Successors of Singular Cardinals and Measurability

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It is shown, starting from a model in which $\kappa < \lambda$, $\kappa$ is $2^\lambda$ supercompact, and $\lambda$ is a measurable cardinal, how to force and obtain a model in which the Axiom of Choice is false and in which the successor of a singular cardinal is measurable. © 1985 Academic Press, Inc.

Perhaps the most important of the large cardinal axioms studied by set theorists is the axiom which asserts the existence of a measurable cardinal. First introduced by Ulam [15] in the 1930s, this axiom has been studied extensively by set theorists for the last 20 years or so. The consequences of this axiom are both deep and well known as can be seen by the work of Silver [13], Rowbottom [12], and others.

One of the more interesting aspects of measurable cardinals is that, assuming the Axiom of Choice, they are quite "large." It is known, for example, that any measurable cardinal $\kappa$ must be the $\kappa$th strongly inaccessible cardinal, the $\kappa$th Ramsey cardinal, the $\kappa$th Ramsey limit of Ramsey cardinals, etc. These properties of a measurable cardinal, though, all heavily use the Axiom of Choice. Thus, one begins to suspect that if the Axiom of Choice is assumed not to hold, the above "largeness" properties of a measurable cardinal may not hold either.

Around 1967, work began which showed that when the Axiom of Choice is false, measurable cardinals can be quite accessible. Solovay showed (see [5]) that assuming the Axiom of Determinateness, $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are measurable cardinals. Then, Jech [2] and Takeuti [14], working independently of one another, showed that assuming $\delta$ was a regular cardinal and $\kappa > \delta$ was a measurable cardinal, it was consistent in an inner model where the Axiom of Choice is false for $\delta^+$ to be a measurable cardinal.

One assumption that was absolutely essential in Jech’s and Takeuti’s

* The author wishes to express his thanks to Aki Kanamori for helpful and interesting conversations on the subject matter of this paper, particularly concerning the argument used in the proof of Theorem 2. The author also wishes to express his thanks to the referee of this paper for the numerous suggestions and corrections made which considerably improved presentation of the material contained herein.

228
propositions was the regularity of $\delta$. This was because all of these proofs used the "Lévy Collapse," a notion of forcing which changes an inaccessible cardinal into the successor of a regular cardinal, yet which cannot be used to collapse to the successor of a singular cardinal. Thus, the question as to how one would force to obtain a model in which the Axiom of Choice were false and in which the successor of a singular cardinal were measurable remained open. That such a result was obtainable was suggested by the situation assuming the Axiom of Determinateness; indeed, Martin had shown [8] that assuming AD, $\aleph_{\omega+1}$ was a measurable cardinal.

The purpose of this paper is to show that it is consistent, assuming some very strong hypotheses, for the successor of a singular cardinal to be measurable. Specifically, we have the following two theorems.

**Theorem 1.** Assume that $V = "\text{ZFC} + \kappa$ and $\lambda$ are cardinals such that $\kappa < \lambda$, $\lambda$ is a measurable cardinal, and $\kappa$ is $2^{\lambda}$ supercompact." Then there is a model, $N$, for the theory "ZF + AC, $\kappa$ is a strong limit cardinal + cof($\kappa$) = $\omega + \kappa$ is a Rowbottom cardinal (in fact, $\kappa$ carries a Rowbottom filter) + $\kappa^+$ is a measurable cardinal."

**Theorem 2.** Assume the same hypotheses as in Theorem 1. Then there is a model, $M$, for the theory "ZF + $\aleph_\omega$ is a strong limit cardinal + $\aleph_\omega$ is a Rowbottom cardinal (in fact, $\aleph_\omega$ carries a Rowbottom filter) + $\aleph_{\omega+1}$ is a measurable cardinal."

Theorem 2 will follow Theorem 1 via a routine collapsing argument. Before beginning the proofs of these two theorems, however, we shall briefly digress to give our notation and mention certain useful facts.

1. **Preliminaries**

Our notation is fairly standard. We work in ZF. Lower case Greek letters $\alpha$, $\beta$, $\gamma$, ..., will be used to denote ordinals, with the letters $\kappa$, $\lambda$, and $\delta$ generally being reserved for cardinals. For ultrafilters and measures, we use the letters $U$ and $\mu$.

For $\kappa$ a cardinal, $\kappa^{+\alpha}$ will denote the $\alpha$th least cardinal $>\kappa$. The cofinality of $\kappa$, cof($\kappa$), is the least possible cardinality of an unbounded subset of $\kappa$.

Given a set $x$, we shall let $2^x$ denote the power set of $x$. $|x|$ will be the cardinality of $x$, and $\bar{x}$ is the order type of $x$. Further, for $f$ a function, $f"x$ is the range of $f$ on $x$, and $f\upharpoonright x$ is $f$ restricted to $x$. For $\alpha$ an ordinal, $R(\alpha)$ is the collection of all sets of rank $<\alpha$.

The symbol $\models$ will mean, as usual, "weakly forces," and $\models$ will mean "decides." By convention, we will say that for forcing conditions $p$ and $q$, $p \leq q$ means that $q$ is stronger than $p$.
We will make use of Solovay's Product Lemma for product forcing, so we recall its statement here. Let $P$ and $R$ be partial orderings defined in $V$. Then if $G_1 \subseteq P$, $G_2 \subseteq R$, the following are equivalent:

1. $G_1 \times G_2$ is $V$-generic over $P \times R$.
2. $G_1$ is $V$-generic over $P$ and $G_2$ is $V[G_1]$-generic over $R$.
3. $G_2$ is $V$-generic over $R$ and $G_1$ is $V[G_2]$-generic over $P$.

We will have occasion to use various forms of the Axiom of Choice (AC) throughout this paper. In particular, for $\kappa$ a cardinal, we will make use of AC$_\kappa$, which states that the cartesian product of $K$ many non-empty sets is non-empty.

If $\kappa, \lambda, \alpha, \beta,$ and $\gamma$ are ordinals, let $[\kappa]^{\lambda} = \{f : f$ is a strictly increasing function from $\lambda$ to $\kappa\}$, and let $[\kappa]^{<\lambda}$ (which is sometimes written as $P_\lambda(\kappa)$) be defined as $\bigcup_{\delta<\lambda} [\kappa]^\delta$. We say that $\kappa \rightarrow [\lambda]^{<\gamma}_{\beta,\gamma}$ if given any partition $F : [\kappa]^{<\alpha} \rightarrow \beta$, there is a set $X, \bar{X} = \lambda$ so that $|F''[X]^{<\alpha}| \leq \gamma$. $\kappa$ is said to be Rowbottom if $\forall \lambda < \kappa [\kappa \rightarrow [\kappa]^{\kappa,\omega}]$. $\kappa$ is said to carry a Rowbottom filter if there is a filter $F$ on $\kappa$ such that for all $F : [\kappa]^{<\omega} \rightarrow \lambda$, where $\lambda < \kappa$, there is some $X \in F$ such that $|F''[X]^{<\omega}| \leq \omega$.

