JOURNAL OF ALGEBRA 23, 181–213 (1972)

Pfister Forms and K-Theory of Fields*

RICHARD ELMAN† AND T. Y. LAM‡

Department of Mathematics, University of California, Berkeley, California 94720

Communicated by A. Fröhlich

INTRODUCTION

The object of this paper is to investigate quadratic forms of the type $\varphi = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$, which we shall call *n-fold Pfister forms*. These forms were first studied systematically by Pfister [7], who showed that the class of Pfister forms essentially coincides with the class of the so-called strongly multiplicative forms (over a field F of characteristic different from two). For $n = 2$, we have $\varphi = \langle 1, a_1, a_2, a_1a_2\rangle$, which is the norm form of the quaternion algebra $(-a_1, -a_2/F)$. It has been known for some time that the isomorphism type of the quaternion algebra $(-a_1, -a_2/F)$ and the isometry type of the norm form $\langle 1, a_1, a_2, a_1a_2\rangle$ are in natural one-to-one correspondence, so that one determines the other, and vice versa. For n-fold Pfister forms with $n > 2$, however, no classification theory has been available so far. In his paper [4], by the method of Stiefel–Whitney classes, Milnor showed that the n-fold Pfister form $\varphi = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$ determines an invariant $l(-1)^{t-n}l(-a_1) \cdots l(-a_n)$ in the algebraic K-group k_tF, where $t = 2^{n-1}$. Our main theorem in this paper is to establish that *$l(-a_1) \cdots l(-a_n) \in k_nF$ is a complete invariant of the isometry type of the n-fold Pfister form φ above* (see Theorem 3.2). The techniques used in the proof of this theorem have also various applications to a question raised by Milnor [4], asking whether k_nF is isomorphic to $I^nF/I^{n+1}F$, where $I(F)$ denotes the ideal of all even-dimensional forms in the Witt ring $W(F)$. In particular, we will be able to show that this is indeed the case, if k_nF has at most 64 elements.

In the first section, we set up the basic notations in this paper, review some familiar facts about quadratic forms, and then establish some elementary properties of 2-fold Pfister forms. Theorem 1.8 about the relationship between

* This is a detailed exposition of the results announced in *Bull. Amer. Math. Soc.* **77** (1971), 971–974.

† This work is based partly on the first author's Ph.D. thesis at the University of California, Berkeley. Current address: Department of Mathematics, Rice University, Houston, TX 77001.

‡ Supported in part by NSF Grant GP-20532.

k_2F, 2-fold Pfister forms and quaternion algebras is the impetus toward further results.

In Section 2, we define the notion of chain-p-equivalence, and prove the technical result (Theorem 2.6) which will become the main tool of the subsequent sections. We also obtain various new results about "factorizing" Pfister forms generalizing existing theorems in the literature (see Theorem 2.7 and the concluding remark of Section 2).

Section 3 is occupied with the proof of the Main Theorem, which, in particular, says that *chain-p-equivalence of n-fold Pfister forms coincides with ordinary isometry of such forms.* We also obtain an interpretation of the level of a field F in terms of the algebraic K-theory of F. If the level of F is $s = 2^m$, we show that the cardinality of $W(F)$ is no smaller than $2 \cdot 2^{m(m+1)(m+2)/6}$ (Theorem 3.7).

In Section 4, we introduce the notion of "linkage" of Pfister forms, and establish an interesting relationship between linkage and the Witt index (Proposition 4.4, Theorem 4.5). Linkage criteria are given for pairs and triples of Pfister forms.

Section 5 studies the structure of k_nF, and the relationship between k_nF and the subsequent groups $k_{n+j}F$. *A rough estimate shows that if* $\dim_{Z_2} k_nF = r$, then $\dim_{Z_2} k_{n+1}F \leqslant r(r+1)/2$ (Proposition 5.1). We obtain also a sufficient condition for $\dim_{Z_2} k_{n+1}F$ to be actually bounded by $\dim_{Z_2} k_nF$ (Theorem 5.10). Using the notion of linkage, we show that: if $n > 1$ and $\dim_{Z_2} k_nF = m \leqslant 6$, then $\dim_{Z_2} k_{n+1}F \leqslant m$. To conclude Section 5, we prove the following stable structure theorem for k_*F: *if* $n > 1$, *and* $k_nF/l(-1)k_{n-1}F$ *has finite* Z_2-*dimension equal to* d, *then, with* $t =$ *the integral part of* $d/(n-1)$, *we have* $k_{n+t+i}F = l(-1)^i k_{n+t}F$ *for all* $i \geqslant 0$ (Theorem 5.13).

The final section applies the results described above to study the quotient groups $I^nF/I^{n+1}F$ in relation to k_nF. *If every element of* k_nF *can be expressed as the sum of at most three generators, these two groups are shown to be isomorphic.* This conclusion applies, in particular, to the case when k_nF has at most 64 elements (Corollary 6.2). *If* k_2F *has at most* 64 *elements, we are able to show also that* k_2F *imbeds into the Brauer group of* F.

The techniques used in this paper seem to suggest that one might try to determine I^nF as abelian groups by taking the n-fold Pfister forms as generators, and finding all relations which hold for these forms. For $n = 2$, this can be achieved without difficulty, by essentially known techniques. The higher powers I^nF ($n > 2$) are, however, more involved, and we have been able to determine the full set of relations among n-fold Pfister forms only in rather special cases. A successful determination of all such relations in general is, of course, likely to resolve Milnor's question asking whether k_nF is isomorphic to $I^nF/I^{n+1}F$.

Our present paper is independent of Pfister's work [7, 8]. In fact, the methods we introduce lead to new simple proofs of several familiar facts about multiplicative forms (see Corollaries 2.3, 2.4) proved first by Pfister in [7].

In conclusion, we wish to express our hearty thanks to Arason and Pfister, who sent us the preprint of their work [1], from which some of the ideas in Section 3 of this paper developed.*

1. Terminology and Basic Facts

In this section, we set up the definitions and notations to be used in this paper, and state some well-known facts about quadratic forms. The primary result here is Theorem 1.8, which will be the motivation of the subsequent sections.

By a field F, we shall always mean a field of characteristic different from two. We call V a *quadratic space* over a field F if V is a finite-dimensional F-vector space equipped with a *nondegenerate* symmetric bilinear form $B : V \times V \to F$. By the *quadratic form* q of the quadratic space (V, B), we mean the map $q : V \to F$ given by $q(v) = B(v, v)$, $v \in V$. We shall identify a quadratic space with its quadratic form when no confusion can arise. We shall say two quadratic spaces are *isometric* if there exists a vector space isomorphism between them which preserves the quadratic forms. We denote isometries by the symbol $\cong$. Let $\langle a \rangle$ denote the one-dimensional quadratic space $F \cdot e$ where the inner product of e with itself is $a \in F - \{0\}$. It is well-known that every n-dimensional quadratic space V is isometric to an orthogonal sum $\langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$. We shall henceforth denote such an orthogonal sum by $\langle a_1, a_2, \ldots, a_n \rangle$.

A quadratic form q is called *isotropic* if there exists $v \neq 0$ in V such that $q(v) = 0$. Otherwise, q is called *anisotropic*. All binary isotropic forms are isometric to $\langle 1, -1 \rangle$. This is called the *hyperbolic plane*, and will be denoted by $\mathbf{H}$. A *hyperbolic space* means, more generally, an orthogonal sum of a number of copies of $\mathbf{H}$. The *Witt decomposition theorem* says that any quadratic form q is decomposable into $q_a \perp q_h$, where q_a is anisotropic, and $q_h \cong m\mathbf{H}$ is hyperbolic. Here, the isometry type of q_a is uniquely determined by q, so we speak of q_a as the "anisotropic part" of q. The integer m above is also uniquely determined by q, and will be called the "*Witt index*" of q.

For a field F, let $\hat{W}(F)$ denote the *Witt-Grothendieck group* of isometry classes of quadratic forms over F (see [8], or [7]). This consists of formal differences $M - N$ of quadratic spaces, where $M - N = M' - N'$ if and only if $M \perp N' \cong M' \perp N$. The Kronecker product of quadratic spaces (forms), denoted by $\otimes$, induces a commutative ring structure on $\hat{W}(F)$. We

* *Added in proof.* After this paper was submitted for publication, we learned that Arason has also obtained independently some of the results in Section 3, and Theorem 4.8.

shall often express this product by q_1q_2 (instead of $q_1 \otimes q_2$), when no confusion can arise. Since **ZH** is an ideal in $\hat{W}(F)$, we may form the *Witt ring* $W(F) = \hat{W}(F)/\mathbf{ZH}$.

The following elementary result is well-known, and will be used implicitly throughout this paper.

PROPOSITION 1.1. (1) *Two forms q_1 and q_2 are equal in $W(F)$ if and only if their anisotropic parts $(q_1)_a$ and $(q_2)_a$ are isometric.* (2) *If q_1 and q_2 have the same dimension, then q_1 is equal to q_2 in the Witt ring $W(F)$ if and only if $q_1 \simeq q_2$.* (3) *The elements of $W(F)$ are in one-to-one correspondence with all the isometry classes of anisotropic forms.*

Let $I(F)$ denote the ideal of all even-dimensional forms in the Witt ring $W(F)$. We shall be interested in its powers $I^n(F)$. In order to study these ideals, we introduce the following class of quadratic forms, which will play a crucial role. For an n-tuple of nonzero elements $(a_1, ..., a_n)$ of F, we write $\langle\langle a_1, ..., a_n\rangle\rangle$ to denote the 2^n-dimensional quadratic form $\otimes_{i=1}^{n} \langle 1, a_i\rangle$, and shall refer to it as an *n-fold Pfister form*. (A 0-fold Pfister form is, by convention, taken to be $\langle 1\rangle$). These are essentially the strongly multiplicative forms studied by Pfister extensively in [7]. Clearly, the n-fold Pfister forms provide a system of generators for the abelian group I^nF.

Since any n-fold Pfister form φ represents 1, we may write $\varphi \simeq \langle 1\rangle \perp \varphi'$. We shall call φ' the *pure subform* of φ; this terminology is justified since the isometry type of φ' is uniquely determined by φ, according to Witt's cancellation theorem [10]. *In the balance of this paper, we shall always write φ' to denote the pure subform of a Pfister form φ.*

Let $\dot{F}$ denote the multiplicative group of F. For any quadratic form q over F, we define

$$D_F(q) = \{a \in \dot{F} \mid q \text{ represents } a\},$$

and

$$G_F(q) = \{a \in \dot{F} \mid \langle a\rangle q \simeq q\} = \{a \in \dot{F} \mid \langle a\rangle q = q \in W(F)\}.$$

The latter is always a subgroup of $\dot{F}$.

