Europ. J. Combinatorics (1985) **6**, 1–6

The Asymptotic Number of Irreducible Partitions*

EDWARD A. BENDER, ANDREW M. ODLYZKO AND L. BRUCE RICHMOND

A partition of $[1, n] = \{1, \ldots, n\}$ is called *irreducible* if no proper subinterval of $[1, n]$ is a union of blocks. We determine the asymptotic relationship between the numbers of irreducible partitions, partitions without singleton blocks, and all partitions when the block sizes must lie in some specified set.

1. STATEMENT OF RESULTS

A partition of $[1, n] = \{1, \ldots, n\}$ is called *irreducible* if no proper subinterval of $[1, n]$ is a union of blocks. Let $\mathcal{B}$ be a set of positive integers, $a_n = a_n(\mathcal{B})$ the number of partitions of $[1, n]$ with block lengths in $\mathcal{B}$, f_n the number of those which have no singleton blocks, and i_n the number which are irreducible. The asymptotic behavior of the a_n, f_n, and i_n depends in a complicated way on the set $\mathcal{B}$. We will show, however, that there are simple relations between these functions.

Let $\mathcal{B}^* = \mathcal{B} - \{1\}$. (Possibly $\mathcal{B}^* = \mathcal{B}$.) Thus $f_n(\mathcal{B}) = a_n(\mathcal{B}^*)$. Since $0 \leq i_n(\mathcal{B}) \leq f_n(\mathcal{B})$, and $a_n(\mathcal{B}^*) = 0$ if n is not a multiple of $d = \gcd(\mathcal{B}^*)$, we will restrict our attention to those n which are multiples of d. Let

$$f(x) = \textstyle\sum_{b \in \mathcal{B}^*} x^b / b! \quad \text{and} \quad F(x) = e^{f(x)}. \tag{1.1}$$

Then $F(x)$ is the exponential generating function for f_n[2, ex. 10].

THEOREM 1. *Suppose $\mathcal{B}^* \neq \emptyset$. Let $d = \gcd(\mathcal{B}^*)$. In what follows, $n \to \infty$ through multiples of d. We have*

$$i_n \sim \begin{cases} f_n, & \text{if } \mathcal{B}^* \neq \{2\}; \\ f_n / e, & \text{if } \mathcal{B}^* = \{2\}. \end{cases} \tag{1.2}$$

If $1 \notin \mathcal{B}$, then $f_n = a_n$, while if $1 \in \mathcal{B}$, then

$$i_n \sim \begin{cases} d a_n / e^{r_n}, & \text{if } \mathcal{B}^* \neq \{2\}, \\ 2 a_n / e^{n^{1/2} + 3/4}, & \text{if } \mathcal{B}^* = \{2\}, \end{cases} \tag{1.3}$$

where r_n is the unique positive real number that satisfies $n = r_n f'(r_n)$.

COROLLARY 1. *If $1 \in \mathcal{B}$ and $\mathcal{B}$ omits at most a finite set of positive integers, then*

$$i_n \sim a_n (\log n) / n. \tag{1.4}$$

COROLLARY 2. *If $\max \mathcal{B} = B < \infty$, $1 \in \mathcal{B}$, and $\mathcal{B} \neq \{1, 2\}$, then, as $n \to \infty$ through multiples of d, we have*

$$i_n \sim \begin{cases} d a_n \exp[-((B-1)! n)^{1/B}], & \text{if } \mathcal{B} - 1 \notin \mathcal{B}; \\ d a_n \exp[-((B-1)! n)^{1/B} + (B-1)/B], & \text{if } B - 1 \in \mathcal{B}. \end{cases} \tag{1.5}$$

In [3, ex. 3] a functional equation of Beissinger [1] was used to prove the theorem for $\mathcal{B} = \{k\}$ by means of formal power series techniques. Here we use results of Hayman [5] on coefficients of entire functions to obtain more complete results. The convergence in

* Research by E. A. Bender and L. B. Richmond was supported by NSERC under grant A4067

0195-6698/85/010001+06 $02.00/0

Corollary 1 is very slow. For ordinary partitions ($\mathscr{B}$ the set of positive integers) and $n=500$ we found that $(a_n \log n)/ni_n \approx 1{\cdot}4$, which is quite far from 1. This is due mainly to the poor convergence of n/log n to e^{r_n} since $a_n\, e^{-r_n}/i_n = 1{\cdot}05$ at $n=500$.