The standard notion of forcing, due to Levy, for collapsing a strongly inaccessible cardinal $\lambda$ to the successor of a regular cardinal $\kappa$ will be useful, so we briefly recall its definition and some of its properties. Let $\text{Col}(\kappa, \lambda) = \{f : f$ is a function from $\kappa \times \lambda$ into $\lambda$ such that $|\text{dom}(f)| < \kappa$ and such that $(a, \beta) \in \text{dom}(f) \Rightarrow f(\langle a, \beta \rangle) < \beta\}$, and let $\text{Col}(\kappa, \lambda)$ be ordered by inclusion. Then any set of compatible conditions of length $<\kappa$ has an upper bound.

Finally, we assume that the reader is completely familiar with the notions of measurable and supercompact cardinals. For definitions and facts about these cardinals, we refer the reader to [4] and [11].

2. THE MAIN THEOREMS

We turn now to the proofs of Theorems 1 and 2. Before beginning the proof of Theorem 1, though, we feel that a few intuitive remarks are in order. The proof of Theorem 1 will use a generalized version of Prikry's notion of forcing [10] for changing the cofinality of a measurable cardinal to $\omega$. (Similar, though more complicated notions of forcing have been used by Magidor.) What we would like to accomplish is to simultaneously change the cofinalities of a particular set of cardinals whose sup is some measurable $\lambda$ to $\omega$, causing all but the least of them to be collapsed. Then, when we pass to a certain inner model, we will have that the least of these cardinals $\kappa$ is a cardinal of cofinality $\omega$, and that all of the rest of the cardinals in this particular set remain collapsed. $\lambda$, however, will remain measurable, and
hence will become \( \kappa^+ \). The partial ordering which we are about to use will precisely accomplish this task.

*Proof of Theorem 1.* The proof of Theorem 1 will be via a sequence of lemmas. Let \( V \models "\text{ZFC} + \kappa \text{ and } \lambda \text{ are cardinals such that } \kappa < \lambda, \lambda \text{ is a measurable cardinal, and } \kappa \text{ is } 2^\lambda \text{ supercompact}," \) and assume that \( \lambda \) is the least measurable cardinal \( > \kappa \). Let \( \hat{\mathcal{U}} \) be a normal ultrafilter on \( P_\kappa(2^\lambda) \). Let \( U = \hat{\mathcal{U}} \cap \lambda \), i.e., for any \( A \subseteq P_\kappa(\lambda) \), say that \( A \in U \) iff \( A = \{ p \cap \lambda : p \in B \} \) for some \( B \in \hat{\mathcal{U}} \). It is well known (and easy to show) that \( U \) is a normal ultrafilter on \( P_\kappa(\lambda) \). It is also true that \( \hat{\mathcal{U}} \) may be chosen so that \( U \) has the (Menas) partition property, and we assume that this has been done. (For a proof of this fact, and for the definition of the partition property, see [9].)

Since the ultrapower \( V_p^{P_\kappa(2^\lambda)/U} \) is closed under \( 2^\lambda \) sequences, we know that \( V_p^{P_\kappa(2^\lambda)/U} \models "\lambda \text{ is the least measurable cardinal } > \kappa." \) Now any ordinal \( \alpha < 2^\lambda \) is represented in the above ultrapower by the function \( f(\alpha) = p \cap \alpha \) (see [11]), so by the fundamental theorem on ultraproducts, we have that \( V_p^{P_\kappa(2^\lambda)/U} \models \"[\lambda]_p = \{ p \in P_\kappa(2^\lambda) : p \cap \alpha \text{ is the least measurable cardinal } > p \cap \kappa \} \in \hat{\mathcal{U}}, \) so \( \hat{\mathcal{B}}^\kappa \) is the least measurable cardinal \( > p \cap \kappa \) in \( \hat{\mathcal{U}} \) (note that \( \hat{\mathcal{B}}^\kappa \subseteq \hat{\mathcal{A}}^\kappa \)). Hence, we have \( \hat{\mathcal{A}}^\kappa = \hat{\mathcal{B}}^\kappa \cap \lambda = \{ p \in P_\kappa(\lambda) : p \cap \lambda \text{ is the least measurable } > p \cap \kappa \} \subseteq \hat{\mathcal{U}} \).

The partial ordering \( P \) which will be used in the proof of Theorem 1 is now defined as follows: \( P \) is the set of all sequences of the form \( \langle p_1, ..., p_n, A \rangle \) where:

1. Each \( p_i \in A^\kappa \).
2. \( p_1 \subseteq ... \subseteq p_n \), where \( p_i \subseteq p_j \) means \( p_i \subseteq p_j \) and \( \bar{p}_i < \bar{p}_j \cap \kappa \).
3. \( A \subseteq A^\kappa, A \in U, \) and \( p \in A \Rightarrow p_i \subseteq p \).

For \( \pi = \langle p_1, ..., p_n, A \rangle \), \( \pi' = \langle q_1, ..., q_m, B \rangle \), say that \( \pi \leq \pi' \) iff:

1. \( n \leq m \).
2. \( p_i = q_i \) for \( i \leq n \).
3. \( q_{n+1}, ..., q_m \in A \).
4. \( B \subseteq A \).

Let \( G \) be \( V \)-generic on \( P \). Let \( r = \langle p_n : n < \omega \rangle \), where \( p_n \in r \) iff \( \exists \pi \in G[\pi = \langle p_1, ..., p_n, ..., p_m, A \rangle] \). Standard density arguments show that \( r \) codes a cofinal \( \omega \) sequence through \( \lambda \). For \( \beta \in [\kappa, \lambda) \) (the interval \( \{ \beta : \kappa \leq \beta < \lambda \} \), \( \beta \) a regular cardinal, let \( r \upharpoonright \beta = \langle p_n \cap \beta : n < \omega \rangle \).