If L is an extension field of K, then any K-quadratic form q gives rise to an L-quadratic form $q_L = L \otimes_K q$. We will need the following well-known facts about anisotropic forms and extension fields which we quote without proof. (See [1, 7 and 9]).

LEMMA 1.2. *Let $L = K(\sqrt{d})$ be a quadratic extension of K. Let τ be an anisotropic form over K such that $L \otimes_K \tau = 0$ in $W(L)$. Then there exists a form τ_1 over K such that τ is isometric to $\langle 1, -d\rangle \otimes \tau_1$ over K. Furthermore, if x_1 is a transcendental element over K, then $x_1^2 - d$ belongs to $G_{K(x_1)}(K(x_1) \otimes_K \tau)$.*

LEMMA 1.3. *Let τ be an anisotropic form over F. Let $X = (x_1, \ldots, x_m)$ be a set of m independent transcendental elements over F, and $F(X) = F(x_1, \ldots, x_m)$. Then $F(X) \otimes_F \tau$ is anisotropic over $F(X)$. If $F(X) \otimes_F \tau$ represents the value $\varphi(X)$ over $F(X)$ where φ is an m-dimensional form over F, then τ is isometric to $\varphi \perp q$ over F for some (anisotropic) form q. In particular,* $\dim_F \tau \geqslant m$.

We shall now state and prove the following theorem which is implicit in the paper of Arason–Pfister [1]. It will be needed repeatedly in the sequel.

THEOREM 1.4. *Let φ be an n-fold Pfister form and q an anisotropic form over F. If q lies in the principal ideal $W(F) \cdot \varphi$, then there exists a form σ such that $q \cong \varphi \cdot \sigma$. If $a \in D_F(q)$, then σ can be chosen so that $a \in D_F(\sigma)$ also.*

Proof. Let $X = (x_1, \ldots, x_{2^n})$ be a set of 2^n-independent transcendental elements over F. Write $X' = (x_2, \ldots, x_{2^n})$, $K = F(X')$ and $d = -\varphi'(X') \in K$. (Recall that φ' denotes the pure subform of φ). Let L be the quadratic extension $K(\sqrt{d})$ of K. Then $\varphi(\sqrt{d}, X') = d + \varphi'(X') = 0$ in L, so $L \otimes_K \varphi$ is isotropic over L. By a well-known theorem on Pfister forms (see Corollary 2.3 below), $L \otimes_K \varphi_K = 0$ in $W(L)$. The hypothesis $q \in W(F) \cdot \varphi$ now implies that $L \otimes_K (q_K) = 0 \in W(L)$. By Lemma 1.3, q_K is anisotropic over K. We may therefore apply Lemma 1.2, concluding that $x_1^2 - d \in G_{K(x_1)}(K(x_1) \otimes_F q)$. Now $K(x_1) = F(X)$ and $x_1^2 - d = \varphi(X)$. Take any $a \in D_F(q)$; then $F(X) \otimes_F \langle a \rangle q$ represents the value $\varphi(X)$ over $F(X)$. By Lemma 1.3, $q \cong \langle a \rangle \varphi \perp q_1$ for some form q_1 over F. Now q_1 is still anisotropic, belongs to $W(F) \cdot \varphi$, and has dimension less than that of q. The proof proceeds by induction. Q.E.D.

We wish to investigate in this paper the relationship between the Pfister forms and the algebraic K-theory of fields. In [4] Milnor defined for any field F the algebraic K-theory groups $K_n(F)$ as follows. Let $\mathscr{F}_n(F)$ be the free abelian group on symbols $l(a_1)l(a_2) \cdots l(a_n)$ where $a_i \in \dot{F}$. Let $\mathscr{R}_n(F)$ be the subgroup of $\mathscr{F}_n(F)$ generated by the following two types of elements:

R1. $l(a_1) \cdots l(a_n)$ where $a_i + a_{i+1} = 1$ for some i such that $1 \leqslant i \leqslant n-1$.

R2. $l(a_1) \cdots l(a_i b_i) \cdots l(a_n) - l(a_1) \cdots l(a_i) \cdots l(a_n)$
$\quad - l(a_1) \cdots l(b_i) \cdots l(a_n), \qquad 1 \leqslant i \leqslant n.$

Then $K_n(F)$ is defined to be the quotient group $\mathscr{F}_n(F)/\mathscr{R}_n(F)$. By $K_*(F)$ we shall mean the graded ring $(\mathbf{Z}, K_1F, K_2F, \ldots)$ with the obvious multiplication (for details, see [4]).

LEMMA 1.5. (1) $l(a)l(a) = l(a)l(-1)$ *for all* $a \in \dot{F}$. (2) *If* $a_1 + \cdots + a_n = 0$ *or* 1, *then* $l(a_1) \cdots l(a_n) = 0$ *in* K_nF (see [4]).

In this paper we are primarily interested in the quotient groups $K_nF/2K_nF$, and shall call these groups k_nF. The elements $l(a_1) \cdots l(a_n) \in k_nF$ will be called the *generators* of k_nF. We have the following properties for these generators:

LEMMA 1.6. (1) *If σ is a permutation of $\{1, 2,..., n\}$, then $l(a_1) \cdots l(a_n) = l(a_{\sigma(1)}) \cdots l(a_{\sigma(n)})$ in k_nF.* (2) *$l(x^2a_1)l(a_2) \cdots l(a_n) = l(a_1) \cdots l(a_n)$ in k_nF.* (3) *If x, y and $z = x + y$ are all nonzero, then $l(x)l(y) = l(z)l(-xy)$ in k_2F. (See* [4] *for* (1) *and* (2), *and* [2] *for the last statement).*

In [4] Milnor defined a map $s_n: k_nF \to I^nF/I^{n+1}F$ by the rule

$$s_n(l(a_1) \cdots l(a_n)) = \langle\langle -a_1 ,..., -a_n\rangle\rangle (\mathrm{mod}\, I^{n+1}F),$$

and proved the following.

THEOREM 1.7. *The maps s_n are group epimorphisms. Furthermore, s_1 and s_2 are isomorphisms.*

In Section 4 of [4], Milnor raised the question whether the maps s_n are isomorphisms for all n. In studying this problem, it is clear that the Pfister forms will play a crucial role. Using Theorem 1.7, we shall first state the following relationship between k_2F and 2-fold Pfister forms. This result is more or less well-known (for example, a special case of it has appeared in [3]), but we shall include some indications of its proof, for completeness.

THEOREM 1.8. *The following statements are equivalent:*

(1) *The quaternion algebras $(a_1 , a_2/F)$ and $(b_1 , b_2/F)$ are isomorphic.*

(2) *$\langle\langle -a_1 , -a_2\rangle\rangle$ and $\langle\langle -b_1 , -b_2\rangle\rangle$ are isometric 2-fold Pfister forms.*

(3) $\langle\langle -a_1 , -a_2\rangle\rangle \equiv \langle\langle -b_1 , -b_2\rangle\rangle \bmod I^3F$.

(4) *$l(a_1)l(a_2) = l(b_1)l(b_2)$ in k_2F.*

Proof. The 2-fold Pfister form $\langle\langle -a_1 , -a_2\rangle\rangle = \langle 1, -a_1 , -a_2 , a_1a_2\rangle$ is precisely the norm form of the quaternion algebra $(a_1 , a_2/F)$. The equivalence of (1) and (2) therefore follows from 57 : 8 of [6]. The implication (2) $\Rightarrow$ (3) is obvious, and the implication (3) $\Rightarrow$ (4) follows from Theorem 1.7. Hence we only have to show that (4) implies (1). But by the universal property of k_2F, the rule $l(x)l(y) \mapsto (x, y/F)$ gives a group homomorphism from k_2F to the Brauer group of F. Therefore, (1) follows from (4), and the proof is complete.

The theorem has the following technical corollaries which will be of use in the sequel.

COROLLARY 1.9. *If $x, y \in F$ and $a_1x^2 + t^2 \neq 0$, then*

$$\langle\langle a_1 , a_2\rangle\rangle \cong \langle\langle a_1 , a_2(a_1x^2 + t^2)\rangle\rangle.$$

In particular, $\langle\langle a_1 , a_2\rangle\rangle \cong \langle\langle a_1 , a_1a_2\rangle\rangle$.

Proof. Theorem 1.8 shows that it suffices to prove that

$$l(-a_1)l(-a_2) = l(-a_1)l(-a_2(a_1x^2 + t^2)) \text{ in } k_2F.$$

Suppose $t \neq 0$. Then in k_2F,

$$\begin{aligned} &l(-a_1)\, l(-a_2(a_1x^2 + t^2)) \\ &\quad = l(-a_1)\, l(-a_2) + l(-a_1)\, l(a_1x^2 + t^2) \\ &\quad = l(-a_1)\, l(-a_2) + l\left(-a_1\left(\frac{x}{t}\right)^2\right) l\left(a_1\left(\frac{x}{t}\right)^2 + 1\right) \\ &\quad = l(-a_1)\, l(-a_2). \qquad \text{(second term} = 0 \text{ by Lemma 1.5).} \end{aligned}$$

If $t = 0$, *then in* k_2F,

$$\begin{aligned} l(-a_1)l(-a_2a_1x^2) &= l(-a_1)l(-a_1a_2) \\ &= l(-a_1)l(-a_2) + l(-a_1)l(a_1) \\ &= l(-a_1)l(-a_2) \qquad \text{by Lemma 1.5(2).} \end{aligned}$$

COROLLARY 1.10. *If* $a_1x^2 + a_2t^2 \neq 0$, *then*

$$\langle\langle a_1, a_2\rangle\rangle \cong \langle\langle a_1x^2 + a_2t^2, a_1a_2\rangle\rangle.$$

Proof. Again, it suffices to prove that

$$l(-a_1)l(-a_2) = l(-a_1x^2 - a_2t^2)l(-a_1a_2) \text{ in } k_2F.$$

Set $z = -a_1x^2 - a_2t^2 \in \dot{F}$. If $x \neq 0$ and $t \neq 0$, then in k_2F

$$l(-a_1)l(-a_2) = l(-a_1x^2)l(-a_2t^2) = l(z)l(-a_1a_2)$$

by Lemma 1.6(3). If $x = 0$ or $t = 0$, we get the result directly from the last statement of the preceding corollary.

