Computers and Mathematics with Applications 60 (2010) 1607–1615

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Accurate analytical solutions to oscillators with discontinuities and fractional-power restoring force by means of the optimal homotopy asymptotic method

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ARTICLE INFO

Article history:
Received 14 March 2010
Received in revised form 8 May 2010
Accepted 27 June 2010

Keywords:
Nonlinear oscillators
Fractional-power restoring force
Optimal homotopy asymptotic method

ABSTRACT

In this paper a new approach combining the features of the homotopy concept with an efficient computational algorithm which provides a simple and rigorous procedure to control the convergence of the solution is proposed to find accurate analytical explicit solutions for some oscillators with discontinuities and a fractional power restoring force which is proportional to sign(x). A very fast convergence to the exact solution was proved, since the second-order approximation lead to very accurate results. Comparisons with numerical results are presented to show the effectiveness of this method. Four numerical applications prove the accuracy of the method, which works very well for the whole range of initial amplitudes. The obtained results prove the validity and efficiency of the method, which can be easily extended to other strongly nonlinear problems.

1. Introduction

In various fields of science and engineering, nonlinear evolution equations, as well as their analytic and numerical solutions, are fundamentally important. Considerable attention has been directed towards the study of strongly nonlinear oscillations and several methods have been used to find approximate solutions to such problems. Perturbation methods are well established tools to study various aspects of non-linear problems [1–3]. However, the use of perturbation theory in many important practical problems is either invalid or simply breaks down for parameters beyond a certain specified range. Therefore, new analytical techniques should be developed to overcome these shortcomings. Such new techniques should work over a greater range of parameters. There are some known attempts in this direction. Some extensions of the Lindstedt–Poincare method to strongly nonlinear systems have been proposed [4–6]. In [6], a new parameter was introduced which remains small regardless of the magnitude of the original parameter. In this way, a strongly non-linear system with a large parameter is transformed into a small parameter system.

Another powerful tool in solving nonlinear problems proves to be the harmonic balance method [1–3], which is a procedure for determining periodic solutions by using a truncated series; but in order to obtain a consistent solution one needs either to know a great deal about the solution a priori or to carry enough terms in the solution and check to order of the coefficients of all the neglected harmonics, as Nayfeh mentioned in [1]. An approach which combines the harmonic balance method and linearization of the non-linear oscillation equation was reported in [7].

There also exists a wide range of literature dealing with the analytical determination of approximate solutions for nonlinear problems using a mixture of methodologies [8–15].

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doi:10.1016/j.camwa.2010.06.042

The purpose of this paper is to construct accurate approximate periodic solutions and frequencies for non-linear oscillators with a fractional-power restoring force by applying an optimal homotopy approach, namely the Optimal Homotopy Asymptotic Method (OHAM). This kind of nonlinear oscillator has been studied up to now using different methodologies [16–21]. The most significant feature of the new proposed approach is the optimal control of the convergence of solutions by means of a particular convergence-control function $h(\tau, p)$, which ensures a very fast convergence when its components are optimally determined. Different from some other methods, the validity of the OHAM is independent of whether or not there exist small parameters in the considered non-linear equations.

In order to develop the application of the method, we consider a differential equation with a single-term positive-power nonlinear oscillator with a fractional-power restoring force:

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \mathrm{sign}(u)\,|u|^{\alpha} = 0 \tag{1}$$

with initial conditions

$$u(0) = A, \qquad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = 0. \tag{2}$$

Our attention here is restricted primarily to rational powers less than unity. Then the function $\mathrm{sign}(u)$ is defined as:

$$\mathrm{sign}(u) = \begin{cases} 1 & u > 0 \\ -1 & u \le 0. \end{cases} \tag{3}$$

There exists no small parameter in the equation, so traditional perturbation methods cannot be applied directly in this case.

2. Basic idea of the proposed method [22–28]

The concept is to couple the homotopy perturbation method and a computational algorithm intended to optimally control the convergence of the solution through an auxiliary function $h(\tau, p)$ which depends on a number of initially unknown parameters. The algorithm used to identify these unknown parameters can be based on the least squares method, collocation method, Galerkin method and so on, but the least squares method is always the first option. A similar treatment was used in [29], where an optimal variational iteration procedure was suggested combining the features of the He's variational iteration method with a computational algorithm which minimizes a residual functional. The main solution procedure briefly described above is completely different from the homotopy analysis method [11].