The model \( N \) which shall be a witness to the conclusions of Theorem 1 will be the least model of \( ZF \) which extends \( V \) and contains each \( r \upharpoonright \beta \) for \( \beta \) a regular cardinal in \( [\kappa, \lambda) \) [but not the \( \lambda \) sequence of the \( r \upharpoonright \beta \)'s]. More
formally, we define $N$ as follows: Let $L$ be the forcing language associated with $P$, and let $L_1 \subseteq L$ be the ramified sublanguage of $P$ which contains a unary predicate symbol $V$ (interpreted as $V(v) \iff v \in V$), symbols $m$ for each $m \in V$, and symbols $r \upharpoonright \beta$ for each $\beta$ a regular cardinal in $[\kappa, \lambda)$. $N$ is then defined inductively as follows:

$$N_0 = \emptyset$$

$$N_\delta = \bigcup_{\alpha < \delta} N_\alpha$$ for $\delta$ a limit ordinal

$$N_{\alpha+1} = \{ x \subseteq N_\alpha : x \in V[G] \text{ and } x \text{ can be defined using a forcing term } \tau \in L_1 \text{ of rank } \leq \alpha \}$$

$$N = \bigcup_{\alpha \in \text{Ordinals}} N_\alpha.$$ 

Standard arguments show that $N \models ZF$. Also, note that there is a natural classification of the formulas of $L_1$ defined in the following fashion: For $\alpha \in [\kappa, \lambda)$, $\alpha$ a regular cardinal (in $V$), say that $\varphi \in L_{1, \alpha}$ iff $\alpha > \max(\{r : r \upharpoonright \gamma \text{ appears in } \varphi\})$. Standard arguments show that each $m$ for $m \in V$ may be chosen so that it is invariant under any automorphism of $P$, and that formulas $\varphi \in L_{1, \alpha}$ may be assumed to be invariant under any automorphism of $P$ which is fixed to $\alpha$, i.e., which is generated by a function which is the identity on $\alpha$.

We now begin the proofs of the lemmas needed to show that $N$ is our desired model.

**Lemma 1.1.** Let $\varphi \in L$, $\pi \in P$, $\pi = \langle p_1, \ldots, p_n, A \rangle$. Then there is $B \subseteq A$, $B \in U$ such that $\langle p_1, \ldots, p_n, B \rangle \models \varphi$.

**Proof of Lemma 1.1.** The proof of this lemma is virtually the same as in ordinary Prikry forcing. We define a partition $f : [A]^{<\omega} \rightarrow 3$ (where in this case, $[A]^{<\omega} = \{ q_1, \ldots, q_k : k \in \omega \text{ and } q_1 \subseteq \cdots \subseteq q_k \}$ as follows:

$$f(q_1 \subseteq \cdots \subseteq q_k) = \begin{cases} 0 & \text{if } \exists B \subseteq A[\langle p_1, \ldots, p_n, q_1, \ldots, q_k, B \rangle \models \varphi] \\ 1 & \text{if } \exists B \subseteq A[\langle p_1, \ldots, p_n, q_1, \ldots, q_k, B \rangle \models \neg \varphi] \\ 2 & \text{otherwise.} \end{cases}$$

As $U$ has the partition property, let $B \subseteq A$ be homogeneous for $f$. Then as usual, we can show that $\langle p_1, \ldots, p_n, B \rangle \models \varphi$. ☐

**Lemma 1.2.** $N \models "\kappa \text{ is a strong limit cardinal.}"$
**Proof of Lemma 1.2.** Lemma 1.1 immediately implies that in $V[G]$, there are no new bounded subsets of $\kappa$. Hence, $V[G] \models \text{"}\kappa \text{ is a strong limit cardinal."}$

**Lemma 1.3.** $N \models \text{"} \text{cof}(\kappa) = \omega.\text{"}$

**Proof of Lemma 1.3.** Standard density arguments show that $r \uparrow \kappa$ codes a cofinal $\omega$ sequence through $\kappa$. However, clearly $r \uparrow \kappa \in N$. 

**Lemma 1.4.** $N \models \text{"} \lambda \leq \kappa^+.\text{"}$

**Proof of Lemma 1.4.** Lemma 1.4 is proven by showing that no ordinal $\delta \in (\kappa, \lambda)$ which is a cardinal in $V$ remains a cardinal in $N$. To show this, we let $\beta > \delta$, $\beta < \lambda$ be such that $\beta$ is a regular cardinal in $V$. We then show that in $V[r \uparrow \beta]$, $\delta$ is not a cardinal. As $V[r \uparrow \beta] \subseteq N$, the collapsing function for $\beta$ present in $V[r \uparrow \beta]$ will also be present in $N$, so $N \models \text{"} \delta \text{ is not a cardinal.}\text{"}$

Proceeding with the proof, let $\delta \in (\kappa, \lambda)$ be a cardinal in $V$, and let $\beta$ be as above. We show that there are no cardinals in the interval $(\kappa, \beta)$ in $V[r \uparrow \beta]$. To do this, let $\alpha$ be the least cardinal of $V$ in this interval which is a cardinal in $V[r \uparrow \beta]$.

**Case 1.** $\alpha$ is a regular cardinal in $V$. In this case, as $\alpha \leq \beta$, we know that $r \uparrow \alpha$ is present in $V[r \uparrow \beta]$. Now as $\alpha$ is regular, again the standard density arguments show that $r \uparrow \alpha$ codes a cofinal $\omega$ sequence through $\alpha$. Thus $V[r \uparrow \beta] \models \text{"} \text{cof}(\alpha) = \omega.\text{"}$ But as $\alpha$ has been chosen to be the least ordinal which remains a cardinal in $V[r \uparrow \beta]$, $V[r \uparrow \beta] \models \text{"} \alpha = \kappa^+.\text{"}$ ($\kappa$ is still a cardinal in $V[r \uparrow \beta]$ by Lemma 1.2 and the fact $V[r \uparrow \beta] \subseteq N$). However, as $V[r \uparrow \beta] \models \text{ZFC}$, we cannot possibly have $V[r \uparrow \beta] \models \text{"} \alpha = \kappa^+,\text{"}$ as $\alpha$ has cofinality $\omega$ in this model.

**Case 2.** $\alpha$ is a singular cardinal in $V$. In this case, again by the leastness of $\alpha$, we again have $V[r \uparrow \beta] \models \text{"} \alpha = \kappa^+.\text{"}$ But as $V \models \text{"} \alpha \text{ is singular,}\text{"}$ $V[r \uparrow \beta] \models \text{"} \alpha \text{ is singular,}\text{"}$ so we reach the same contradiction as before since a successor cardinal cannot be singular in a model of $\text{ZFC}$.

The above shows that no cardinal $\delta \in (\kappa, \lambda)$ remains a cardinal in $N$, so Lemma 1.4 is proven.

We now proceed to show that not only does $\lambda$ remain a cardinal in $N$, but that it remains measurable as well. To do this, we first fix $\mu \in V$, $\mu$ a measure on $\kappa$.