2. CHAIN-p-EQUIVALENCE

The element $l(a) \in k_1F = \dot{F}/\dot{F}^2$ is of course the complete invariant of the 1-fold Pfister form $\langle\langle -a\rangle\rangle = \langle 1, -a\rangle$. By Theorem 1.8, $l(a_1)l(a_2) \in k_2F$ is also a complete invariant for the isometry class of the 2-fold Pfister form $\langle\langle -a_1, -a_2\rangle\rangle$. One naturally asks if an analogous result will hold for the isometry class of an n-fold Pfister form. Motivated by this, one is led to the following definition. Let $\langle\langle a_1, \ldots, a_n\rangle\rangle$ and $\langle\langle b_1, \ldots, b_n\rangle\rangle$ be two n-fold Pfister forms. We shall say that they are *simply-p-equivalent* if there exist two indices i and j such that

(a) $\langle\langle a_i, a_j\rangle\rangle \cong \langle\langle b_i, b_j\rangle\rangle$ (see Theorem 1.8 for equivalent conditions), and

(b) $a_k = b_k$ whenever k is different from i and j.

[Note: In the condition (a) above, if i is equal to j, the expression $\langle\langle a_i, a_j\rangle\rangle$ is understood to be just $\langle\langle a_i\rangle\rangle$.] More generally, we say that two n-fold Pfister forms φ and σ are *chain-p-equivalent* if there exists a sequence of n-fold Pfister forms $\varphi_0, \varphi_1, \ldots, \varphi_m$ such that $\varphi_0 = \varphi$, $\varphi_m = \sigma$, and that each φ_i is simply-p-equivalent to φ_{i-1} $(0 \leqslant i \leqslant m-1)$. Chain-$p$-equivalence is clearly an equivalence relation on all n-fold Pfister forms; it will be denoted by the symbol $\approx$. If $n \leqslant 2$, this new symbol $\approx$ is clearly synonymous with $\cong$.

From definition, and Theorem 1.8, we may record

PROPOSITION 2.1. *If* $\langle\langle -a_1, \ldots, -a_n\rangle\rangle \approx \langle\langle -b_1, \ldots, -b_n\rangle\rangle$, *then* $l(a_1) \cdots l(a_n) = l(b_1) \cdots l(b_n)$ *in* k_nF.

Our goal in this and the next section is to prove that *chain-p-equivalence of n-fold Pfister forms coincides with ordinary isometry of such forms.* This fact (Main Theorem 3.2 below) is a rather surprising analog of Witt's classical chain equivalence theorem [10].

As the first step toward our goal, we prove the following important proposition:

PROPOSITION 2.2. *Suppose* $\varphi = \langle\langle a_1, \ldots, a_n\rangle\rangle$ *is an n-fold Pfister form* $(n \geqslant 1)$, *and* $b \in D_F(\varphi')$. *(Recall that* $\varphi \cong \langle 1\rangle \perp \varphi'$*). Then there exist nonzero elements* $b_2, \ldots, b_n$ *of F such that* φ *is chain-p-equivalent to* $\langle\langle b, b_2, \ldots, b_n\rangle\rangle$.

Remark. If the word "chain-p-equivalent" above is replaced by the word "isometric", the proposition is a known result, which, for example, has appeared in Scharlau's notes [9].

Proof of Proposition 2.2. The proof is by induction on n. If $n = 1$, then $\varphi = \langle\langle a_1\rangle\rangle = \langle a_1, 1\rangle$. Since $b \in D_F(\varphi') = D_F(\langle a_1\rangle)$, we have $\langle b\rangle \cong \langle a_1\rangle$, and the result follows. Now assume the result for $(n-1)$-fold Pfister forms. Let $\tau = \langle\langle a_1, \ldots, a_{n-1}\rangle\rangle \cong \langle 1\rangle \perp \tau'$. Then $\varphi \cong \tau\langle a_n, 1\rangle \cong \langle a_n\rangle\tau \perp \tau$, so $\varphi' \cong \langle a_n\rangle\tau \perp \tau'$. Since, by hypothesis, $b \in D_F(\varphi')$, there exist $x \in D_F(\tau) \cup \{0\}$ and $y \in D_F(\tau') \cup \{0\}$ such that $b = a_n x + y$. We may further write $x = x_0 + t^2$ where $x_0 \in D_F(\tau') \cup \{0\}$. Since $b \neq 0$, x and y cannot be both equal to zero.

Case 1. If $x = 0$, then $0 \neq b = y \in D_F(\tau')$. By inductive hypothesis, there exist nonzero $d_2, \ldots, d_{n-1}$ such that $\tau \approx \langle\langle y, d_2, \ldots, d_{n-1}\rangle\rangle$. Thus

$$\varphi \approx \langle\langle y, d_2, \ldots, d_{n-1}, a_n\rangle\rangle = \langle\langle b, d_2, \ldots, d_{n-1}, a_n\rangle\rangle$$

and we are done.

Case 2. Suppose $x \neq 0$. We claim that $\varphi \approx \langle\langle a_1, \ldots, a_{n-1}, xa_n \rangle\rangle$. There is nothing to prove if $x_0 = 0$, for then $x = t^2$. So we may suppose that $x_0 \in D_F(\tau')$. By inductive hypothesis, there exist $c_2, \ldots, c_{n-1}$ in $\dot{F}$ such that

$$\tau \approx \langle\langle x_0, c_2, \ldots, c_{n-1} \rangle\rangle;$$

thus

$$\begin{aligned} \varphi &\approx \langle\langle x_0, c_2, \ldots, c_{n-1}, a_n \rangle\rangle \\ &\approx \langle\langle x_0, c_2, \ldots, c_{n-1}, (x_0 + t^2) a_n \rangle\rangle \qquad \text{(Corollary 1.9)} \\ &\approx \langle\langle a_1, \ldots, a_{n-1}, xa_n \rangle\rangle \end{aligned}$$

proving our claim. If $y = 0$, then the last entry xa_n above is just b and we are done. So we may assume that $y \in D_F(\tau')$. Again our inductive hypothesis implies that $\tau \approx \langle\langle y, d_2, \ldots, d_{n-1} \rangle\rangle$ where $d_i \in \dot{F}$, $2 \leqslant i \leqslant n - 1$. Hence

$$\begin{aligned} \varphi &\approx \langle\langle y, d_2, \ldots, d_{n-1}, xa_n \rangle\rangle \\ &\approx \langle\langle y + xa_n, d_2, \ldots, d_{n-1}, xya_n \rangle\rangle \qquad \text{(by Corollary 1.10)} \\ &= \langle\langle b, d_2, \ldots, d_{n-1}, xya_n \rangle\rangle. \end{aligned}$$

Q.E.D.

This proposition has the following interesting consequences:

COROLLARY 2.3. *If $\varphi = \langle\langle a_1, \ldots, a_n \rangle\rangle$ is isotropic, then φ is hyperbolic. (This is a well-known fact, usually proved by Pfister's theory of multiplicative forms* [7].)

Proof. Since φ contains a hyperbolic plane, we obtain $-1 \in D_F(\varphi')$ by Witt's cancellation theorem. By the Proposition, $\varphi \approx \langle\langle -1, \ldots \rangle\rangle$, so φ is a hyperbolic form.

COROLLARY 2.4. *For a Pfister form φ, $D_F(\varphi) = G_F(\varphi)$. In particular, φ satisfies the "strongly multiplicative" property* [7], *namely, over the transcendental extension $K = F(X)$, there is an isometry $\varphi(X) \cdot \varphi_K \cong \varphi_K$.*

Proof. It is clear that $G_F(\varphi) \subset D_F(\varphi)$, since φ represents 1. For the reverse inclusion, let $c \in D_F(\varphi)$ so that $c = t^2 + b$, where $b \in D_F(\varphi') \cup \{0\}$. We may clearly assume that $b \neq 0$. Therefore, by the Proposition, $\varphi \approx \langle\langle b, b_2, \ldots, b_n \rangle\rangle$. Since $c \in D_F(\langle 1, b \rangle)$, we have $\langle 1, b \rangle \cong \langle c, cb \rangle$. Consequently,

$$\begin{aligned} \langle c \rangle \varphi &\cong \langle c \rangle \langle 1, b \rangle \langle\langle b_2, \ldots, b_n \rangle\rangle \\ &\cong \langle c \rangle \langle c, cb \rangle \langle\langle b_2, \ldots, b_n \rangle\rangle \\ &\cong \langle\langle b, b_2, \ldots, b_n \rangle\rangle \cong \varphi. \end{aligned}$$

Q.E.D.

COROLLARY 2.5. *Let* $\varphi = \langle\langle a_1, \ldots, a_n \rangle\rangle$ *and* $a \in D_F(\varphi)$. *Then*

$$\langle\langle a_1, \ldots, a_n, b \rangle\rangle \approx \langle\langle a_1, \ldots, a_n, ab \rangle\rangle.$$

In particular $\langle\langle a_1, \ldots, a_n, a \rangle\rangle$ *is isometric to* 2φ, *and* $\langle\langle a_1, \ldots, a_n, -a \rangle\rangle$ *is hyperbolic.*

Proof. Write $a = a_0 + t^2$ where $a_0 \in D_F(\varphi') \cup \{0\}$. We may assume that $a_0 \in D_F(\varphi')$ (there is nothing to prove if $a_0 = 0$). By the Proposition, there exist nonzero $a_2', \ldots, a_n'$ such that $\varphi \approx \langle\langle a_0, a_2', \ldots, a_n' \rangle\rangle$. Hence

$$\begin{aligned} \langle\langle a_1, \ldots, a_n, b \rangle\rangle &\approx \langle\langle a_0, a_2', \ldots, a_n', b \rangle\rangle \\ &\approx \langle\langle a_0, a_2', \ldots, a_n', b(a_0 + t^2) \rangle\rangle \\ &\approx \langle\langle a_1, a_2, \ldots, a_n, ab \rangle\rangle. \end{aligned}$$

The last statement of the corollary follows immediately from this, by setting $b = \pm 1$. Q.E.D.

The following generalization of Proposition 2.2 is the main tool in the study of chain-p-equivalence.

THEOREM 2.6. *If* $\tau = \langle\langle a_1, \ldots, a_r \rangle\rangle$ $(r \geqslant 0)$, $\gamma = \langle\langle b_1, \ldots, b_s \rangle\rangle$ $(s \geqslant 1)$ *and* $c_1 \in D_F(\tau\gamma')$, *then there exist nonzero elements* $c_2, \ldots, c_s$ *in* F *such that*

$$\langle\langle a_1, \ldots, a_r, b_1, \ldots, b_s \rangle\rangle \approx \langle\langle a_1, \ldots, a_r, c_1, \ldots, c_s \rangle\rangle.$$

In particular, if $\tau\gamma'$ *represents* -1, *then* $\tau\gamma$ *is hyperbolic.*

Proof. We prove this theorem by induction on s. If $s = 1$, then $c_1 \in D_F(\langle b_1 \rangle \tau)$, so $c_1 = b_1 x$ where $x \in D_F(\tau)$. Corollary 2.5 implies that $\langle\langle a_1, \ldots, a_r, b_1 \rangle\rangle \approx \langle\langle a_1, \ldots, a_r, b_1 x \rangle\rangle = \langle\langle a_1, \ldots, a_r, c_1 \rangle\rangle$. By induction, we may assume the result for $\langle\langle a_1, \ldots, a_r, b_1, \ldots, b_{s-1} \rangle\rangle$. Let $\sigma = \langle\langle b_1, \ldots, b_{s-1} \rangle\rangle$ so $\gamma = \sigma\langle b_s, 1 \rangle = \langle b_s \rangle \sigma \perp \sigma$ and $\gamma' \cong \langle b_s \rangle \sigma \perp \sigma'$. Therefore $\tau\gamma' \cong \langle b_s \rangle \tau\sigma \perp \tau\sigma'$. Since $c_1 \in D_F(\tau\gamma')$, there exist $x \in D_F(\tau\sigma) \cup \{0\}$ and $y \in D_F(\tau\sigma') \cup \{0\}$ such that $c_1 = b_s x + y$. If $x \neq 0$ and $y \neq 0$, we obtain the result in the following two steps.