A similar optimal approach based on the homotopy technique was recently reported in [30], where the authors present a new analytical technique that combines He's homotopy perturbation method and the least squares method, called OHPM. There are significant differences between OHAM and OHPM. The main difference is the construction of the homotopy, which in OHAM involves the auxiliary function $h(\tau, p)$ and, in case of oscillatory movements, also involves an arbitrary auxiliary parameter λ which is determined using the principle of minimal sensitivity; while in OHPM the construction of the homotopy is the same as in He's homotopy perturbation method [31–33]. Instead, in the frame of OHPM the nonlinear operator is expanded in a series with respect to the parameter p and a number of auxiliary functions are introduced within the coefficients of this truncated power series. These auxiliary functions depend on a number of unknown parameters which are optimally determined to provide a way to control the convergence of the solution.

In order to show the basics of OHAM, we consider a nonlinear ODE of the form

$$\ddot{u}(t) + f(t, u(t)) = 0 \tag{4}$$

where the dot denotes the derivative with respect to time and f is in general a nonlinear term. Initial conditions are:

$$u(0) = A, \qquad \dot{u}(0) = 0. \tag{5}$$

The Eq. (4) describes a system oscillating with an unknown period T. If we switch to a scalar time $\tau = 2\pi t/T = \omega t$, under the transformation

$$\tau = \omega t \tag{6}$$

we can rewrite Eqs. (4) and (5) in the form:

$$\omega^2 u''(\tau) + f(\tau, u(\tau)) = 0 \tag{7}$$

$$u(0) = A, \qquad u'(0) = 0 \tag{8}$$

where the prime denotes derivative with respect to τ.

By the homotopy technique, we construct a homotopy in a more general form:

$$H(\phi(\tau, p), h(\tau, p)) = (1 - p)L(\phi(\tau, p)) - h(\tau, p)N[\phi(\tau, p), \Omega(\lambda, p)] = 0 \tag{9}$$

where L is a linear operator:

$$L(\phi(\tau,p)) = \omega_0^2\left[\frac{\partial^2\phi(\tau,p)}{\partial t^2} + \phi(\tau,p)\right] \tag{10}$$

while N is a nonlinear operator:

$$N\left[\phi(\tau,p),\Omega(\lambda,p)\right] = \Omega^2(\lambda,p)\frac{\partial^2\phi(\tau,p)}{\partial\tau^2} + \lambda\phi(\tau,p) + f(\tau,\phi(\tau,p)) - p\lambda\phi(\tau,p) \tag{11}$$

where $p \in [0, 1]$ is the embedding parameter, $h(\tau, p)$ is an auxiliary function so that $h(\tau, 0) = 0$, $h(\tau, p) \neq 0$ for $p \neq 0$ and λ is an arbitrary parameter. From Eq. (8) we obtain the initial conditions:

$$\phi(0,p) = A, \qquad \left.\frac{\partial\phi(\tau,p)}{\partial\tau}\right|_{\tau=0} = 0. \tag{12}$$

Obviously when $p = 0$ and $p = 1$, it holds that:

$$\phi(\tau,0) = u_0(\tau), \qquad \phi(\tau,1) = u(\tau), \qquad \Omega(\lambda,0) = \omega_0, \qquad \Omega(\lambda,1) = \omega \tag{13}$$

where $u_0(\tau)$ is the initial approximation of $u(\tau)$. Therefore, as the embedding parameter p increases from 0 to 1, $\phi(\tau, p)$ varies from the initial approximation $u_0(\tau)$ to the solution $u(\tau)$, so does $\Omega(\lambda, p)$ from the initial approximation ω_0 to the exact frequency ω. Expanding $\phi(\tau, p)$ and $\Omega(\lambda, p)$ in series with respect to the parameter p, one has respectively

$$\phi(\tau,p) = u_0(\tau) + pu_1(\tau) + p^2u_2(\tau) + \cdots \tag{14}$$

$$\Omega(\lambda,p) = \omega_0 + p\omega_1 + p^2\omega_2 + \cdots. \tag{15}$$

If the initial approximation $u_0(\tau)$ and the auxiliary function $h(\tau, p)$ are properly chosen so that the above series converges at $p = 1$, one has:

$$u(\tau) = u_0(\tau) + u_1(\tau) + u_2(\tau) + \cdots \tag{16}$$

$$\omega = \omega_0 + \omega_1 + \omega_2 + \cdots. \tag{17}$$

We propose the auxiliary function $h(\tau, p)$ of the form:

$$h(\tau,p) = pC_1 + p^2C_2 + \cdots + p^mC_m(\tau) \tag{18}$$

where C_i, $i = 1, 2, \ldots, m$, can be simple constants or functions depending on τ and on some constants. It is very important to properly choose this function because the convergence of the solution greatly depends on that. More details are presented in [22].