When $\mathscr{B}=\{1,2,3,\dots\}$, we obtain the Bell numbers for $a_n(\mathscr{B})$. These have been studied asymptotically by the saddle point method. See deBruijn [4, (6.2.6)] for a derivation. Hayman [5] defined a class of functions to which he was able to apply the saddle point method to obtain asymptotics. His class includes (1.1) for many $\mathscr{B}$'s. We show that arguments like Hayman's can be used to cover all $\mathscr{B}$'s. This is the content of the technical lemmas in the next section. In Section 3 we use a combinatorial inequality to deduce (1.2a). We then apply the results of Section 2 to deduce (1.3) and the Corollaries.

As asymptotic formula for a_n easily follows, but, as in the case of the Bell numbers, is usually not easily used. We state it for completeness:

$$a_n \sim \frac{d \exp(\sum s_n/k!)n!}{s_n^n(2\pi \sum ks_n/(k-1)!)^{1/2}}, \tag{1.6}$$

where all sums are over $k \in \mathscr{B}$ and $s_n > 0$ is determined by $n = \sum s_n^k/(k-1)!$.

In this paper, the symbol C stands for a positive real constant, not necessarily the same at each occurence.

2. Two Lemmas

The lemmas in this section contain the necessary analytic results.

LEMMA 1. *Let $\mathscr{N}$ be a set of positive integers with greatest common divisor one and let g be a positive function on the positive integers with $g(n)/g(n-1) \geq Cn$. Define*

$$h(z) = \sum_{n\in\mathscr{N}} \frac{z^n}{g(n)} \quad \textit{and} \quad \sum_{n\geq 0} H_n z^n = H(z) = e^{h(z)}.$$

Then $H_n/H_{n+1} \sim s_n$, where s_n is the positive root of $n = s_n h'(s_n)$.

PROOF. Define, following Hayman [5],

$$\begin{aligned} A(r) &= \frac{dh(r)}{d \log r} = \sum_{n\in\mathscr{N}} \frac{nr^n}{g(n)}, \\ B(r) &= \frac{dA(r)}{d \log r} = \sum_{n\in\mathscr{N}} \frac{n^2 r^n}{g(n)} = O(e^{Cr}). \end{aligned} \tag{2.1}$$

If $H(z)$ is admissible in the sense of Hayman, then we are done by [5, cor. IV]. Unfortunately this is not always true. We will show that the conditions for admissibility hold except for $\theta \in \Theta$, where $\Theta = \Theta(r)$ is a set such that

$$\mu(\Theta) \sup_{\theta\in\Theta} |H(r\, e^{i\theta})| = o(H(r)/B(r)^{1/2}),$$

where μ denotes Lebesgue measure. It is easy to verify that Hayman's proofs carry over to this situation. If $\mathscr{N}$ is finite, then $H(z)$ is admissible [5, thm. X], and we are done. Hence suppose that $\mathscr{N}$ is infinite. Clearly $B(r) \to \infty$ as $r \to \infty$, giving Hayman's condition [5; (2.6)] for admissibility.

Since $h(z)$ is an entire function,

$$h(r\, e^{i\theta}) = h(r) + i\theta A(r) - \frac{\theta^2}{2} B(r) + O(\theta^3 \sum n^3 r^n/g(n)).$$

When $n \geq Cr$, it readily follows from Stirling's formula that $r^n/g(n) < 2^{-n}$ and so

$$h(r\,e^{i\theta}) = h(r) + i\theta A(r) - \frac{\theta^2}{2} B(r) + O(\theta^3 r^3 h(r)), \quad \text{for} \quad |\theta| \leq \pi.$$

Since $h(r)$ and $B(r)$ grow faster than any fixed power of r,

$$H(r\,e^{i\theta}) \sim H(r)\,e^{i\theta A(r) - \theta^2 B(r)/2}$$

uniformly for $|\theta| \leq r^2/B(r)^{1/2}$. This proves Hayman's condition [5, (2.4)] for admissibility.