Step 1. $\langle\langle a_1, \ldots, a_r, b_1, \ldots, b_s \rangle\rangle \approx \langle\langle a_1, \ldots, a_r, b_1, \ldots, b_s x \rangle\rangle$ by Corollary 2.5.

Step 2. By induction there exist nonzero elements $c_2, \ldots, c_{s-1}$ such that

$$(*) \qquad \langle\langle a_1, \ldots, a_r, b_1, \ldots, b_{s-1} \rangle\rangle \approx \langle\langle a_1, \ldots, a_r, y, c_2, \ldots, c_{s-1} \rangle\rangle.$$

Hence by Step 1,

$$\begin{aligned} &\langle\langle a_1, \ldots, a_r, b_1, \ldots, b_{s-1}, b_s \rangle\rangle \\ &\quad \approx \langle\langle a_1, \ldots, a_r, b_1, \ldots, b_{s-1}, b_s x \rangle\rangle \\ &\quad \approx \langle\langle a_1, \ldots, a_r, y, c_2, \ldots, c_{s-1}, b_s x \rangle\rangle \\ &\quad \approx \langle\langle a_1, \ldots, a_r, c_1, c_2, \ldots, c_{s-1}, b_s xy \rangle\rangle \qquad \text{by Corollary 1.10.} \end{aligned}$$

We are now left with the case when one of x, y is zero. If $y = 0$, then $0 \neq c_1 = b_s x$, and Step 1 provides the necessary proof. If $x = 0$, then $c_1 = y$, and from $(*)$,

$$\langle\langle a_1, ..., a_r, b_1, ..., b_s \rangle\rangle \approx \langle\langle a_1, ..., a_r, c_1, ..., c_{s-1}, b_s \rangle\rangle$$

which completes the proof.

Using this theorem together with Theorem 1.4, we can now state the following result.

THEOREM 2.7. *Suppose φ, σ and ρ are, respectively, n-fold, s-fold and r-fold Pfister forms, where $s \geqslant 0$ and $r \geqslant 1$. Suppose further there exists a form q such that $\varphi' \simeq \rho'\sigma \perp q$. Then there exists an $(n - (r + s))$-fold Pfister form μ such that $\varphi \simeq \rho\sigma\mu$.*

Proof. Again we prove the theorem by induction.

Step 1. Assume $r = 1$, $\rho = \langle\langle x \rangle\rangle = \langle x, 1 \rangle$, and induct on $s \geqslant 0$. For $s = 0$, the conclusion follows from Proposition 2.2. For $s \geqslant 1$, write $\sigma = \gamma\langle 1, a \rangle$, where γ is an $(s - 1)$-fold Pfister form. Then $\varphi' \simeq \langle x \rangle\sigma \perp q \simeq \langle x \rangle\gamma \perp \langle xa \rangle\gamma \perp q$. By the inductive hypothesis, there exists an $(n - s)$-fold Pfister form μ such that $\varphi \simeq \langle 1, x \rangle\gamma\mu$. Expanding this, we get

$$\varphi \simeq \langle 1, x \rangle\gamma\mu' \perp \langle x \rangle\gamma \perp \gamma.$$

On the other hand,

$$\varphi \simeq \varphi' \perp \langle 1 \rangle \simeq \langle x \rangle\gamma \perp \langle xa \rangle\gamma \perp (q \perp \langle 1 \rangle).$$

Witt's cancellation theorem therefore yields

$$(**) \qquad \langle 1, x \rangle\gamma\mu' \perp \gamma \simeq \langle xa \rangle\gamma \perp (q \perp \langle 1 \rangle).$$

Subcase 1. *Assume here that $q \perp \langle 1 \rangle$ is anisotropic.* From the above equation, we have $q \perp \langle 1 \rangle \in W(F) \cdot \gamma$. Hence by Theorem 1.4 we may write $q \perp \langle 1 \rangle \simeq \gamma \perp q_1$ for some q_1. By $(**)$ and the cancellation theorem, we obtain $\langle 1, x \rangle\gamma\mu' \simeq \langle xa \rangle\gamma \perp q_1$. Since γ represents 1, we have $xa \in D_F(\langle 1, x \rangle\gamma \cdot \mu')$. Theorem 2.6 now implies that $\varphi \simeq \langle 1, x \rangle\gamma\langle 1, xa \rangle\eta$ where η is an $(n - (s + 1))$-fold Pfister form. But Corollary 1.9 shows that $\langle\langle x, xa \rangle\rangle \simeq \langle\langle x, a \rangle\rangle$. Thus $\varphi \simeq \langle 1, x \rangle \cdot \langle 1, a \rangle \cdot \gamma\eta = \langle 1, x \rangle\sigma\eta$, as desired.

Subcase 2. *Assume here that $q \perp \langle 1 \rangle$ is isotropic.* Since

$$\varphi = \langle x \rangle\sigma \perp (q \perp \langle 1 \rangle),$$

we conclude (from Corollary 2.3) that $\varphi = 0 \in W(F)$. If $s < n - 1$, then the conclusion of Theorem 2.7 is trivial. We may therefore assume that $s = n - 1$, so $\dim(\langle x\rangle\sigma) = \dim(q \perp \langle 1\rangle) = 2^{n-1}$. But then $\varphi = 0 \in W(F)$ implies (by Proposition 1.1) that $q \perp \langle 1\rangle \simeq \langle -x\rangle\sigma$. Hence again by Corollary 2.3, σ is also hyperbolic. We have $\varphi \simeq \rho\sigma(\simeq 2^{n-1}\mathbf{H})$ in this case.

Step 2. In the preceding step, we have shown that the theorem is valid for $r = 1$ (and for all s). For $r > 1$, we may assume, as inductive hypothesis, that the result is true for $r - 1$ (and for all s). Write $\rho = \tau\langle 1, x\rangle$ where τ is an $(r-1)$-fold Pfister form. Hence $\rho' \simeq \tau'\langle 1, x\rangle \perp \langle x\rangle$. From the original hypothesis we obtain

$$\varphi' \simeq \rho'\sigma \perp q \simeq \langle x\rangle\sigma \perp \tau'(\langle 1, x\rangle\sigma) \perp q.$$

From the inductive hypothesis, there exists an $(n - (r + s))$-fold Pfister form μ such that $\varphi \simeq \tau\langle 1, x\rangle\sigma\mu = \rho\sigma\mu$, as was to be shown.

Remark. The following special cases are noteworthy:

(1) $(r = 1, s = 1)$. If φ is an n-fold Pfister form $(n \geqslant 2)$ such that φ' contains a binary subform isometric to $\langle x, y\rangle$, then $\varphi \simeq \langle\langle x, y\rangle\rangle\mu$ for a suitable $(n-2)$-fold Pfister form μ. (The theorem applies here with $\rho = \langle 1, x\rangle$ and $\sigma = \langle 1, y/x\rangle$.)

(2) $(r = 1, \rho = \langle 1, 1\rangle)$. If φ' contains a subform isometric to an s-fold Pfister form σ, then $\varphi \simeq 2\sigma\mu$ for a suitable $(n - s - 1)$-fold Pfister form μ.

(3) $(s = 0)$. If φ contains a subform isometric to ρ, then $\varphi \simeq \rho\mu$ for a suitable $(n - r)$-fold Pfister form μ. The only known proof of this utilizes Pfister's theory of multiplicative forms [7].

3. The Main Theorem

In this section we generalise Theorem 1.8 to arbitrary n-fold Pfister forms, and prove that chain-p-equivalence of Pfister forms coincides with ordinary isometry of such forms. The main tools have been developed in the last section (Proposition 2.2 and Theorem 2.6).

We first record here the beautiful "Hauptsatz" of Arason–Pfister in their recent work [1] which we shall use.

Hauptsatz 3.1. *If φ is an anisotropic form (of positive dimension) over F such that $\varphi \in I^nF$, then* $\dim \varphi \geqslant 2^n$.

Corollary to the Hauptsatz. *Let φ, γ be two quadratic forms which represent a common value $a \in \dot{F}$. Suppose further that* $\dim \varphi + \dim \gamma \leqslant 2^{n+1} + 1$. *Then*

$$\varphi \equiv \gamma (\mathrm{mod}\, I^{n+1}F) \Rightarrow \varphi = \gamma \in W(F).$$

Proof. Since $a \in D_F(\varphi) \cap D_F(\gamma)$, there exist forms φ_0 and γ_0 such that $\varphi \cong \langle a \rangle \perp \varphi_0$ and $\gamma \cong \langle a \rangle \perp \gamma_0$. Consider $q = \varphi_0 \perp \langle -1 \rangle \gamma_0$. Since

$$\varphi \perp \langle -1 \rangle \gamma \cong \langle a, -a \rangle \perp \varphi_0 \perp \langle -1 \rangle \gamma_0 \cong \mathbf{H} \perp q,$$

the hypothesis $\varphi \equiv \gamma \pmod{I^{n+1}F}$ implies that $q \in I^{n+1}F$. Let $q \cong q_a \perp q_h$ be a Witt decomposition of q, where q_a is the anisotropic part and q_h hyperbolic. Then, in the Witt ring, $q_a = q \in I^{n+1}F$. If q_a is not the zero form, then by the Hauptsatz above, we have inequalities

$$2^{n+1} \leqslant \dim q_a \leqslant \dim q = \dim \varphi + \dim \gamma - 2.$$

This implies that $\dim \varphi + \dim \gamma \geqslant 2^{n+1} + 2$, a contradiction to our hypothesis. Thus q_a must be the zero form; in other words $q = 0 \in W(F)$. It follows that $\varphi = \gamma \in W(F)$. Q.E.D.