**Lemma 1.5.** Let $x$ be a set of ordinals so that $x \in N$. Then for some $\beta < \lambda$, $x \in V[r \uparrow \beta]$.

**Proof of Lemma 1.5.** Let $r$ be a term which denotes $x$, and let $\delta$ be an
ordinal so that \( \pi = \langle p_1, \ldots, p_n, F \rangle \models \tau \subseteq \delta. \) Since \( x \in N, \) let \( \alpha \in (\kappa, \lambda) \) be so that \( \tau \in L_{1, \alpha} \) and so that \( \alpha > \text{sup}(p_1 \cup \ldots \cup p_n). \)

Let \( \gamma < \delta \) be given and suppose that \( \pi' = \langle p_1, \ldots, p_n, A \rangle \models \gamma \in \tau. \) Without loss of generality, assume that \( \pi' \models \gamma \subseteq \tau \), the case wherein \( \pi' \models \gamma \not\subseteq \tau \) is symmetric. Assume further that \( \langle p_1, \ldots, p_n, B \rangle \) is such that \( [A]^{<\omega} \cap \alpha = [B]^{<\omega} \cap \alpha \), i.e., \( \langle q_1 \cap \alpha, \ldots, q_k \cap \alpha \rangle : k \in \omega, q_1 \subseteq \cdots \subseteq q_k, \) and \( q_1, \ldots, q_k \in A \rangle = \langle q_1 \cap \alpha, \ldots, q_k \cap \alpha \rangle : k \in \omega, q_1 \subseteq \cdots \subseteq q_k, \) and \( q_1, \ldots, q_k \in B \). We first make the following

Claim. \( \langle p_1, \ldots, p_n, B \rangle \models \gamma \subseteq \tau. \)

Proof of Claim. If not, then let \( q_1, \ldots, q_k \in B \) be so that for some \( C \in U \)
\( s = \langle p_1, \ldots, p_n, q_1, \ldots, q_k, C \rangle \models \gamma \not\subseteq \tau. \) As \( [A]^{<\omega} \cap \alpha = [B]^{<\omega} \cap \alpha \), we can let \( q'_1, \ldots, q'_k \in A \) be so that for \( j = 1, \ldots, k, q'_j \cap \alpha = q_j \cap \alpha. \) Then, for \( C' = A - \{q'_1, \ldots, q'_k\}, t = \langle p_1, \ldots, p_n, q'_1, \ldots, q'_k \rangle \models \gamma \subseteq \tau. \)

Subclaim. There is an automorphism \( \psi \) such that \( \psi(s) \) is compatible with \( t \) and such that \( \psi(s) \models \gamma \not\subseteq \tau. \)

Proof of Subclaim. Let \( j \) be arbitrary, where \( 1 \leq j \leq k. \) Since \( q_j, q'_j \in A^\ast, \) we know that \( |q_j| = \) the least measurable \( \geq q_j \cap \kappa \) and \( |q'_j| = \) the least measurable \( \geq q'_j \cap \kappa. \) Since \( q_j \cap \alpha = q'_j \cap \alpha \) and \( \alpha > \kappa, q_j \cap \kappa = q'_j \cap \kappa. \) Hence, \( |q_j| = |q'_j|. \) We can thus inductively define a function \( h \) on \( q_k. \) First, we note that for \( 1 \leq j \leq k, \) since \( q_j \cap \alpha = q'_j \cap \alpha, |q_j| = |q'_j|, \) \( |q_j - \alpha| = |q'_j - \alpha|. \) Also, we note that for \( 1 \leq j \leq k, |(q_j - \alpha) - p_n| = |(q'_j - \alpha) - p_n|, \) since \( p_n \subseteq q_j, p_n \subseteq q'_j. \)

Now, let \( h_1 : (q_1 - \alpha) - p_n \rightarrow (q'_1 - \alpha) - p_n \) be a 1–1 onto map, and assume that \( h_{j - 1} \) has been defined for \( j - 1 < k. \) We know that \( |(q_j - \alpha) - (q_{j - 1} - p_n)| = |(q'_j - \alpha) - (q'_{j - 1} - p_n)|; \) this follows from the above and from the fact that \( |q_j| = |q'_j| \) and \( |q_j| > |q'_{j - 1}|. \) Hence, we let \( h_j : (q_j - \alpha) - (q_{j - 1} - p_n) \rightarrow (q'_j - \alpha) - (q'_{j - 1} - p_n) \) be a 1–1 onto map, and let \( h = \bigcup_{j=1}^{k} h_j. \) We note that \( h : (q_k - \alpha) - p_n \rightarrow (q'_k - \alpha) - p_n \) is a 1–1 onto map, and that \( h \upharpoonright (q_j - \alpha) - p_n \) is a 1–1 onto map from \( (q_j - \alpha) - p_n \) to \( (q'_j - \alpha) - p_n. \)

Extend \( h \) to a 1–1 onto map of \( \lambda, h^\ast, \) which we define as follows: \( h^\ast = h \) on \( (q_k - \alpha) - p_n. \) On \( p_n \) and on \( q_k \cap \alpha, h^\ast \) is the identity. Finally, as \( |q_k| = |q'_k| < \lambda, q_k \cap \alpha = q'_k \cap \alpha, h^\ast \upharpoonright \lambda - q_k \) is taken to be any 1–1 onto map from \( \lambda - q_k \) to \( \lambda - q'_k \) which is the identity on \( \alpha. \)

\( h^\ast \) now naturally generates automorphism \( \psi \) of \( P, \) defined as \( \psi(\langle r_1, \ldots, r_m, E \rangle) = \langle h^\ast r_1, \ldots, h^\ast r_m, h^\ast E \rangle. \)

By definition, it almost immediately follows that \( \psi \) as just defined is an automorphism; the only thing which is not immediately clear is that \( E \in U \Rightarrow h^\ast \upharpoonright E \in U. \) To establish this fact, we argue as in [7]. The argument proceeds by showing that the normality of \( U \) immediately implies that \( \{ p \in P_\kappa(\lambda) : h^\ast p = p \} \in U. \) If this is false, then \( \{ p \in P_\kappa(\lambda) : h^\ast p \neq p \} \in U, \)
so there must exist some \( \beta \in p \) such that either \( h^*(\beta) \in p \) or \( h^{*-1}(\beta) \in p \). By normality, there is some \( E_0 \in U \) on which the \( \beta \) as defined above is constant. But then, by the fineness of \( U \), since \( \{ p \in E_0 : \{ \beta, h^*(\beta), h^{*-1}(\beta) \} \in p \} \in U \), we have an immediate contradiction. Thus, \( E \in U \Rightarrow k^*E \in U \), so \( \psi \) is indeed an automorphism.

\[
\psi(s) = (h^*p_1, \ldots, h^*q_k, h^*C).
\]

By the definition of \( h^* \), \( h^*p_n = p_n \) and for \( j = 1, \ldots, k, h^*q_j = q_j \). Thus, \( \psi(s) = (p_1, \ldots, p_n, q_1, \ldots, q_k, h^*C) \), which is clearly compatible with \( t \). And, as \( h^* \) is the identity on \( \alpha \), since \( \tau \in L_{1, \alpha} \) has been chosen to be invariant under any automorphism which is the identity on \( \alpha \), \( \psi(s) \models \gamma \in \tau \).