It remains to consider $r^2/B(r)^{1/2} \leq |\theta| \leq \pi$. In the remainder of the proof, we assume that r is sufficiently large. Let $\nu = \nu_r$ and $M(r)$ be such that a maximum term in the sum (2.1) for $B(r)$ occurs at $n = \nu$ and has value $M(r)$. By Stirling's formula, $M(r) = O((Cr/\nu)^\nu)$. Since each term in the sum is unbounded as $r \to \infty$, $\nu = O(r)$. One also has $B(r) = O(\nu M(r))$ and so $M(r) \geq CB(r)/r$. let

$$z = r\,e^{i\theta}, \quad \text{where} \quad \theta = \alpha + \frac{2\pi j}{\nu} \quad \text{and} \quad |\alpha| \leq \frac{\pi}{\nu}.$$

We have

$$h(r) - \Re h(z) \geq \frac{r^\nu}{g(v)}(1 - \cos \nu\alpha) \geq C\frac{\nu^2 r^\nu}{g(\nu)}\alpha^2 \geq C\frac{B(r)\alpha^2}{r}.$$

If $|\alpha| \geq r^2/B(r)^{1/2}$, it follows from this and the fact that $B(r) = O(e^{Cr})$ that $H(z) = o(H(r)/B(r)^{1/2})$. Thus we may assume that $|\alpha| < r^2/B(r)^{1/2}$ and $j \neq 0$. Let $k > 3$ be a fixed element of $\mathcal{N}$ and let $\mathcal{M}$ be a fixed finite subset of $\mathcal{N}$ with greatest common divisor 1. If jk/ν is not an integer, then

$$h(r) - \Re h(z) \geq Cr^k(1 - \cos k\theta) \geq \frac{Cr^k}{\nu^2} \geq Cr^{k-2},$$

and so we are done by the bound in (2.1). If jk/ν is an integer, then j/ν is a multiple of $1/k$. Let

$$\Theta = \left\{\alpha + \frac{2\pi j}{k} : 0 < j < k, |\alpha| < r^2/B(r)^{1/2}\right\}.$$

Clearly $\mu(\Theta) \leq Cr^2 B(r)^{1/2}$. For every $\theta \in \Theta$ there is an associated j. Since $\gcd \mathcal{M} = 1$, we can associate with j (and hence with θ) an $m \in \mathcal{M}$ such that jm/k is nonzero modulo 1. Then

$$h(r) - \Re h(z) \geq \frac{Cr^m}{k^2} \geq Cr$$

and so $\sup|H(z)| \leq H(r)e^{-Cr}$.

LEMMA 2. *If r_u is defined by $u = r_u f'(r_u)$ for all positive real u, then*
(a) $\delta \log u \leq r_u = O(u^{1/3})$, *when $\mathcal{B}^* \neq \{2\}$ for some $\delta > 0$;*
(b) $v/u \leq (r_u/u)/(r_v/v) \leq (v/u)^{1/2}$ *when $v \leq u$;*
(c) $f_n/f_{n+d} \sim r_n^d/n^d$ *when n is a multiple of $d = \gcd(\mathcal{B}^*)$.*

PROOF. Define $a(r) = rf'(r)$ and $b(r) = ra'(r)$. Choose $k \in \mathcal{B}^*$ with $k \geq 3$ if possible. Since

$$r\,e^r = \sum_{b=1}^{\infty} r^b/(b-1)! > a(r) > r^k/(k-1)!,$$

(a) follows. From $a(r_t)=t$, we obtain

$$\frac{\mathrm{d}r_t}{\mathrm{d}t}=\frac{1}{a'(r_t)}=\frac{r_t}{b(r_t)}>0,$$

and so

$$-\frac{1}{t}<\frac{\mathrm{d}\log(r_t/t)}{\mathrm{d}t}=\frac{1}{b(r_t)}-\frac{1}{t}\leqslant-\frac{1}{2t},$$

since $b(r_t)\geqslant 2a(r_t)=2t$. Integrating over $v\leqslant t\leqslant u$ yields (b).