The results of the mth-order approximations are given by:

$$\tilde{u}(\tau) \approx u_0(\tau) + u_1(\tau) + \cdots + u_m(\tau) \tag{19}$$

$$\tilde{\omega} \approx \omega_0 + \omega_1 + \cdots + \omega_m. \tag{20}$$

Substituting Eqs. (14) and (15) into Eq. (11) yields:

$$N(\phi,\Omega) = N_0(u_0,\omega_0,\lambda) + pN_1(u_0,u_1,\omega_0,\omega_1,\lambda) + p^2N_2(u_0,u_1,u_2,\omega_0,\omega_1,\omega_2,\lambda) + \cdots. \tag{21}$$

If we substitute Eqs. (21) and (18) into Eq. (9) and we equate to zero the coefficients of the same powers of p, we obtain the following linear equations:

$$L(u_0) = 0, \qquad u_0(0) = A, \qquad u_0'(0) = 0 \tag{22}$$

$$\begin{gathered} L(u_i) - L(u_{i-1}) - \sum_{j=1}^{i} C_jN_{i-j}(u_0,u_1,\ldots,u_{i-j},\omega_0,\omega_1,\ldots,\omega_{i-j},\lambda) = 0, \\ u_i(0) = u_i'(0) = 0, \quad i = 1,2,\ldots,m-1 \end{gathered} \tag{23}$$

$$L(u_m) - L(u_{m-1}) - \sum_{j=1}^{m-1} C_jN_{m-j} - C_m(\tau)N_0 = 0, \qquad u_m(0) = u_m'(0) = 0. \tag{24}$$

We notice that $\omega_0, \omega_1, \ldots, \omega_n$, can be determined avoiding the presence of secular terms in the Eqs. (23) and (24).

The frequency ω depends upon the arbitrary parameter λ and we can apply the so-called "principle of minimal sensitivity" [34] in order to fix the value of λ. We do this imposing that:

$$\frac{\mathrm{d}\omega}{\mathrm{d}\lambda} = 0. \tag{25}$$

At this moment, the mth-order approximation given by Eq. (19) depends on the parameters $C_1, C_2, \ldots, C_{m-1}$ and also on the function $C_m(\tau)$.

If $R(\tau, C_1, C_2, \ldots, C_q)$ is the residual obtained substituting the mth-order approximation (19) into Eq. (7):

$$R(\tau, C_1, C_2, \ldots, C_q) = \tilde{\omega}^2 \tilde{u}''(\tau) + f(\tau, \tilde{u}(\tau)) \tag{26}$$

and if the functional J is given by the integral:

$$J(C_1, C_2, \ldots, C_q) = \int_a^b R^2(\tau, C_1, C_2, \ldots, C_q) d\tau \tag{27}$$

where a and b are values from the domain of Eq. (41), then the parameters $C_1, C_2, \ldots, C_q$ can be optimally determined from the following equations:

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \cdots = \frac{\partial J}{\partial C_q} = 0 \tag{28}$$

where q is the total number of constants.

We notice that the parameters $C_1, C_2, \ldots, C_q$ involved in the convergence-control function $h(\tau, p)$ could be also identified via various methods, such as the collocation method, the least square method, the Galerkin method and so on.

One remarks that OHAM contains the auxiliary function $h(\tau, p)$, which provides us with a simple way to adjust and optimally control the convergence of the solution through determination of optimal values for some parameters C_i. As an important advantage, one observes that instead of an infinite series, the OHAM searches for only a few terms (mostly two or three terms).