The fact that the subclaim is true now immediately yields a contradiction as \( \psi(s) \) and \( t \) are compatible, \( \psi(s) \models \gamma \in \tau \) and \( t \models \gamma \in \tau \). This contradiction immediately proves our claim.

Let now, for \( D \subseteq P_\kappa(\lambda), D \upharpoonright \alpha = \{ p \cap \alpha : p \in D \} \). It is routine to check that

\[
G \uparrow \alpha = \{ (r_1 \cap \alpha, \ldots, r_m \cap \alpha, D \uparrow \alpha) : (r_1, \ldots, r_m, D) \in P \}
\]

is \( V \)-generic on \( P \uparrow \alpha = \{ (r_1 \cap \alpha, \ldots, r_m \cap \alpha, D \uparrow \alpha) : (r_1, \ldots, r_m, D) \in P \} \), where \( P \uparrow \alpha \) is ordered using the obvious analogue to the ordering on \( P \). Working in \( V[G \uparrow \alpha] = V[r \uparrow \alpha] \), if we now define assuming that \( \pi \uparrow \alpha = \langle p_1, \ldots, p_n, F \uparrow \alpha \rangle \in G \uparrow \alpha \) a set \( y \) by \( \gamma \in y \) iff \( \exists t' \supseteq \pi [t' = \langle p_1, \ldots, p_n, r_1, \ldots, r_m, D \rangle, \langle p_1, \ldots, p_n, r_1 \cap \alpha, \ldots, r_m \cap \alpha, D \uparrow \alpha \rangle \subseteq G \uparrow \alpha \), and \( t' \models \gamma \in \tau \] then we make as our final

Claim. \( x = y \).

Proof of Claim. If \( y \in x \), then this implies that some \( \langle p_1, \ldots, p_n, r_1, \ldots, r_m, D \rangle \in G \) forces "\( \gamma \in \tau \)". Thus, by definition of \( G \uparrow \alpha \), \( \langle p_1, \ldots, p_n, r_1 \cap \alpha, \ldots, r_m \cap \alpha, D \uparrow \alpha \rangle \in G \uparrow \alpha \).

Hence, \( \gamma \in y \), so \( x \subseteq y \). If \( \gamma \in y \), then let \( \langle p_1, \ldots, p_n, r_1, \ldots, r_m, D \rangle \in P \) be so that \( \langle p_1, \ldots, p_n, r_1, \ldots, r_m, D \rangle \models "\gamma \in \tau \" \) and \( \langle p_1, \ldots, p_n, r_1 \cap \alpha, \ldots, r_m \cap \alpha, D \uparrow \alpha \rangle \in G \uparrow \alpha \). By the definition of \( G \uparrow \alpha \) we can let \( \langle p_1, \ldots, p_n, r_1, \ldots, r_m', D' \rangle \in G \) be so that for each \( i = 1, \ldots, m, r_i \cap \alpha = r'_i \cap \alpha \). By the proof of the first claim, this condition forces "\( \gamma \in \tau \)". Thus, \( \gamma \in x \), \( y \subseteq x \), and \( x = y \).

Setting \( \beta = \alpha \) yields the proof of Lemma 1.5.

Lemma 1.6. Let \( \mu' = \{ x \subseteq \lambda : x \subseteq N \) and \( x \) contains a \( \mu \) measure 1 set \}. Then \( \mu' \) is a \( \lambda \) additive normal measure on \( \lambda (= \kappa^+) \) in \( N \).

Proof of Lemma 1.6. By Lemma 1.5, if \( x \subseteq N \), then for some \( \beta < \lambda \), \( x \in V[r \uparrow \beta] \). By the definition of \( P \uparrow \beta \) and the inaccessibility of \( \lambda \) in \( V \), \( |P \uparrow \beta| < \lambda \). Thus, by the Lévy–Solovay arguments \( [6], V[r \uparrow \beta] = \"\lambda \) is a measurable cardinal," and the measure \( \mu^\beta = \{ x \subseteq \lambda : x \in V[r \uparrow \beta] \) and \( x \) contains a \( \mu \) measure 1 set \} is a \( \lambda \) additive normal measure on \( \lambda \) in \( V[r \uparrow \beta] \).
Hence, $N \models \text{"Either } x \text{ or } \lambda - x \text{ contains a } \mu \text{ measure 1 set."}$ If $N \models \text{"} f : \lambda \rightarrow \lambda \text{ is a regressive function,"}$ then as $f$ may be coded by a set of ordinals in $N$, for some $\beta < \lambda$, $f \in V[r \upharpoonright \beta]$. Thus, by the above, $V[r \upharpoonright \beta] \models \text{"} f \text{ is constant on a } \mu \text{ measure 1 set,"}$ so $N \models \text{"} \mu \text{ is normal."}$ Finally, if $N \models \text{"} \langle x_\alpha : \alpha < \gamma < \lambda \rangle \text{ is a sequence of subsets of } \lambda \text{ so that each } x_\alpha \text{ contains a } \mu \text{ measure 1 set,"}$ then we let $y \subseteq \lambda$, $y \in N$ code $\langle x_\alpha : \alpha < \gamma < \lambda \rangle$. For a $\beta$ so that $y \in V[r \upharpoonright \beta]$ we have that $V[r \upharpoonright \beta] \models \text{"} \langle x_\alpha : \alpha < \gamma < \lambda \rangle \text{ is a sequence of subsets of } \lambda \text{ so that each } x_\alpha \text{ contains a } \mu \text{ measure 1 set."}$ Hence, by the above, $V[r \upharpoonright \beta] \models \text{"} \mu^\beta(\bigcap_{\alpha < \gamma} x_\alpha) = 1," \text{ i.e., } V[r \upharpoonright \beta] \models \text{"} \bigcap_{\alpha < \gamma} x_\alpha \text{ contains a } \mu \text{ measure 1 set."}$ This immediately implies that $N \models \text{"} \mu'(\bigcap_{\alpha < \gamma} x_\alpha) = 1."$ This last statement proves Lemma 1.6.