MAIN THEOREM 3.2. *The following statements are equivalent*:

(1) $\langle\!\langle -a_1, ..., -a_n \rangle\!\rangle$ *and* $\langle\!\langle -b_1, ..., -b_n \rangle\!\rangle$ *are chain-p-equivalent.*

(2) $l(a_1) \cdots l(a_n) = l(b_1) \cdots l(b_n)$ *in* k_nF.

(3) $\langle\!\langle -a_1, ..., -a_n \rangle\!\rangle \equiv \langle\!\langle -b_1, ..., -b_n \rangle\!\rangle \pmod{I^{n+1}F}$.

(4) $\langle\!\langle -a_1, ..., -a_n \rangle\!\rangle$ *and* $\langle\!\langle -b_1, ..., -b_n \rangle\!\rangle$ *are isometric.*

(5) *There exist nonzero elements a and b in F such that* $\langle a \rangle \cdot \langle\!\langle -a_1, ..., -a_n \rangle\!\rangle$ *is isometric to* $\langle b \rangle \cdot \langle\!\langle -b_1, ..., -b_n \rangle\!\rangle$.

In particular, $l(a_1) \cdots l(a_n) \in k_nF$ *is a complete invariant for the isometry class of the Pfister form* $\langle\!\langle -a_1, ..., -a_n \rangle\!\rangle$.

Proof. Observe that if φ is an n-fold Pfister form, then $\langle a \rangle \varphi \equiv \varphi \pmod{I^{n+1}F}$. Thus (5) $\Rightarrow$ (3). Also, trivially, (4) $\Rightarrow$ (5). Consequently, it is sufficient to show that the first four statements are equivalent. This will be done in the following order: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). To simplify notations, we write $\varphi = \langle\!\langle -a_1, ..., -a_n \rangle\!\rangle$, $\gamma = \langle\!\langle -b_1, ..., -b_n \rangle\!\rangle$.

(1) $\Rightarrow$ (2). This is just Proposition 2.1.

(2) $\Rightarrow$ (3). This follows from Theorem 1.7.

(3) $\Rightarrow$ (4). This follows from the Corollary to the Hauptsatz since φ and γ both represent 1, and $\dim \varphi + \dim \gamma = 2^{n+1}$.

(4) $\Rightarrow$ (1). Assuming $\varphi \cong \gamma$, we claim that for any integer r, $0 \leqslant r \leqslant n$,

There exist $c_{r+1}, \ldots, c_n \in \dot{F}$ *such that*

(A_r):

$$\varphi \approx \langle\langle -b_1, \ldots, -b_r, -c_{r+1}, \ldots, -c_n \rangle\rangle.$$

If this is established, then setting $r = n$, the statement (A_n) implies the desired conclusion that $\varphi \approx \gamma$. Now we prove (A_r) inductively on r. There is nothing to prove in case $r = 0$. Assume inductively that (A_r) is true, where $r < n$. We must proceed to verify (A_{r+1}). Set $\tau = \langle\langle -b_1, \ldots, -b_r \rangle\rangle$, $\rho = \langle\langle -b_{r+1}, \ldots, -b_n \rangle\rangle$, and $\eta = \langle\langle -c_{r+1}, \ldots, -c_n \rangle\rangle$. Then the latter is an s-fold Pfister form, where $s = n - r$. We have, from the various hypotheses,

$$\tau\rho \cong \gamma \cong \varphi \cong \tau\eta.$$

In other words, $\tau \perp \tau\rho' \cong \tau \perp \tau\eta'$. By the cancellation theorem, it follows that $\tau\rho' \cong \tau\eta'$. But then

$$-b_{r+1} \in D_F(\rho') \subset D_F(\tau\rho') = D_F(\tau\eta').$$

By Theorem 2.6 we obtain

$$\langle\langle -b_1, \ldots, -b_r, -c_{r+1}, \ldots, -c_n \rangle\rangle \approx \langle\langle -b_1, \ldots, -b_r, -b_{r+1}, -c'_{r+2}, \ldots, -c_n' \rangle\rangle,$$

where $c_j' \in \dot{F}$ $(r + 2 \leqslant j \leqslant n)$. From this and the inductive hypothesis (A_r), we obtain $\varphi \approx \langle\langle -b_1, \ldots, -b_r, -b_{r+1}, -c'_{r+2}, \ldots, -c_n' \rangle\rangle$ which establishes the truth of (A_{r+1}). The proof of the Main Theorem is now complete.

We shall now record a number of corollaries of the Main Theorem.

COROLLARY 3.3. *$\langle\langle -a_1, \ldots, -a_n \rangle\rangle$ is isotropic (hyperbolic) if and only if $l(a_1) \cdots l(a_n) = 0$ in k_nF. In particular, $k_nF = 0$ if and only if $I^nF = 0$, if and only if every n-fold Pfister form is isotropic (hyperbolic).*

COROLLARY 3.4. *Let $\mathscr{F}_n'(F)$ be the free abelian group on symbols $t(a_1) \cdots t(a_n)$ where $a_i \in \dot{F}$ $(1 \leqslant i \leqslant n)$, and let $\mathscr{R}_n'(F)$ be the subgroup of $\mathscr{F}_n'(F)$ generated by the following two types of elements:*

(R′1) *$t(a_1) \cdots t(a_n)$ where $\langle\langle -a_1, \ldots, -a_n \rangle\rangle$ is isotropic.*

(R′2) $t(a_1) \cdots t(a_ib_i) \cdots t(a_n) - t(a_1) \cdots t(a_i) \cdots t(a_n) - t(a_1) \cdots t(b_i) \cdots t(a_n)$.

Then the quotient group $k_n'(F) = \mathscr{F}_n'(F)/\mathscr{R}_n'(F)$ is canonically isomorphic to k_nF.

Proof. One checks easily that the natural maps $k_n'F \to k_nF$ given by $t(a_1) \cdots t(a_n) \mapsto l(a_1) \cdots l(a_n)$ and $k_nF \to k_n'F$ given by $l(a_1) \cdots l(a_n) \mapsto t(a_1) \cdots t(a_n)$ are well-defined homomorphisms and hence isomorphisms.

In [4], Milnor showed that the element -1 is a sum of squares in F if and only if every positive dimensional element of the ring K_*F is nilpotent. Corollary 3.3 allows us to refine this result to the following.

COROLLARY 3.5. *The element -1 is a sum of squares in F if and only if every positive dimensional element of the ring k_*F is nilpotent. Furthermore, if $s = 2^m$ is the level of F (the level of a field is always a power of 2 or infinity), then m is precisely the smallest integer such that $l(-1)^{m+1} = 0$ in k_*F. In this case $l(-1)^{m+2} = 0$ in K_*F, and for any $a \in \dot{F}$, $l(a)^{m+3} = 0$ in K_*F.*

Proof. If -1 is a sum of squares in F, then every positive dimensional element of k_*F is nilpotent, since this is true already in K_*F. Conversely, suppose -1 is not a sum of squares. Then for any n, the n-fold Pfister form $\langle\langle 1,..., 1\rangle\rangle$ is anisotropic. Corollary 3.3 therefore implies that $l(-1)^n \neq 0$ in k_nF. Next, suppose the level of F is $s = 2^m$. Then, the m-fold Pfister form $\varphi = \langle\langle 1,..., 1\rangle\rangle$ is anisotropic, and the $(m + 1)$-fold Pfister form $\varphi\langle 1, 1\rangle$ is isotropic. Hence in k_*F, $l(-1)^m \neq 0$, and $l(-1)^{m+1} = 0$, according to Corollary 3.3. Lifting to $K_{m+1}F$, we may write $l(-1)^{m+1} = 2\alpha$, $\alpha \in K_{m+1}F$. But $2l(-1) = l(1) = 0 \in K_1F$, so $l(-1)^{m+2} = 0$ in $K_{m+2}F$. Finally, if $a \in \dot{F}$, by Lemma 1.5(1),

$$l(a)^{m+3} = l(a)l(-1)^{m+2} = 0 \text{ in } K_{m+3}F.$$

This completes the proof of Corollary 3.5.

We shall conclude this section with another application of the Main Theorem to fields F of finite level $s = 2^m$. This result (Theorem 3.7 below) provides a lower bound on the Z_2-dimensions of the vector spaces $k_{m-r+1}F$, where $1 \leqslant r \leqslant m$. For $r = m$, we shall get $\dim_{Z_2} k_1F \geqslant m(m + 1)/2$, a fact established first by Kaplansky. Thus, our theorem may be viewed as an extension of Kaplansky's result. We shall need the following simple lemma.

LEMMA 3.6. *For any given nonnegative integer s, $l(-1)^s l(x) = 0$ in $k_{s+1}F$ if and only if $x \in D_F(2^s\langle 1\rangle)$.*

Proof. If $x \in D_F(2^s\langle 1\rangle)$, then $2^s\langle 1, -x\rangle$ is isotropic; by Corollary 2.3 and Theorem 3.2, we get $l(-1)^s l(x) = 0$. Conversely, if $l(-1)^s l(x) = 0$, the Main Theorem implies that $2^s\langle 1, -x\rangle$ is hyperbolic. By Witt's cancellation theorem, we obtain $2^s\langle x\rangle \simeq 2^s\langle 1\rangle$; in particular $x \in D_F(2^s\langle 1\rangle)$. Q.E.D.

THEOREM 3.7. *Suppose F has finite level $s = 2^m$. Then,*

(1) $\dim_{Z_2} k_{m-r+1}F \geqslant r(r + 1)/2$, *where* $1 \leqslant r \leqslant m$.

(2) $\dim_{Z_2} I^{m-r+1}F/I^{m-r+2}F \geqslant r(r + 1)/2$, *where* $1 \leqslant r \leqslant m$.

(3) *The subgroup of the Brauer group of F generated by the quaternion algebras has Z_2-dimension no smaller than $m(m - 1)/2$.*

(4) *The cardinality of $W(F)$ is no smaller than $2 \cdot 2^{m(m+1)(m+2)/6}$.*

(*This lower bound is much more satisfactory than the existing bounds in the literature.*)

Proof. (1) To simplify the notation, we write $D_j = D_F(2^j\langle 1\rangle)/\dot{F}^2$. Thus, k_1F admits a filtration $0 = D_0 \subset D_1 \subset \cdots \subset D_m \subset D_{m+1} = k_1F$. By writing down -1 as a sum of 2^m squares and looking at the subsums of this expression [8, Satz 18(d)], it is easy to see that, for $j \leqslant m$, $\dim_{Z_2} D_j/D_{j-1} \geqslant m - j + 1$. Now, multiplication by $l(-1)^{m-r}$ defines a Z_2-linear map $f: k_1F \to k_{m-r+1}F$. By Lemma 3.6, ker (f) is precisely D_{m-r}. Consequently,

$$\begin{aligned}\dim_{Z_2} k_{m-r+1}F &\geqslant \dim_{Z_2} k_1F/D_{m-r} \geqslant \dim_{Z_2} D_m/D_{m-r}\\ &= \dim_{Z_2} D_m/D_{m-1} + \cdots + \dim_{Z_2} D_{m-r+1}/D_{m-r}\\ &\geqslant 1 + 2 + \cdots + r = r(r+1)/2.\end{aligned}$$

(2) The above argument actually shows that $\dim_{Z_2} l(-1)^{m-r}k_1F \geqslant r(r+1)/2$. By the Main Theorem, the restriction of s_{m-r+1} (see Theorem 1.7) to $l(-1)^{m-r}k_1F$ is a monomorphism from $l(-1)^{m-r}k_1F$ to $I^{m-r+1}F/I^{m-r+2}F$. (2) follows immediately from this observation.