3. Application of OHAM to oscillators with fractional-power restoring force

For Eq. (1), the nonlinear operator (11) is given by the equation:

$$\begin{aligned} N[\phi(\tau, p), \Omega(\lambda, p)] = {} & \Omega^2(\lambda, p)\phi''(\tau, p) + \lambda\phi(\tau, p) + \operatorname{sign}(\phi(\tau, p))\, |\phi(\tau, p)|^\alpha \\ & - p\lambda\phi(\tau, p). \end{aligned} \tag{29}$$

Eq. (21) can be written as

$$\omega_0^2(u_0'' + u_0) = 0, \qquad u_0(0) = A, \qquad u'(0) = 0 \tag{30}$$

and has the solution

$$u_0(\tau) = A\cos\tau. \tag{31}$$

In our example, $f(t, u(t)) = \operatorname{sign}(u(t))\, |u(t)|^\alpha$, where u is given by Eq. (16). In the following we have taken into account the identity:

$$f(u) = f(u_0 + pu_1 + p^2u_2 + \cdots) = f(u_0) + pu_1 f'(u_0) + p^2\left[u_2 f'(u_0) + \frac{1}{2}u_1^2 f''(u_0)\right] + 0(p^3) \tag{32}$$

where for example

$$f'(u_0) = \alpha \operatorname{sign}(u_0)\, |u_0|^{\alpha-1}. \tag{33}$$

Taking into account Eq. (32), the first term in Eq. (21) is given by:

$$N_0(u_0, \omega_0, \lambda) = \omega_0^2 u_0'' + \lambda u_0 + \operatorname{sign}(u_0)\, |u_0|^\alpha. \tag{34}$$

For $i = 1$ into Eq. (23), we obtain the equation in u_1:

$$\begin{aligned} & \omega_0^2(u_1'' + u_1) - \omega_0^2(u_0'' + u_0) - C_1[\omega_0^2 u_0'' + \lambda u_0 + \operatorname{sign}(u_0)\, |u_0|^\alpha] = 0, \\ & u_1(0) = u_1'(0) = 0. \end{aligned} \tag{35}$$

Using Eq. (31), we obtain the following Fourier series expansions:

$$\operatorname{sign}(u_0)\, |u_0|^\alpha = A^\alpha(a_{1\alpha}\cos\tau + a_{3\alpha}\cos 3\tau + \cdots) \tag{36}$$

where

$$a_{2k+1\alpha} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} (\cos\tau)^\alpha \cos(2k+1)\tau\, d\tau, \quad k = 0, 1, 2, \ldots. \tag{37}$$

Substituting Eqs. (31) and (36) into Eq. (35), we have:

$$\omega_0^2(u_1''+u_1)-C_1(-A\omega_0^2+\lambda A+a_{1\alpha}A^\alpha)\cos\tau-C_1A^\alpha a_{3\alpha}\cos 3\tau-C_1A^\alpha a_{5\alpha}\cos 5\tau-C_1A^\alpha a_{7\alpha}\cos 7\tau-\cdots=0. \quad (38)$$

No secular terms in u_1 requires eliminating contributions proportional to $\cos\tau$ in Eq. (38):

$$\omega_0^2=\lambda+a_{1\alpha}A^\alpha,\quad \lambda\geq 0. \quad (39)$$

The solution of Eq. (38) can be written

$$u_1(\tau)=\frac{C_1a_{3\alpha}A^\alpha}{8\omega_0^2}(\cos\tau-\cos 3\tau)+\frac{C_1a_{5\alpha}A^\alpha}{24\omega_0^2}(\cos\tau-\cos 5\tau)+\frac{C_1a_{7\alpha}A^\alpha}{48\omega_0^2}(\cos\tau-\cos 7\tau)+\cdots. \quad (40)$$

For $m=2$ into Eq. (24) and if we consider the simplest case $C_m^*(\tau)=C_2$ (constant), then the equation in u_2 has the form:

$$\begin{aligned}&\omega_0^2(u_2''+u_2)-\omega_0^2(u_1''+u_1)-C_1[\omega_0^2u_1''+2\omega_0\omega_1u_0''+\lambda(u_1-u_0)+\alpha\,\mathrm{sign}(u_0)\,|u_0|^{\alpha-1}\,u_1]\\&\quad -C_2[\omega_0^2u_0''+\lambda u_0+\mathrm{sign}(u_0)\,|u_0|^\alpha]=0,\qquad u_2(0)=u_2'(0)=0.\end{aligned} \quad (41)$$

