effects in $(O(\hbar)$ taken into account. We then have shown that the dynamics of MS particles in the presence of an external field should be influenced by the κ-parameter but also by the presence of a Berry curvature which is itself κ dependent.

In this connection, let us pause to mention the significant (and possibly more ambitious) work [22] that attempts to deal with the full κ-Minkowski quantum field theory. Considering the $*$-product for the fields, the authors of [22] demonstrate that if properly interpreted, the interaction vertices have the requisite symmetry under the interchange of momenta of identical incoming particles. Indeed, the κ-Minkowski $*$-product is much more involved that (Moyal) $*$-product that appears when the non-commutativity is constant in nature. It is true that we have so far only looked at the quantum mechanics of a DSR model but the nature of the energy–momentum conservation laws [13] inherently possess the above mentioned symmetry. This seems to indicate that the DSR in Magueijo–Smolin base might be a better alternative that DSR in other bases in the context of formulating the DSR quantum field theory.

The next task we wish to pursue is the effect of full electromagnetic interactions on the DSR particle model considered here. This is a non-trivial extension since electromagnetism involves a $U(1)$ gauge symmetry and one has to appropriately generalize the concept of gauge invariance in non-commutative spacetime. This feature probably did not show up in the present restricted setup.

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Appendix A

We will compare the MS Dirac equation obtained here in (14) with that derived in [6]. It is to be noted that [6] uses the bicrossproduct base whereas we have used the MS base.

It is well known in the DSR community that there are distinct formulations (or bases) of the DSR regarding the modified dispersion relation, transformation properties and phase space algebra. The most popular are known as bicrossproduct base [4], MS base [5] and standard base [16]. The different bases are connected by non-linear transformation and [17] but the different bases are inequivalent with drastically different physical consequences due to the non-linearity in the transformations.

The DSR Dirac equation obtained in [6] is

$$\left(\gamma^{\mu} D_{\mu} - \omega\right) u = 0, \tag{A.1}$$

where

$$D_0 = \frac{e^{\lambda E'} - \cosh(\lambda\omega)}{\sinh(\frac{\omega}{\kappa})}, \tag{A.2}$$

$$D_i = \frac{e^{\frac{E'}{\kappa}}}{\kappa \sinh(\lambda\omega)} p'_i. \tag{A.3}$$

Here (E', p'_i) are energy and momenta in bicrossproduct basis and they satisfy the dispersion law [4],

$$\frac{2}{\lambda^2} \cosh(\lambda E') - (\vec{p}')^2 e^{\lambda E'} = 2\kappa^2 \cosh(\lambda\omega). \tag{A.4}$$

The relations connecting the above set to the MS set of variables (that we have used) are [17],

$$p'_i = p_i; \qquad E' = -\frac{1}{2\lambda} \log\left(1 - 2\lambda E + \lambda^2 \vec{p}^2\right), \qquad E = \frac{1}{2\lambda}\left(1 - e^{-2\lambda E'} + \lambda^2 \vec{p}'^2\right); \qquad \lambda m = \tanh(\lambda\omega). \tag{A.5}$$

It is straightforward to check that using (A.5) on (A.1) will generate (14), the MS–Dirac equation derived in our Letter.

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