(3) Let Q denote the subgroup of the Brauer group of F generated by quaternion algebras. By the universal property of k_2F, the rule $l(x)l(y) \mapsto (x, y/F) \in Q$ gives a homomorphism $g: k_2F \to Q$. By Theorem 1.8, the restriction of g to $l(-1)k_1F$ is a monomorphism. The proof of (1) shows that $\dim_{Z_2} l(-1)k_1F \geqslant m(m-1)/2$, so (3) follows immediately from this.

(4) We may assume that $W(F)$ is finite. Then,

$$\begin{aligned}\text{card } W(F) &\geqslant \text{card } \frac{W(F)}{I(F)} \cdot \text{card } \frac{I(F)}{I^2F} \cdot \cdots \cdot \text{card } \frac{I^mF}{I^{m+1}F}\\ &\geqslant 2 \cdot \prod_{r=1}^{m} 2^{r(r+1)/2} = 2 \cdot 2^e,\end{aligned}$$

where $e = \sum_{r=1}^{m} r(r+1)/2 = m(m+1)(m+2)/6$. This completes the proof.

Remark. Let d be the Z_2-dimension of the subgroup of the Brauer group of F generated by the quaternion algebras. If $d \leqslant 2$, then the lower bound $d \geqslant m(m-1)/2$ in (3) shows that the level of F is at most 4 (if finite). If in fact $d = 1$, one sees easily that $k_3F = 0$, and so $I^3F = 0$ (assuming finite level); further, as observed by Kaplansky, every five dimensional form over F is isotropic.

In Section 5, we shall study upper bounds on the Z_2-dimensions of k_nF, using different techniques.

4. Linkage of Pfister Forms

In this section we introduce the notion of "linkage" of Pfister forms, and establish an interesting relationship between linkage and the Witt index which, in particular, will yield a linkage criterion for pairs of Pfister forms. The main results are: Proposition 4.4, Theorems 4.5 and 4.8.

Definition 4.1. Let $\varphi_1, \ldots, \varphi_m$ be a set of n-fold Pfister forms, and r a nonnegative integer. We shall say that $\varphi_1, \ldots, \varphi_m$ are *r-linked* if there exist an r-fold Pfister form σ, and $(n-r)$-fold Pfister forms $\tau_1, \ldots, \tau_m$, such that $\varphi_i \cong \sigma\tau_i$, $1 \leqslant i \leqslant m$. We shall call such σ an *r-linkage* for $\varphi_1, \ldots, \varphi_m$. If, further, $\varphi_1, \ldots, \varphi_m$ are r-linked but not $(r+1)$-linked, then we write $r = i(\varphi_1, \ldots, \varphi_m)$ and call the integer r the *linkage number* of the set $\varphi_1, \ldots, \varphi_m$. If this linkage number $i(\varphi_1, \ldots, \varphi_m) \geqslant n-1$, then we shall simply say that $\varphi_1, \ldots, \varphi_m$ are *linked*. [See (2) of the following remark].

Remark 4.2. (1) By Theorem 1.4 and the remark of Theorem 2.7, it is easy to see that an r-fold Pfister form σ is an r-linkage for $\varphi_1, \ldots, \varphi_m$ if and only if $\varphi_i \in W(F) \cdot \sigma$ $(1 \leqslant i \leqslant m)$, if and only if each φ_i contains a subform isometric to σ.

(2) The linkage number of $\varphi_1, \ldots, \varphi_m$ is equal to n if and only if $\varphi_1, \ldots, \varphi_m$ are pairwise isometric.

Examples 4.3. (1) If F is a real-closed field, then any set of n-fold Pfister forms are linked. In fact, there are only two nonisometric n-fold Pfister forms: the hyperbolic form $2^{n-1}\mathbf{H}$ and the positive definite form $2^n\langle 1\rangle$.

(2) Let F be a global field, and $n \geqslant 3$. Then any set of n-fold Pfister forms over F are linked. In fact, $2^{n-1}\langle 1\rangle$ will be an $(n-1)$-linkage. This can be proved by using the Appendix of [4], or the simplified approach in [2]. See also Example 6.4(1) below.

We shall be primarily interested in the linkage number of pairs (or possibly triples) of n-fold Pfister forms. The following proposition computes the linkage number in terms of the Witt index, and provides the basic motivation of the notion of linkage.

Proposition 4.4. *Suppose φ and γ are n-fold Pfister forms, and r a nonnegative integer. Let $q = \varphi \perp \langle -1\rangle\gamma$. Then φ and γ are r-linked if and only if the Witt index of q is $\geqslant 2^r$. Further, if r is precisely the linkage number $i(\varphi, \gamma)$, then the Witt index of q is exactly 2^r.*

Proof. Step 1. Suppose φ and γ are r-linked, say $\varphi \cong \sigma\varphi_1$ and $\gamma \cong \sigma\gamma_1$. Then $q \cong \sigma(\langle 1\rangle \perp \varphi_1') \perp \langle -1\rangle\sigma(\langle 1\rangle \perp \gamma_1') \cong \sigma \cdot \mathbf{H} \perp \sigma(\varphi_1' \perp \langle -1\rangle\gamma_1')$. Since $\sigma \cdot \mathbf{H} \cong 2^r\mathbf{H}$, we see that the Witt index of q is $\geqslant 2^r$.

Step 2. Here we assume q contains at least 2^r copies of the hyperbolic plane, and wish to show that φ and γ are r-linked. We do this by induction on r. If $r = 0$, there is nothing to prove. We now assume that $r > 0$, and that we have settled the case for $r - 1$. In particular, φ and γ are $(r-1)$-linked. Let σ_0 be an $(r-1)$-linkage, and write $\varphi \simeq \sigma_0\varphi_1$, $\gamma \simeq \sigma_0\gamma_1$. As in Step 1, $q \simeq 2^{r-1}\mathbf{H} \perp (\sigma_0\varphi_1' \perp \langle -1\rangle\sigma_0\gamma_1')$. However, by hypothesis, q contains at least $2^r\mathbf{H}$, so $\sigma_0\varphi_1' \perp \langle -1\rangle\sigma_0\gamma_1'$ must be isotropic. In particular, there exists an element $c \in D_F(\sigma_0\varphi_1') \cap D_F(\sigma_0\gamma_1')$. Now apply Proposition 2.6 simultaneously to $\varphi \simeq \sigma_0\varphi_1$ and $\gamma \simeq \sigma_0\gamma_1$. We see right away that there exist $(n-r)$-Pfister forms φ_2 and γ_2 such that $\varphi \simeq \sigma_0\langle 1, c\rangle \cdot \varphi_2$ and $\gamma \simeq \sigma_0\langle 1, c\rangle \cdot \gamma_2$. Therefore $\sigma = \sigma_0\langle 1, c\rangle$ is the required r-linkage for φ and γ, with $\varphi \simeq \sigma\varphi_2$ and $\gamma \simeq \sigma\gamma_2$.

Step 3. Suppose the linkage number of φ and γ is precisely equal to r. Then by the argument in Step 2, the form $\sigma\varphi_2' \perp \langle -1\rangle\sigma\gamma_2'$ must be anisotropic! Therefore $q = 2^r\mathbf{H} \perp (\sigma\varphi_2' \perp \langle -1\rangle\sigma\gamma_2')$ is the Witt decomposition of q, showing that the Witt index of q is precisely 2^r.

THEOREM 4.5. *Let φ and γ be n-fold Pfister forms, and x, y be two nonzero elements of F. Then the Witt index of the form $\eta = \langle x\rangle\varphi \perp \langle y\rangle\gamma$ is either zero or equal to 2^r where $r = i(\varphi, \gamma)$.*

Proof. Assuming that η is isotropic, we must show that the Witt index of η is equal to 2^r.

Case 1. φ is isotropic. If γ also happens to be isotropic, then η is hyperbolic (Corollary 2.3), with Witt index 2^n. But in this case $i(\varphi, \gamma) = n$, and the theorem is proved. If γ is anisotropic, then η has Witt index 2^{n-1}, and clearly the linkage number $i(\varphi, \gamma) = n - 1$, also proving the theorem.

Case 2. We may assume now that φ and γ are both anisotropic. Since η is isotropic, there exists a nonzero vector $w = (u, v)$ such that $\eta(w) = x\varphi(u) + y\gamma(v) = 0$. Since one of the vectors u, v must be nonzero, one of $a = \varphi(u)$, $b = \gamma(v)$ and hence both a, b must be nonzero. Therefore, by Corollary 2.4, $\langle a\rangle\varphi \simeq \varphi$ and $\langle b\rangle\gamma \simeq \gamma$, so

$$\langle xa\rangle(\varphi \perp \langle -1\rangle\gamma) \simeq \langle xa\rangle\varphi \perp \langle yb\rangle\gamma \simeq \langle x\rangle\varphi \perp \langle y\rangle\gamma = \eta.$$

Now from Proposition 4.4, the Witt index of η equals 2^r, where $r = i(\varphi, \gamma)$. Q.E.D.

Remark. One might hope to generalize Theorem 4.5 to the case of more than two Pfister forms by saying that if an orthogonal sum $\langle x_1\rangle\varphi_1 \perp \cdots \perp \langle x_s\rangle\varphi_s$ is isotropic, then it has Witt index equal to a power of 2. Unfortunately this is not the case; for example, over the real field, if φ_1 is the positive definite

n-fold Pfister form and $\varphi_2, \varphi_3, \varphi_4$ are the hyperbolic n-fold Pfister forms, then the Witt index of $\varphi_1 \perp \varphi_2 \perp \varphi_3 \perp \varphi_4$ is $3 \cdot 2^{n-1}$. Similar counter-examples can also be constructed over global fields.

For a pair of Pfister forms whose orthogonal sum is isotropic, we have the following analog of Proposition 4.4.