**Lemma 1.7.** $N \models \text{"} \kappa \text{ is a Rowbottom cardinal that carries a Rowbottom filter."}$

*Proof of Lemma 1.7.* By Lemmas 1.1–1.3, let $\langle \delta_n : n < \omega \rangle$ be an $\omega$ sequence in $V[r \upharpoonright \kappa]$ (and hence also in $V[r \upharpoonright \beta]$ for $\beta \in (\kappa, \lambda)$ and in $N$) of cardinals so that each of the above structures is a model for the sentence “Each $\delta_n$ is a measurable cardinal with normal measure $\mu_n$.” In $N$, define a (Prikry) filter $F$ on $\kappa$ by $A \in F$ iff $\exists n \forall m \geq n [A \cap \delta_m \text{ contains a } \mu_m \text{ measure 1 set}],$ and for $\beta \in (\kappa, \lambda)$ let $F^\beta$ be the (Prikry) filter in $V[r \upharpoonright \beta]$ defined on $\kappa$ in the exact same fashion. The claim is that $N \models \text{"} F \text{ is a Rowbottom filter on } \kappa.$” That $N \models \text{"} F \text{ is a filter}"$ is not too difficult to see and is left to the reader.

To see that $N \models \text{"} F \text{ is Rowbottom,"}$ let $f \in N$ be a Rowbottom partition of $[\kappa]^{<\omega}$, and as $f$ may be coded by a set of ordinals, let $\beta \in (\kappa, \lambda)$ be so that $f \in V[r \upharpoonright \beta]$. As $V[r \upharpoonright \beta] \models \text{"} Each \ k_n \text{ is measurable and } \bigcup_{n < \omega} \delta_n = k, \text{ a theorem of Prikry [10] shows that } V[r \upharpoonright \beta] \models \text{"} There is a set } A \in F^\beta \text{ so that } A \text{ is Rowbottom homogeneous for } f,"$ As $F^\beta \subseteq F$, $A \in F$ and $N \models \text{"} F \text{ is Rowbottom."}$ This proves Lemma 1.7.

**Lemma 1.8.** $N \models AC_\kappa.$

*Proof of Lemma 1.8.* Let $N \models \text{"} \langle x_\beta : \beta < \kappa \rangle \text{ is a } \kappa \text{ sequence of non-empty sets,"}$ and let $\tau \in V$ be a term which denotes this sequence so that $\tau \in L_{1,\alpha}$ for some $\alpha < \lambda$. Without loss of generality assume that $\tau$ is a function on $\kappa$ so that for $\beta < \kappa$, $\tau(\beta)$ is an element of $L_{1,\alpha}$ which denotes $x_\beta$. Thus, it will be the case that $| \tau : \beta < \kappa | \neq \emptyset$. Therefore, using $AC_\kappa$ in $V$, let for $\beta < \kappa$, $\sigma_\beta$ be so that $| \tau : \beta \in \tau(\beta)$. As each $\tau(\beta)$ will always denote an element of $N$, we can assume that $\sigma_\beta \in L_{1,\gamma(\beta)}$ for some $\gamma(\beta)$ which depends upon $\beta$. Let $f \in V$ be so that for $\beta < \kappa$, $f(\beta) = \sigma_\beta$.

Let $\gamma = \bigcup_{\beta < \kappa} \gamma(\beta) \cup \alpha$. As $\lambda$ is regular in $V$, $\gamma < \lambda$. Thus, every occurrence of $r \upharpoonright \gamma(\beta)$ in $\sigma_\beta$ can be replaced by an occurrence of $r \upharpoonright \gamma$, making it possible using $f$ to write a term $g \in L_{1,\gamma}$ which is a function on $\kappa$ so that for each
\( \beta < \kappa, |\neg \neg \forall \alpha (\beta < \kappa). \) As \( g \) can be evaluated in \( N \) using \( r \upharpoonright \gamma \), \( g \) will denote in \( N \) an AC sequence for \( \langle \chi^\beta : \beta < \kappa \rangle \). This proves Lemma 1.8.

Lemmas 1.1–1.8 complete the proof of Theorem 1.

We turn now to the proof of Theorem 2. The proof uses a collapsing argument the use of which was suggested to us by Kanamori.

**Proof of Theorem 2.** Let \( V \) and \( N \) be as in Theorem 1. Thus, in \( N \), we have a cardinal \( \kappa \) such that \( \text{cof}(\kappa) = \omega \) and \( \kappa^+ \) is measurable. Also, as \( V \) and \( N \) possess the same bounded subsets of \( \kappa \), as we have previously observed in Lemma 1.7, there is an \( \omega \) sequence of measurable cardinals whose limit is \( \kappa \) present in \( N \).

We let \( \langle \delta_i : i < \omega \rangle \) be such an \( \omega \) sequence. Let \( P_0 \) be the partial ordering \( \text{Col}(\omega_1, \delta_0) \), and for \( i \geq 1 \), let \( P_i \) be the partial ordering \( \text{Col}(\delta_{i-1}, \delta_i) \). Let \( \hat{P} = \prod_{i \in \omega} P_i \). The partial ordering \( P \) is taken as being \( \{ p \in \prod_{i \in \omega} P_i : \) The \( i \)th coordinate of \( p, p_i, \) is non-empty only finitely often \( \} \). For \( p, q \in P \), we say that \( p \leq q \) iff \( \forall i[p_i \leq q_i] \).

Note that for any \( n \in \omega \) we may view \( P \) as \( P_0^* \times P^n \), where \( P_0^* = \prod_{i \leq n} P_i \) (ordered componentwise) and \( P^n = \{ p \in \prod_{i > n} P_i : \) The \( i \)th coordinate of \( p \) is non-empty only finitely often \( \} \) (also ordered component wise). Let \( G \) be \( N \)-generic on \( P \). It follows by the above remarks that \( G_n = G \upharpoonright P_0^* \) (the projection of \( G \) onto its first \( n + 1 \) coordinates) is \( N \)-generic on \( P_0^* \).