Having in view Eqs. (41) and (40), we can write the identities:

$$\mathrm{sign}(u_0)\,|u_0|^{\alpha-1}\,(\cos\tau-\cos 3\tau)=2A^{-1}\mathrm{sign}(u_0)\,|u_0|^\alpha\,(1-\cos 2\tau) \quad (42)$$

$$\mathrm{sign}(u_0)\,|u_0|^{\alpha-1}\,(\cos\tau-\cos 5\tau)=2A^{-1}\mathrm{sign}(u_0)\,|u_0|^\alpha\,(\cos 2\tau-\cos 4\tau) \quad (43)$$

$$\mathrm{sign}(u_0)\,|u_0|^{\alpha-1}\,(\cos\tau-\cos 7\tau)=2A^{-1}\mathrm{sign}(u_0)\,|u_0|^\alpha\,(1-\cos 2\tau+\cos 4\tau-\cos 6\tau). \quad (44)$$

Substitution of Eqs. (31), (36), (40), (42), (43) and (44) into Eq. (41) yields:

$$\begin{aligned}\omega_0^2(u_2''+u_2)&=\left[2AK_1\frac{\lambda-\omega_0^2}{\omega_0^2}-2A\omega_0\omega_1-\lambda A+\frac{2AK_2}{\omega_0^2}\right]\cos\tau\\&+\left[\frac{C_1^2A^\alpha a_{3\alpha}}{8}\left(9-\frac{\lambda}{\omega_0^2}\right)+\frac{\alpha C_1^2A^{2\alpha-1}\beta_3}{48\omega_0^2}+(C_1+C_2)\,A^\alpha a_{3\alpha}\right]\cos 3\tau\\&+\left[\frac{C_1^2A^\alpha a_{5k}}{24}\left(25-\frac{\lambda}{\omega_0^2}\right)+\frac{\alpha C_1^2A^{2\alpha-1}\beta_5}{48\omega_0^2}+(C_1+C_2)A^\alpha a_{5\alpha}\right]\cos 5\tau\\&+\left[\frac{C_1^2A^\alpha a_{7\alpha}}{48}\left(49-\frac{\lambda}{\omega_0^2}\right)+\frac{\alpha C_1^2A^{2\alpha-1}\beta_7}{48\omega_0^2}+(C_1+C_2)A^\alpha a_{7\alpha}\right]\cos 7\tau+\cdots\end{aligned} \quad (45)$$

where

$$\begin{aligned}K_1&=\frac{C_1A^{\alpha-1}(6a_{3\alpha}+2a_{5\alpha}+a_{7\alpha})}{96}\\K_2&=\frac{C_1A^{2\alpha-2}(6a_{1\alpha}a_{3\alpha}-6a_{3\alpha}^2+2a_{1\alpha}a_{5\alpha}-2a_{5\alpha}^2+a_{1\alpha}a_{7\alpha}-a_{7\alpha}^2)}{96}\\\beta_3&=-6a_{1\alpha}a_{3\alpha}+12a_{3\alpha}^2-6a_{3\alpha}a_{5\alpha}+2a_{5\alpha}^2+a_{3\alpha}+a_{7\alpha}-3a_{5\alpha}a_{7\alpha}+a_{7\alpha}^2\\\beta_5&=-6a_{3\alpha}^2+12a_{3\alpha}a_{5\alpha}-2a_{1\alpha}a_{5\alpha}+2a_{3\alpha}a_{5\alpha}-7a_{3\alpha}a_{7\alpha}+3a_{5\alpha}a_{7\alpha}-a_{7\alpha}^2\\\beta_7&=-8a_{3\alpha}a_{5\alpha}+2a_{5\alpha}^2-a_{1\alpha}a_{7\alpha}+13a_{3\alpha}a_{7\alpha}-a_{5\alpha}a_{7\alpha}+2a_{7\alpha}^2.\end{aligned} \quad (46)$$

The secular term in the solution of u_2 can be eliminated from Eq. (45) if

$$\omega_1=\frac{K_1(\lambda-\omega_0^2)}{\omega_0^3}-\frac{\lambda}{2\omega_0}+\frac{K_2}{\omega_0^3}. \quad (47)$$

From Eqs. (20) and (47) we obtain the frequency in the form:

$$\tilde{\omega}=\omega_0+\frac{K_1(\lambda-\omega_0^2)}{\omega_0^3}-\frac{\lambda}{2\omega_0}+\frac{K_2}{\omega_0^3} \quad (48)$$

where ω_0 is given by Eq. (39).