We now prove (c). Let $h(z)=f(z^{1/d})$ in Lemma 1. Then $H(z)=F(z^{1/d})$, $\mathcal{N}=\mathcal{B}^*/d$, and $g(n)=(nd)!$. Hence by [5, cor. IV],

$$\frac{f_{dk}/(dk)!}{f_{dk+d}/(dk+d)!}\sim s_k,$$

where s_k is defined by $k=s_k h'(s_k)$. Since $sh'(s)=s^{1/d}f'(s^{1/d})/d$, we have $s_k=r_{dk}^d$.

3. Proof of the Theorem

We first prove (1.2). If $\mathcal{B}^*=\{2\}$, we may apply Kleitman's result [6] or [3, ex. 3]. Suppose $\mathcal{B}^*\neq\{2\}$. Since a singleton block is a subinterval, $i_n\leqslant f_n$. The number of partitions with no fixed points and proper subinterval $[j, j+k-1]$ is $f_k f_{n-k}$. Since $1<j<n$ and $f_1=0$,

$$0\leqslant f_n-i_n\leqslant n\sum_{k=2}^{n-2} f_k f_{n-k}\leqslant 2n\sum_{k=2}^{n/2} f_k f_{n-k}. \tag{3.1}$$

We may restrict the index of the sum further to multiples of d since all other terms are zero. By Lemma 2(c) and then (b),

$$\frac{f_{k+d}f_{n-k+d}}{f_k f_{n-k}}<C\frac{r_{n-k}^d/(n-k)^d}{r_k^d/k^d}\leqslant C\left(\frac{k}{n-k}\right)^{d/2}, \quad \text{for} \quad k\leqslant n/2. \tag{3.2}$$

Let m be the least k for which $f_k\neq 0$. Taking a product of (3.2) over $m\leqslant k<i$ in multiples of d and rearranging we obtain

$$\frac{f_i f_{n-i}}{f_m f_{n-m}}<\frac{C^i}{\binom{n-m}{i-m}^{1/2}}.$$

Combining this with (3.1) we obtain $f_n-i_n=O(nf_{n-m})$, which is $O(r_n^m f_n/n^{m-1})$. Since $m\geqslant 2$, this is $o(f_n)$ by Lemma 2(a). This proves (1.2).

Consider (1.3a). Clearly

$$a_n=\sum_{k=0}^{n-2}\binom{n}{k}f_{n-k}. \tag{3.3}$$

For $k\leqslant n^{2/5}=N$ we have $\binom{n}{k}\sim n^k/k!$ and, by Lemma 2(c) and (b), $f_{n-k}\sim f_n(r_n/n)^k$. Thus

$$\sum_{k\leqslant N}\binom{n}{k}f_{n-k}\sim f_n\sum_{k:\,d|k} r_n^k/k!\sim f_n\,\mathrm{e}^{r_n}/d.$$

For $k \geq N$, we have by Lemma 2(c), the left half of (b) and (a), respectively,

$$\frac{\binom{n}{k} f_{n-k}}{\binom{n}{k-d} f_{n-k+d}} \leq O(((n-k)/k)^d (r_{n-k}/(n-k))^d) = O((r_n/N)^d) = o(1).$$

Thus the sum over $k \geq N$ in (3.3) is negligible, proving (1.3a).