The model \( M \) which shall witness Theorem 2 is the least model of \( ZF \) which extends \( N \) and contains each \( G_n \) (but again, not the \( \omega \) sequence of the \( G_n \)'s). As in Theorem 1, we can talk about \( M \) using a ramified sublanguage \( L_1 \subseteq L \) of the forcing language with respect to \( P \), where \( L_1 \) contains a predicate symbol \( N \) (interpreted as \( N(m) \leftrightarrow m \in N \)), symbols \( m \) for each \( m \in N \), and symbols \( G_n \) for each \( n \in \omega \). \( M \) is then defined inductively as follows:

\[
M_0 = \emptyset
\]

\[
M_\delta = \bigcup_{\alpha < \delta} M_\alpha \text{ for } \delta \text{ a limit ordinal}
\]

\[
M_{\alpha + 1} = \{ x \subseteq M_\alpha : x \in N[G] \text{ and } x \text{ can be defined using a forcing term } \tau \in L_1 \text{ of rank } \leq \alpha \}.
\]

\[
M = \bigcup_{\alpha \in \text{Ordinals}} M_\alpha.
\]

As in Theorem 1, standard arguments show that \( M \models ZF \). Again, there is a natural classification of the formulas of \( L_1 \) defined as follows: For any \( \phi \in L_1 \), say that \( \phi \in L_{1,n} \) iff \( n > \max(|m : G_m \text{ appears in } \phi|) \). Standard arguments allow us to assume each \( m \) for \( m \in N \) is invariant under any
automorphism of $P$ and that any formula $\phi \in L_{1,n}$ is invariant under an
automorphism which fixes $G_m$ for $m \leq n - 1$, i.e., which is generated by a
function which is the identity on $\delta_{n-1}$.

We now begin the proofs of the lemmas needed to show that $M$ is our
desired model.

**Lemma 2.1.** Let $\phi \in L_{1,n}$. If $\langle p, q \rangle \models \phi$, where $p \in P_{n-1}^* \text{ and } q \in P^{n-1}$,
then $\langle p, \emptyset \rangle \models \phi$.

**Proof of Lemma 2.1.** Assume that $\langle p, q \rangle \models \phi$ but $\langle p, \emptyset \rangle \not\models \phi$. Then there
is some $\langle p', q' \rangle \supseteq \langle p, \emptyset \rangle$ such that $\langle p', q' \rangle \models \neg \phi$.

For any $m \geq n$, there is an automorphism $\pi_m$ of $P_m$ which is the identity
on $\delta_{n-1}$ such that $\pi_m(q_m)$ is compatible with $q_m$; the definition of any such
automorphism may be carried out without using AC. Hence, let us pick for
each $m \geq n$ such an automorphism. These automorphisms may be picked
without using AC since there is some $k$ such that $k' \geq k \Rightarrow q_k = q'_k = \emptyset$; for
any such $k$, we let $\pi_k$ be the identity. For $m \leq n - 1$, we let $\pi_m$ be the
identity.

$\pi = \langle \pi_m : m < \omega \rangle$ is thus an automorphism of $P$ such that $\pi(\langle p', q' \rangle)$ is
compatible with $\langle p', q' \rangle$. But, as $\phi \in L_{1,n}$, we have that $\pi(\langle p', q' \rangle) \models \phi$ and
$\langle p', q' \rangle \models \neg \phi$. This contradiction proves Lemma 2.1. \qed

**Lemma 2.2.** $M \models \langle \kappa = \aleph_\omega \text{ and } (\kappa^+)^N \leq \aleph_\omega + 1 \rangle$.

**Proof of Lemma 2.2.** As each $G_n \in M$, we clearly have that $\kappa \leq \aleph_\omega$ and
$\kappa^+ \leq \aleph_{\omega+1}$; in fact, $\delta_0 \leq \aleph_2$ and $\delta_{i+1} \leq \delta_i^*$. Hence, to show that $\kappa = \aleph_\omega$,
it suffices to show that in $M$, each $\delta_i$ is a cardinal. It will immediately follow
from the fact that $M \models \langle \kappa = \aleph_\omega \rangle$ that $M \models \langle (\kappa^+)^N \leq \aleph_\omega + 1 \rangle$.

To show that each $\delta_i$ remains a cardinal, let $\tau$ be a term such that for
some $\delta_i$ and some $p \in P$, $p \models \langle \tau \subseteq \delta_i \rangle$. As $\tau$ will denote a subset of $\delta_i$ in $M$,
we may assume that $\tau \in L_{1,n}$ for some $n$. But by Lemma 2.1, using the
representation of $p$ as $\langle p_{n-1}^*, p^{n-1} \rangle$, where $p_{n-1}^* \in P_{n-1}^*$ and $p^{n-1} \in P^{n-1}$,
$\langle p_{n-1}^*, \emptyset \rangle \models \langle \tau \subseteq \delta_i \rangle$ and for any $q \geq p$ such that $q \models \langle a \in \tau \rangle$,
$\langle q_{n-1}^*, \emptyset \rangle \models \langle \alpha \in \tau \rangle$. Thus, any subset of $\delta_i$ is in $N[G_m]$ for some $m$, so it
suffices to show that in $N[G_m]$, each $\delta_i$ is a cardinal.

Now if $N = AC$, then the standard facts about the Lévy collapse orderings
would show that any $\delta_i$ remains a cardinal in $N[G_m]$ for any $m$. However, as
we do not have full $AC$ at our disposal, we have to be somewhat careful with
our argument. As $P_m^*$ and $2^{P_m}$ both possess well-orderings, the proof that,
using the proper antichain criteria, regular cardinals in $N | 2^{P_m}$ are still
regular cardinals in $N[G_m]$ may be carried out to show that $\delta_i$ for $i > m$
is still a regular cardinal in $N[G_m]$. To show that each $\delta_i$ for $i \leq m$ remains a
cardinal in $N[G_m]$, we note that as $R(\kappa)^N = R(\kappa)^N$, $R(\kappa) = ZFC$, and
$P_m \in R(\kappa)$, the subsets of each $\delta_i$ for $i \leq m$ present in $N[G_m]$ are the same as
SINGULAR CARDINALS

those in $R(\kappa)[G_m]$. $R(\kappa)[G_m] - \delta_i$ is a cardinal" for $i \leq m$ by the standard facts about forcing with products of Lévy orderings over a model of ZFC. Thus, each $\delta_i$ for $i \leq m$ remains a cardinal in $N[G_m]$. This proves Lemma 2.2.