The parameter λ can be determined applying the "principle of minimal sensitivity". From Eq. (25) we obtain:

$$\lambda=-\frac{1}{2}a_{1\alpha}A^\alpha+\sqrt{\frac{1}{4}a_{1\alpha}^2A^{2\alpha}+6K_2-6K_1a_{1\alpha}A^\alpha}. \quad (49)$$

From Eqs. (49) and (39) it follows that

$$\omega_0^2 = \frac{1}{2}a_{1\alpha}A^\alpha + \sqrt{\frac{1}{4}a_{1\alpha}^2 A^{2\alpha} + 6K_2 - 6K_1 a_{1\alpha}A^\alpha} \tag{50}$$

where K_1 and K_2 are given by Eq. (46).

Now, we can write the solution of Eq. (45):

$$\begin{aligned} u_2(\tau) = & \left[\frac{C_1^2 A^\alpha a_{3\alpha}}{64\omega_0^2}\left(9 - \frac{\lambda}{\omega_0^2}\right) + \frac{\alpha C_1^2 A^{2\alpha-1}\beta_3}{384\omega_0^4} + \frac{(C_1+C_2)A^\alpha a_{3\alpha}}{8\omega_0^2}\right](\cos\tau - \cos 3\tau) \\ & + \left[\frac{C_1^2 A^\alpha a_{5\alpha}}{576\omega_0^2}\left(25 - \frac{\lambda}{\omega_0^2}\right) + \frac{\alpha C_1^2 A^{2\alpha-1}\beta_5}{1152\omega_0^4} + \frac{(C_1+C_2)A^\alpha a_{5\alpha}}{24\omega_0^2}\right](\cos\tau - \cos 5\tau) \\ & + \left[\frac{C_1^2 A^\alpha a_{7\alpha}}{2304\omega_0^2}\left(49 - \frac{\lambda}{\omega_0^2}\right) + \frac{\alpha C_1^2 A^{2\alpha-1}\beta_7}{2304\omega_0^4} + \frac{(C_1+C_2)A^\alpha a_{7\alpha}}{48\omega_0^2}\right](\cos\tau - \cos 7\tau). \end{aligned} \tag{51}$$

In order to determine the second-order approximate solution it is necessary to substitute Eqs. (31), (40) and (51) into the equation:

$$\tilde{u}(\tau) = u_0(\tau) + u_1(\tau) + u_2(\tau). \tag{52}$$

By means of the transformation (6), the second-order approximate solution of Eq. (1) is:

$$\begin{aligned} \tilde{u}(t) = & A\cos\tilde{\omega}t + \left[\frac{C_1^2 A^\alpha a_{3\alpha}}{64\omega_0^2}\left(9 - \frac{\lambda}{\omega_0^2}\right) + \frac{\alpha C_1^2 A^{2\alpha-1}\beta_3}{384\omega_0^4} + \frac{(2C_1+C_2)A^\alpha a_{3\alpha}}{8\omega_0^2}\right](\cos\tilde{\omega}t - \cos 3\tilde{\omega}t) \\ & + \left[\frac{C_1^2 A^\alpha a_{5\alpha}}{576\omega_0^2}\left(25 - \frac{\lambda}{\omega_0^2}\right) + \frac{\alpha C_1^2 A^{2\alpha-1}\beta_5}{1152\omega_0^4} + \frac{(2C_1+C_2)A^\alpha a_{5\alpha}}{24\omega_0^2}\right](\cos\tilde{\omega}t - \cos 5\tilde{\omega}t) \\ & + \left[\frac{C_1^2 A^\alpha a_{7\alpha}}{2304\omega_0^2}\left(49 - \frac{\lambda}{\omega_0^2}\right) + \frac{\alpha C_1^2 A^{2\alpha-1}\beta_7}{2304\omega_0^4} + \frac{(2C_1+C_2)A^\alpha a_{7\alpha}}{48\omega_0^2}\right](\cos\tilde{\omega}t - \cos 7\tilde{\omega}t) \end{aligned} \tag{53}$$

where $\beta_3, \beta_5, \beta_7$ are given by Eq. (46) and $\tilde{\omega}, \lambda, \omega_0$ depends on the constants C_1 and C_2, which will be optimally determined following the procedure described in the previous section.

4. Numerical applications

We illustrate the accuracy of the OHAM by comparing previously obtained approximate solutions with the numerical integration results obtained by means of a fourth-order Runge–Kutta method.