PROPOSITION 4.6. *Suppose φ, γ are n-fold Pfister forms such that $\eta = \varphi \perp \gamma$ is isotropic and $i(\varphi, \gamma) = r \geqslant 2$. Then there exist an $(r-1)$-linkage σ and an element $z \in \dot{F}$ such that $\varphi \cong \sigma\langle 1, z\rangle\tau_1$, $\gamma \cong \sigma\langle 1, -z\rangle\tau_2$ where τ_1, τ_2 are $(n-r)$-fold Pfister forms.*

Proof. We prove this by induction on r. Since $r \geqslant 2$, by Theorem 4.5, $\varphi \perp \gamma$ contains $4\mathbf{H}$. By the cancellation theorem, $\varphi' \perp \gamma'$ is isotropic, so there exists $y \in \dot{F}$ such that $y \in D_F(\varphi')$ and $-y \in D_F(\gamma')$. By Proposition 2.2, $\varphi \cong \langle 1, y\rangle\mu_1$, $\gamma \cong \langle 1, -y\rangle\mu_2$ where μ_1, μ_2 are $(n-1)$-fold Pfister forms. We may proceed now to the inductive step directly because it clearly subsumes the arguments needed for $r = 2$. Therefore, we assume, as inductive hypothesis, that $\varphi \cong \sigma_0\langle 1, z\rangle\beta_1$, $\gamma \cong \sigma_0\langle 1, -z\rangle\beta_2$, where σ_0 is an $(r-2)$-linkage, $z \in \dot{F}$, and β_1, β_2 are $(n-(r-1))$-fold Pfister forms. We have

$$\begin{aligned}\eta = \varphi \perp \gamma &\cong \sigma_0\langle 1, z\rangle \perp \sigma_0\langle 1, -z\rangle \perp \sigma_0\langle 1, z\rangle\beta_1' \perp \sigma_0\langle 1, -z\rangle\beta_2' \\ &\cong 2\sigma_0 \perp 2^{r-2}\mathbf{H} \perp \sigma_0\langle 1, z\rangle\beta_1' \perp \sigma_0\langle 1, -z\rangle\beta_2'.\end{aligned}$$

Adding $2\langle -1\rangle\sigma_0$ to both sides, we get

$$\eta \perp 2\langle -1\rangle\sigma_0 \cong (2^{r-1} + 2^{r-2})\mathbf{H} \perp \sigma_0\langle 1, z\rangle\beta_1' \perp \sigma_0\langle 1, -z\rangle\beta_2'.$$

According to Theorem 4.5, η contains $2^r\mathbf{H}$. Since $r \geqslant 2$, we clearly have $2^r > 2^{r-1} + 2^{r-2}$. Therefore, again by the cancellation theorem,

$$\sigma_0\langle 1, z\rangle\beta_1' \perp \sigma_0\langle 1, -z\rangle\beta_2'$$

is isotropic, so we can find $w \in \dot{F}$ such that $w \in D_F(\sigma_0\langle 1, z\rangle\beta_1')$, and $-w \in D_F(\sigma_0\langle 1, -z\rangle\beta_2')$. By Proposition 2.6 we may write $\varphi \cong \sigma_0\langle 1, z\rangle\langle 1, w\rangle\tau_1$, $\gamma \cong \sigma_0\langle 1, -z\rangle\langle 1, -w\rangle\tau_2$, where τ_1, τ_2 are appropriate $(n-r)$-fold Pfister forms. But $\langle\langle z, w\rangle\rangle \cong \langle\langle z, zw\rangle\rangle$ by Corollary 1.9, and similarly $\langle\langle -z, -w\rangle\rangle \cong \langle\langle -z, wz\rangle\rangle$. Setting $\sigma = \sigma_0\langle\langle wz\rangle\rangle$, we have now $\varphi \cong \sigma\langle 1, z\rangle\tau_1$ and $\gamma \cong \sigma\langle 1, -z\rangle\tau_2$, as desired. Q.E.D.

Remark. In the above argument, we need the assumption $r \geqslant 2$ only insomuch as getting the induction started. One might hope to prove the proposition also for the case $r = 1$. However, the proposition is no longer true in this case. We shall construct a field F, two 2-fold Pfister forms φ, γ(1-linked) for which $\varphi \perp \gamma$ is isotropic, but $\varphi' \perp \gamma'$ is not isotropic. This will then give the required counter-example. To facilitate the construction, we first state a simple lemma.

LEMMA 4.7. *Let F_0 be a field, and q_1, q_2 be anisotropic forms over F_0. Let t be a transcendental element over F_0. Then $q_1 \perp \langle t\rangle q_2$ is an anisotropic form over $K = F_0(t)$.*

This can be easily proven by a simple degree argument analogous to the well-known proof of Springer's theorem on the Witt ring of a local field. (See, e.g., [9, Chap. 4]). Details are left to the reader.

To construct our counterexample, take any field F_0 of characteristic not 2, over which $\langle 1, 1\rangle$ and $\langle 1, -2\rangle$ are both anisotropic (e.g., take F_0 to be the field of rational numbers!). Let $F = F_0(t, s)$ where t, s are independent transcendental elements over F_0. By Lemma 4.7 (applied twice) together with Lemma 1.3, we see that the six-dimensional form

$$\langle 1, 1\rangle \perp \langle t\rangle\langle 1, -2\rangle \perp \langle s\rangle\langle 1, -2\rangle$$

is anisotropic over F. Set $\varphi = \langle\langle -2, t\rangle\rangle$, $\gamma = \langle\langle -2, s\rangle\rangle$ over F. Then

$$\begin{aligned}\varphi' \perp \gamma' &\cong \langle -2, -2, -2t, t, -2s, s\rangle \\ &\cong \langle -2\rangle[\langle 1, 1\rangle \perp \langle t\rangle\langle 1, -2\rangle \perp \langle s\rangle\langle 1, -2\rangle],\end{aligned}$$

and is anisotropic over F. But $\varphi \perp \gamma$ contains $\langle 1, 1, -2, -2\rangle \cong 2\mathbf{H}$. Therefore $\varphi \perp \gamma$ is isotropic, and $i(\varphi, \gamma)$ is equal to one.

Remark. In the statement of Proposition 4.6, if we require σ to be an r-linkage (instead of being an $(r-1)$-linkage), the Proposition will no longer be true, in general. In fact, if F contains a square root of -1, Proposition 4.4 says that we cannot find an r-linkage σ satisfying the conclusion of Proposition 4.6.

We shall now investigate linkage of three n-fold Pfister forms. The following result provides a sufficient condition for $\varphi_1, \varphi_2, \varphi_3$ to be linked.

THEOREM 4.8. *Let $\varphi_i = \langle\langle a_{i1}, \ldots, a_{in}\rangle\rangle$ $(i = 1, 2, 3)$ be three n-fold Pfister forms. Suppose $\varphi_1 + \varphi_2 + \varphi_3$ is an element of $I^{n+1}F$. Then $\varphi_1, \varphi_2, \varphi_3$ are linked (i.e., $i(\varphi_1, \varphi_2, \varphi_3) \geq n-1$). There exist an $(n-1)$-linkage σ, and $x, y \in \dot{F}$ such that*

$$\varphi_1 \cong \sigma\langle 1, -xy\rangle, \qquad \varphi_2 \cong \sigma\langle 1, x\rangle \qquad \text{and} \qquad \varphi_3 \cong \sigma\langle 1, y\rangle.$$

In particular, there exists an isometry

$$\langle -y\rangle\varphi_1 \perp \varphi_3 \cong \varphi_2 \perp 2^{n-1}\mathbf{H}.$$

Proof. We repeat once more the ascent construction in the proof of Theorem 1.4, and pass to the field $L = K(\sqrt{-\varphi_1'(X')})$. Since $L \otimes_F \varphi_1 = 0$ in $W(L)$, we get $L \otimes_F (\varphi_2 + \varphi_3) \in I^{n+1}L$. Theorem 3.2 implies that $L \otimes_F \varphi_2 \cong$

$L \otimes_F \varphi_3$ over L. Take a Witt decomposition of $q = \varphi_2 \perp \langle -1 \rangle \varphi_3$ over F, say, $q \cong q_a \perp q_h$, where q_a is anisotropic, and q_h is hyperbolic.

Case 1. q_a is the zero form. This means that $\varphi_2 \cong \varphi_3$ over F. The hypothesis $\varphi_1 + \varphi_2 + \varphi_3 \in I^{n+1}F$ then implies that $\varphi_1 \in I^{n+1}F$, and so by Hauptsatz 3.1, φ_1 is hyperbolic. The conclusion of Theorem 4.8 is completely trivial in this case.

Case 2. q_a is not the zero form. Since $L \otimes_F q_a = 0 \in W(L)$, the proof of Theorem 1.4 shows that there exist $r \in D_F(q_a)$ and a form τ_0 over F such that $q_a \cong \langle r \rangle \varphi_1 \perp \tau_0$. Hence $q = \varphi_2 \perp \langle -1 \rangle \varphi_3 \cong \langle r \rangle \varphi_1 \perp \tau$ where $\tau = \tau_0 \perp q_h$ has dimension 2^n. From the given hypothesis, we have $\varphi_2 + \langle -1 \rangle \varphi_3 \equiv \langle r \rangle \varphi_1$ (mod $I^{n+1}F$); thus $\tau \in I^{n+1}F$, and Hauptsatz 3.1 implies that $\tau \cong 2^{n-1}\mathbf{H}$. This means that the Witt index of $\varphi_2 \perp \langle -1 \rangle \varphi_3$ is at least 2^{n-1}, and according to Proposition 4.4, $i(\varphi_2, \varphi_3) \geqslant n - 1$. We may thus write $\varphi_2 \cong \sigma\langle 1, x \rangle$, and $\varphi_3 \cong \sigma\langle 1, y \rangle$, where σ is an $(n-1)$-linkage, and $x, y \in \dot{F}$. On the other hand, we have the equation

$$\varphi_2 - \varphi_3 = \sigma(\langle x \rangle - \langle y \rangle) = \langle -y \rangle \sigma \langle 1, -xy \rangle \in W(F).$$

By the implication (5) $\Rightarrow$ (4) in Main Theorem 3.2, we deduce that $\varphi_1 \cong \sigma\langle 1, -xy \rangle$. Consequently σ is an $(n-1)$-linkage for $\varphi_1, \varphi_2, \varphi_3$ and $i(\varphi_1, \varphi_2, \varphi_3) \geqslant n - 1$. The last statement in Theorem 4.6 now becomes obvious. Q.E.D.

We shall say that an extension field L_0 of F is a *splitting field* for a quadratic form φ if $\varphi_{L_0} = 0 \in W(L_0)$.