**Lemma 2.3.** $\models \kappa (= \aleph_\omega)$ is a strong limit cardinal.

**Proof of Lemma 2.3.** It is enough to show that every subset of $\delta_i$ in $M$ lies in $N[G_{i+1}]$, since as in Lemma 2.2, every subset of $\delta_i$ in $N[G_{i+1}]$ will actually be present in $R(\kappa)[G_{i+1}]$, a model of ZFC in which $\kappa$ is the supremum of all of the ordinals. To show that every subset of $\delta_i$ in $M$ lies in $N[G_{i+1}]$, let $A \subseteq \delta_i$, $A \in N$. Let $\tau$ be a term which denotes $A$ such that $\tau \in L_{i,n}$ for some $n$. Then, as in Lemma 2.2, $A \in N[G_{n-1}]$. If $n < i + 1$, then we are done. If $n > i + 1$, then write $P_n$ as $P_{i+1} \times (P_{i+2} \times \cdots \times P_n)$. As $P_{i+2} \times \cdots \times P_n$ is well-orderable in $N$, we can show via the closure conditions that forcing with $P_{i+2} \times \cdots \times P_n$ adds no new subsets to $\delta_i$. An application of the product lemma for product forcing then shows that any subset of $\delta_i$ present in $N[G_i]$ is present in $N[G_{i+1}]$.

We now proceed to show that not only is $(\kappa^+)^N$ still a cardinal in $M$, but that it remains measurable as well. To do this, we first fix $\mu \in N$, $\mu$ a normal measure on $(\kappa^+)^N$.

**Lemma 2.4.** Let $\mu' = \{x \subseteq (\kappa^+)^N: x \in M$ and $x$ contains a $\mu$ measure 1 set}. Then $\mu'$ is a $(\kappa^+)^N$ additive measure on $(\kappa^+)^N$ in $M$, and $M \models \kappa^N = \aleph_{\omega+1}$.

**Proof of Lemma 2.4.** In the proof of Lemma 2.2 it was shown that given a particular $\delta_i$, any subset of it in $M$ was in $N[G_n]$ for some $n$. The exact same proof will show that for any ordinal $\alpha$ and any $x \subseteq \alpha$ so that $x \in M$, $x \in N[G_n]$ for some $n$. And, as noted in Lemma 2.2, for any $n \in \omega$ both $P_n^*$ and $2P_n^*$ possess well-orderings in $N$; indeed, as $P_n^* \in R(\kappa)^N = R(\kappa)^N$ and $N \models \kappa$ is a strong limit cardinal, $N \models |P_n^*| < 2^{P_n^*} < \kappa^+$. Thus, as the Lévy–Solovay arguments [6] for showing that a $\delta$ additive normal measure $\nu$ on a cardinal $\delta$ extends to a $\delta$ additive normal measure $\nu'$ when forcing with a partial ordering that has a well-ordering of size $< \delta$ do not require any use of AC (see [1] for an exposition of a similar argument), $\mu^n = \{x \subseteq (\kappa^+)^N: x \in N[G_n]$ and $x$ contains a $\mu$ measure 1 set} is a $(\kappa^+)^N$ additive normal measure on $(\kappa^+)^N$ in $N[G_n]$. Hence, the same argument as given in Lemma 1.6 can be repeated to show that $\mu'$ is a measure on $(\kappa^+)^N$ in $M$. As $M \models \kappa^N \leq \aleph_{\omega+1}$ and $(\kappa^+)^N$ is a measurable cardinal, $M \models \aleph_{\omega+1} = (\kappa^+)^N$.

**Lemma 2.5.** $M \models \aleph_{\omega}$ carries a Rowbottom filter.
Proof of Lemma 2.5. As in Lemma 1.7, we can assume that $\langle \mu_i: i < \omega \rangle \subseteq V[r \upharpoonright \kappa] \subseteq N$ is an $\omega$ sequence so that $\mu_i$ is a normal measure on $\delta_i$. Assume further that $\langle W_i: i < \omega \rangle \subseteq V[r \upharpoonright \kappa] \subseteq N$ is an $\omega$ sequence so that $W_i$ well orders $\mu_i$. Clearly, both of the above sequences are also elements of $N$.

In $M$, define a filter $F$ on $\kappa (= \kappa_\omega)$ by $A \in F$ iff $\exists m \forall n \forall m \geq n [A \cap \delta_m$ contains a $\mu_m$ measure 1 set]. Again, as in Lemma 1.7 it is not too difficult to see that $M \vDash "F$ is a filter on $\kappa."$ To see that $M \vDash "F$ is Rowbottom," let $f \in M$ be a Rowbottom partition on $[\kappa]^{<\omega}$, and by Lemmas 2.2 and 2.4, let $n$ be so that $f \in N[G_n]$. As noted in Lemma 2.4, $N \models "P_n$ is well-orderable," and by the definition of $P_n$ and the fact that $P_n^* \in R(\kappa)^V = R(\kappa)^N$, $|P_n^*| < \delta_m$ when $m \geq n + 1$. Thus, the Lévy–Solovay arguments [6] applied in $R(\kappa)^N$ show that for each $m \geq n + 1$, $\mu^*_m = \{A \subseteq \tilde{\delta}_m: A \in R(\kappa)^N \cap G_n \}$ and $A$ contains a $\mu_m$ measure 1 set) is a $\delta_m$-additive normal measure on $\delta_m$ in $R(\kappa)^N[G_n]$ and hence has the same properties in $N[G_n]$.

Prikry’s proof [10] that an $\omega$ limit of measurable cardinals is a Rowbottom cardinal which carries a Rowbottom filter uses DC by choosing measure 1 homogeneous sets for certain canonically defined partitions based on earlier measure 1 sets that were chosen. As each $\mu_m^*$ measure 1 set contains a $\mu_m$ measure 1 set for $m \geq n + 1$ we may carry out Prikry’s proof for $f$ in $N[G_n]$ by each time choosing the $W_m$ least $\mu_m$ measure 1 set for the $m$ appropriate to the current stage of the construction. In this way a set $C \subseteq \kappa$ which is homogeneous for $f$ may be constructed in $N[G_n]$ so that for $m \geq n + 1 \mu_m(C \cap \delta_m) = 1$. As $C \in M$ and $C \in F$, the proof of Lemma 2.5 is complete.

Lemmas 2.1–2.5 complete the proof of Theorem 2.

Remark. AC$\omega$ fails in $M$, as by definition of the model $M$, there is no sequence $\langle f_n: n \in \omega \rangle \varepsilon M$, where $f_n$ is a bijection between $\delta_n^+$ and $\delta_{n+1}^+$. Thus, Lemma 2.4, which shows that normal measures on $\kappa_{\omega+1}$ in $M$ exist, is especially significant.

In conclusion to this paper, we mention that, as pointed out to us by Kanamori, strong hypotheses are needed to obtain the conclusions of Theorems 1 and 2. This is since if $\kappa_{\omega+1}$ is a measurable cardinal, $\square \kappa_\omega$ must fail, so by recent work of W. Mitchell, there are inner models with many measurable cardinals.

References

2. T. Jech, $\omega_1$ can be measurable, Israel J. Math. 6 (1968), 363–367.