4.1

In the first case we consider $\alpha = \frac{3}{5}$ and from Eq. (37) we obtain:

$$a_{1\frac{3}{5}} = 1.0872931957; \qquad a_{3\frac{3}{5}} = -\frac{1}{9}a_{1\frac{3}{5}}; \qquad a_{5\frac{3}{5}} = \frac{1}{21}a_{1\frac{3}{5}}; \qquad a_{7\frac{3}{5}} = -\frac{11}{399}a_{1\frac{3}{5}}.$$

(a) For $A = 1$, applying the conditions (28) we obtain

$$C_1 = -0.7682142317; \qquad C_2 = -3.5628152871; \qquad \tilde{\omega} = 1.040588306; \qquad \frac{\lambda}{\omega_0^2} = -0.012538032.$$

The exact frequency in the case $A = 1$ is $\omega_{ex} = 1.04075$ [20] and therefore the relative error between the approximate and the exact frequency is 0.016%.

The second-order approximate solution (53) becomes:

$$\tilde{u}(t) = 1.00699391\cos\tilde{\omega}t - 0.013465301\cos 3\tilde{\omega}t + 0.009104865\cos 5\tilde{\omega}t - 0.002633483\cos 7\tilde{\omega}t. \tag{54}$$

(b) For $A = 5$, we obtain the following expressions:

$$C_1 = -0.768214231; \qquad C_2 = -4.214764755; \qquad \tilde{\omega} = 0.754197242; \qquad \frac{\lambda}{\omega_0^2} = -0.012538032.$$

The exact frequency in this case is $\omega_{ex} = 0.754314435$ [20] and therefore the relative error between the approximate and the exact frequency is 0.015%.

The second-order approximate solution (53) becomes in this case:

$$\tilde{u}(t) = 5.076158387\cos\tilde{\omega}t - 0.113168423\cos 3\tilde{\omega}t + 0.052073171\cos 5\tilde{\omega}t - 0.015063135\cos 7\tilde{\omega}t. \tag{55}$$

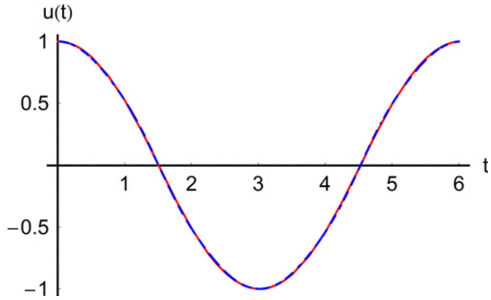

Fig. 1. Comparison between the approximate solution (54) and numerical results in the case $\alpha = 3/5,\ A = 1$. — Numerical simulation, - - - Approximate solution.

4.2

In the second case we consider $\alpha = \frac{1}{3}$. From Eq. (37) we obtain:

$$a_{1\frac{1}{3}} = 1.1595952669; \qquad a_{3\frac{1}{3}} = -\frac{1}{5}a_{1\frac{1}{3}}; \qquad a_{5\frac{1}{3}} = \frac{1}{10}a_{1\frac{1}{3}}; \qquad a_{7\frac{1}{3}} = -\frac{7}{110}a_{1\frac{1}{3}}.$$

(b) For $A = 1$, we find

$$C_1 = -0.812457981; \qquad C_2 = 0.040087495; \qquad \tilde{\omega} = 1.070005112; \qquad \frac{\lambda}{\omega_0^2} = -0.040045851.$$

The exact frequency for $A = 1$ is $\omega_{ex} = 1.07045$ [20] and therefore the relative error between the approximate and the exact frequency is 0.044%.

The second-order approximate solution (53) becomes:

$$\tilde{u}(t) = 1.033439236\cos\tilde{\omega}t - 0.02397233\cos 3\tilde{\omega}t - 0.008055218\cos 5\tilde{\omega}t - 0.001411688\cos 7\tilde{\omega}t. \tag{56}$$

(b) For $A = 5$, we obtain the values:

$$C_1 = -0.812457981; \qquad C_2 = -0.689430076; \qquad \tilde{\omega} = 0.625742785; \qquad \frac{\lambda}{\omega_0^2} = -0.040045851.$$

The exact frequency is $\omega_{ex} = 0.626002957$ [20] and therefore the relative error between the approximate and the exact frequency is 0.042%.