Suppose $\mathscr{B}^* = \{2\}$. Clearly $f_n = 2^{-n/2} n!/(n/2)!$ and a_n is the number of involutions of an n-element set. The former is easily seen to be asymptotic to $2^{1/2}(n/\mathrm{e})^{n/2}$ by Stirling's formula. The number of involutions is well known to be asymptotic to

$$\left(\frac{n}{\mathrm{e}}\right)^{n/2} \frac{\mathrm{e}^{n^{1/2}-1/4}}{2^{1/2}}.$$

This completes the proof of (1.3).

4. Proof of the Corollaries

We begin with a simple observation.

Lemma 3. *Let $P(x)$ be a polynomial of degree $p \leq (\max \mathscr{B}) - 2$ and define s_u by*

$$\sum_{b \in \mathscr{B}^*} b s_u^b / b! + P(s_u) = u.$$

Then $\mathrm{e}^{r_n} \sim \mathrm{e}^{s_n}$.

Proof. Let $\beta \geq p+2$ be in $\mathscr{B}$. Then

$$|P(s_n)| = \left| \sum_{b \in \mathscr{B}^*} b(r_n^b - s_n^b)/b! \right| = |r_n - s_n| \sum_{b \in \mathscr{B}^*} \sum_{i=0}^{b-1} b r_n^i s_n^{b-i-1}/b! \geq |r_n - s_n| \beta s_n^{\beta-1}/\beta!.$$

Since the left side is a polynomial of degree $p < \beta - 1$, we have $r_n - s_n = o(1)$.

For Corollary 1 let $P(x) = \sum ix^i/i!$, the sum ranging over all positive integers $i \notin \mathscr{B}$. Then $s_n\, \mathrm{e}^{s_n} = n$, from which we have

$$\mathrm{e}^{-s_n} = s_n/n \sim (\log n)/n.$$

For Corollary 2 let $P(x) = -\sum ix^i/i!$, the sum ranging over all positive integers $i \in \mathscr{B}^*$ not exceeding $B-2$. Define $g(u)$ by $Bg(u)^B/B! = u$. If $B-1 \notin \in \mathscr{B}^*$, then $s_n = g(n)$ and the Corollary follows. If $B-1 \in \mathscr{B}^*$, then

$$\begin{aligned}
s_n &= g(n - s_n^{B-1}/(B-2)!) \\
&= \left((B-1)!n\left(1 - \frac{s_n^{B-1}}{n(B-2)!}\right)\right)^{1/B} \\
&= ((B-1)!n)^{1/B}\left(1 - \frac{1}{B}\frac{s_n^{B-1}}{(B-2)!n} + O(s_n^{2B-2}/n^2)\right) \\
&= ((B-1)!n)^{1/B}\left(1 - \frac{1}{B}\frac{((B-1)!n)^{(B-1)/B}}{(B-2)!n} + O(N^{-2/B})\right).
\end{aligned}$$

References

1. J. S. Beissinger, Factorization and enumeration of labeled combinatorial objects, Ph.D. dissertation, University of Pennsylvania, 1981.
2. E. A. Bender and J. R. Goldman, Enumerative uses of generating functions, *Indiana Univ. Math. J.* **20** (1971), 753-765.
3. E. A. Bender and L. B. Richmond, An asymptotic expansion for the coefficients of some formal power series II: Lagrange inversion, *Discrete Math.* **50** (1984), 135-142.
4. N. G. deBruijn, *Asymptotic Methods of Analysis*, North-Holland, Amsterdam, 1958.
5. W. K. Hayman, A generalization of Stirling's formula, *J. reine u. angew. Math.* **196** (1956), 67-95.
6. D. J. Kleitman, Proportions of irreducible diagrams, *Studies in Appl. Math.* **49** (1970), 297-299.

Received 19 May 1983 and in revised form 16 April 1984

Edward A. Bender
University of California at San Diego,
La Jolla, CA 92093 U.S.A.
Andrew M. Odlyzko
Bell Laboratories, Murray Hill, NJ 07974, *U.S.A.*
and
Bruce Richmond
University of Waterloo, Waterloo,
Ontario N2L 3G1, Canada