The second-order approximate solution (53) becomes in this case:

$$\tilde{u}(t) = 5.014642562\cos\tilde{\omega}t - 0.008561393\cos 3\tilde{\omega}t - 0.005486728\cos 5\tilde{\omega}t - 0.000594441\cos 7\tilde{\omega}t. \tag{57}$$

5. Results and discussions

Figs. 1–4 show a comparison between the present analytical solutions and the numerical integration results obtained using a fourth-order Runge–Kutta method. One can observe that the second-order approximate analytical results obtained through OHAM are almost identical to the numerical simulation results in all considered cases for various values of the parameters α and A. Moreover, the relative error between the approximate and the exact frequency presented in [20] varies between 0.015% and 0.044%, which proves the accuracy of the method.

6. Conclusions

In the present work we proposed an optimal homotopy approach to obtain approximate analytical solutions for some oscillators with fractional-power restoring force, which is proportional to sign(x). The validity of the proposed procedure, called the Optimal Homotopy Asymptotic Method (OHAM) was demonstrated on some representative examples, and very good agreement was found between the approximate analytic results and numerical simulation results. The proposed procedure is valid even if the nonlinear equation does not contain any small or large parameter.

The OHAM provides us with a simple and rigorous way to optimally control and adjust the convergence of a solution and can give very good approximations in a few terms. The arbitrary parameter λ involved in this procedure is determined by means of the "principle of minimal sensitivity". The convergence of the approximate solution given by OHAM is greatly determined by the convergence-control function $h(\tau, p)$, which involves the presence of some parameters or functions C_i, which are optimally determined. A rigorous computational algorithm is applied to obtain the optimal values of the parameters C_i. Theoretically, the more parameters C_i we choose, the more accurate the solution will be, but since in this case only two parameters C_1 and C_2 led to very accurate results it was not necessary to increase their number. This version of the method proves to have a very fast convergence to the exact solution; so it very rapid, effective and accurate.

This method, which proves to work very well in practice, can be easily applied to other strongly nonlinear problems.

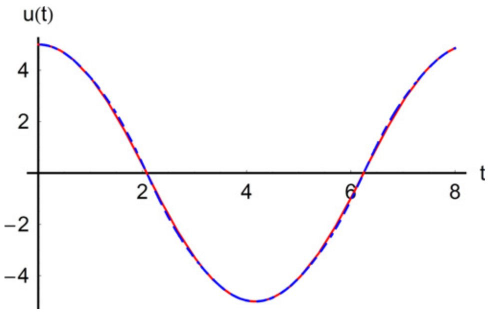

Fig. 2. Comparison between the approximate solution (55) and numerical results in the case $\alpha = 3/5,\ A = 5$. — Numerical simulation, - - - Approximate solution.

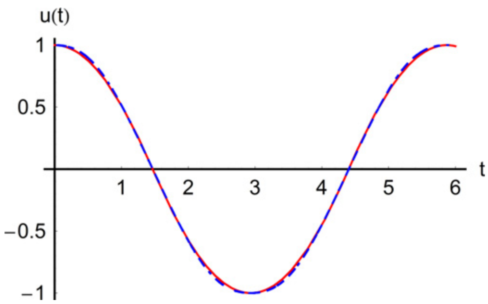

Fig. 3. Comparison between the approximate solution (56) and numerical results in the case $\alpha = 1/3,\ A = 1$. — Numerical simulation, - - - Approximate solution.

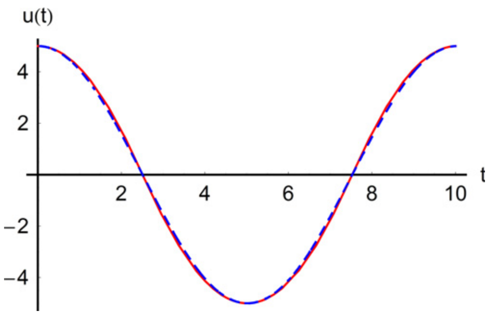

Fig. 4. Comparison between the approximate solution (57) and numerical results in the case $\alpha = 1/3,\ A = 5$. — Numerical simulation, - - - Approximate solution.

Acknowledgement

This work has partly been done as a part of the authors' involvement in the project "Variational approaches to some non-linear vibration problems", supported by the Provincial Secretariat for Science and Technological Development, Autonomous Province of Vojvodina, Republic of Serbia.

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