COROLLARY 4.9. *Retain the notations and hypotheses in Theorem* 4.8. *Then,*

(1) *There exists a quadratic extension L_1 of F which is a common splitting field for φ_1, φ_2 and φ_3.*

(2) *There exists also a field of algebraic functions L_2, quadratic extension of $F(y_2, y_3, ..., y_{2^n-1})$, which is a common splitting field for φ_1, φ_2 and φ_3.*

Proof. Let σ be the $(n-1)$-fold Pfister form which appeared in the conclusion of Theorem 4.8. For (1), we may take any quadratic extension $L_1 = F(\sqrt{-a})$, where $a \in D_F(\sigma')$. For (2), we take $L_2 = F(Y)(\sqrt{-\sigma'(Y)})$, where $Y = (y_2, y_3, ..., y_{2^n-1})$ is a set of $2^{n-1} - 1$ independent transcendental elements over F.

Remark 4.10. This corollary can be regarded as the high dimensional analog of the Zusatz in Section 4 of [8], due to Pfister. See also Theorem 6.1 below.

5. Structure of k_nF

In this section, we shall be interested in the structure of k_nF and the relationship between k_nF and $k_{n+1}F$. Recall that an element in k_nF of the form $l(x_1) \cdots l(x_n)$ is called a *generator* of k_nF. Our first result is the following.

PROPOSITION 5.1. *Suppose k_nF is spanned by r generators $\alpha_i = l(a_{i1}) \cdots l(a_{in})$, $1 \leqslant i \leqslant r$. Then $k_{n+1}F$ is spanned by $\beta_{ij} = l(a_{j1})\alpha_i$ $(1 \leqslant j \leqslant i \leqslant r)$; in particular, $\dim_{Z_2} k_{n+1}F \leqslant r(r+1)/2$. The quotient group $k_{n+1}F/l(-1)k_nF$ is spanned by β_{ij} $(1 \leqslant j < i \leqslant r)$.*

Proof. The group $k_{n+1}F$ is certainly generated by $l(x)\alpha_i$ where $x \in \dot{F}$. But we can expand $l(x)l(a_{i2}) \cdots l(a_{in}) = \sum_j \epsilon_j\alpha_j$ $(\epsilon_j \in Z_2)$. Thus, $l(x)\alpha_i = l(x)l(a_{i1}) \cdots l(a_{in}) = l(a_{i1}) \sum_j \epsilon_j\alpha_j = \sum_j \epsilon_j\beta_{ji}$. Therefore, $k_{n+1}F$ is spanned by β_{ij} $(1 \leqslant i, j \leqslant r)$. Consider now a generator β_{ji} where $j < i$. We have

$$\beta_{ji} = l(a_{i1})\, \alpha_j = l(a_{j1}) \sum_h \epsilon_h{}'\alpha_h = \sum_h \epsilon_h{}'\beta_{hj} \qquad (\epsilon_h{}' \in Z_2).$$

This says that β_{ji} $(j < i)$ is in the span of β_{hj}, $1 \leqslant h \leqslant r$. By an obvious induction, we see that β_{ji} $(j < i)$ actually lies in the span of β_{pq}, $1 \leqslant q \leqslant p \leqslant r$. The upper bound on $\dim_{Z_2} k_{n+1}F$ in the proposition follows immediately from this. Further, $\beta_{ii} = l(-1)\alpha_i$ by Lemma 1.5(1), so $k_{n+1}F/l(-1)k_nF$ is spanned by β_{pq}, $1 \leqslant q < p \leqslant r$. Q.E.D.

COROLLARY 5.2. *If k_nF is finite, then I^mF is a finitely generated abelian group for any $m \geqslant n$.*

Proof. Let $z_i = l(b_{i1}) \cdots l(b_{in})$ $(1 \leqslant i \leqslant s)$ be the finite set of *all* distinct generators in k_nF. Then, by the Main Theorem 3.2, an arbitrary n-fold Pfister form φ must be isometric to one of $\varphi_i = \langle\langle -b_{i1}, \ldots, -b_{in} \rangle\rangle$, $1 \leqslant i \leqslant s$. Consequently, φ_i $(1 \leqslant i \leqslant s)$ form a finite system of generators for the abelian group I^nF. By the Proposition, we know that, for any $m \geqslant n$, k_mF is finite, so I^mF are all finitely generated abelian groups, by the above argument.

The bound on $\dim_{Z_2} k_{n+1}F$ given by Proposition 5.1 is perhaps not the best possible result, since the spanning set α_i used in the argument there was just picked arbitrarily. If we choose this spanning set α_i more carefully, there can be clearly duplications among the β_{ij} $(1 \leqslant j \leqslant i \leqslant r)$. For example, if $a_{11} = a_{21}$ (i.e., α_1 and α_2 have the same leading factor), then $\beta_{i1} = \beta_{i2}$ for all i, and we are able to cut down the number of generators needed to span $k_{n+1}F$. To investigate this elimination procedure more systematically, we formulate the following notion of linkage of generators in k_nF, which is the natural analog of Definition 4.1.

DEFINITION 5.3. Let $z_i = l(b_{i1}) \cdots l(b_{in})$ $(1 \leqslant i \leqslant s)$ be a set of generators in k_nF, and r a non-negative integer. We say that the z_i's are r-linked if the corresponding Pfister forms $\varphi_i = \langle\langle -b_{i1}, ..., -b_{in}\rangle\rangle$ $(1 \leqslant i \leqslant s)$ are r-linked. We say that a generator $l(c_1) \cdots l(c_r) \in k_rF$ is an r-linkage for the z_i's if the r-fold Pfister form $\sigma = \langle\langle -c_1, ..., -c_r\rangle\rangle$ is an r-linkage for the φ_i's. By the Main Theorem 3.2, this is the case if and only if we can express each z_i in the form $z_i = l(c_1) \cdots l(c_r) l(d_{i,r+1}) \cdots l(d_{in})$. We define also the linkage number of the z_i's to be $i(z_1, ..., z_s) = i(\varphi_1, ..., \varphi_s)$. Finally, if $i(z_1, ..., z_s) \geqslant n-1$, we shall say that $z_1, ..., z_s$ are linked.

LEMMA 5.4. *Two generators z_1, z_2 in k_nF are linked if and only if each element in the span of z_1, z_2 is equal to a generator.*

Proof. The "only if" part is obvious. To prove the converse, express $z_1 + z_2$ as a generator z in k_nF. Let φ_1, φ_2, and φ be the n-fold Pfister forms corresponding to z_1, z_2, and z. Applying the homomorphism $s_n: k_nF \to I^nF/I^{n+1}F$, we obtain a congruence $\varphi_1 + \varphi_2 \equiv \varphi \pmod{I^{n+1}F}$. By Theorem 4.8, we conclude that $i(z_1, z_2) = i(\varphi_1, \varphi_2) \geqslant i(\varphi_1, \varphi_2, \varphi) \geqslant n-1$.

Q.E.D.

If H is a Z_2-subspace (= subgroup) in k_nF, the codimension of H is defined to be $\dim_{Z_2} k_nF/H$. We prove next:

PROPOSITION 5.5. *Let H be a Z_2-subspace in k_nF of codimension $\geqslant 2$. Then there exists a linked pair of generators which are Z_2-linearly independent in k_nF/H.*

Proof. For any integer s, $0 \leqslant s \leqslant n-1$, we claim that there exist two generators which are s-linked, and are Z_2-linearly independent in k_nF/H. (The case $s = n-1$ clearly implies the Proposition). We prove this by induction on s. There is nothing to prove when $s = 0$. For the inductive step, assume that we have found $\alpha = l(x_1) \cdots l(x_s) l(a_{s+1}) \cdots l(a_n)$ and $\beta = l(x_1) \cdots l(x_s) l(b_{s+1}) \cdots l(b_n)$ which are Z_2-linearly independent in k_nF/H, where $s < n-1$. Consider the two generators

$$\gamma = l(x_1) \cdots l(x_s) l(b_{s+1}) l(a_{s+2}) \cdots l(a_n),$$
$$\delta = l(x_1) \cdots l(x_s) l(a_{s+1}) l(b_{s+2}) \cdots l(b_n).$$

If $\gamma \notin H$, then either α, γ are Z_2-independent in k_nF/H, or else β, γ are Z_2-independent in k_nF/H. Since $i(\alpha, \gamma)$ and $i(\beta, \gamma)$ are both $\geqslant s+1$, we will have completed the inductive step. We may thus assume that $\gamma \in H$, and similarly we may assume also that $\delta \in H$. Now we have

$$\alpha \equiv \alpha + \gamma = l(x_1) \cdots l(x_s) l(a_{s+1} b_{s+1}) l(a_{s+2}) \cdots l(a_n) \pmod{H},$$
$$\beta \equiv \beta + \delta = l(x_1) \cdots l(x_s) l(a_{s+1} b_{s+1}) l(b_{s+2}) \cdots l(b_n) \pmod{H}.$$

Thus, $\alpha' = \alpha + \gamma$ and $\beta' = \beta + \delta$ are Z_2-independent in k_nF/H, and $i(\alpha', \beta') \geqslant s + 1$. This completes the proof of the Proposition.

Remark 5.6. Let H be an arbitrary subspace in k_nF. If the image of $l(c_1) \cdots l(c_m)k_{n-m}F$ in k_nF/H has Z_2-dimension $\geqslant 2$, the above argument shows also that there exist generators

$$l(c_1) \cdots l(c_m) \cdots l(c_{n-1})l(a) \qquad \text{and} \qquad l(c_1) \cdots l(c_m) \cdots l(c_{n-1})l(b)$$

which are Z_2-independent in k_nF/H.

COROLLARY 5.7. *If* $\dim_{Z_2} k_nF$ *is* $2r$ *or* $2r - 1$, *then each element in* k_nF *can be expressed as a sum of r generators.*

Proof. Assume that k_nF has Z_2-dimension $2r$. Applying the proposition step by step, we may construct a Z_2-basis for k_nF consisting of generators $\alpha_1, \beta_1, ..., \alpha_r, \beta_r$, where $i(\alpha_i, \beta_i) \geqslant n - 1$. The desired conclusion now follows from Lemma 5.4. The proof for the case $\dim_{Z_2} k_nF = 2r - 1$ is identical.

Our next objective is to give a sufficient condition for $\dim_{Z_2} k_{n+1}F$ to be $\leqslant \dim_{Z_2} k_nF$. We shall first prove the following technical lemma.

LEMMA 5.8. *Let* $H = \sum_j l(a_j)k_{n-1}F \subset k_nF$, *where* $a_j \in \dot{F}$, *and let* $s \geqslant 0$ *be an integer. Suppose a generator* $z = l(-1)^s l(c_1) \cdots l(c_{n+1-s})$ *does not belong to* $\sum_j l(a_j)k_nF + l(-1)^{s+1}k_{n-s}F$. *Let* $\gamma_i = l(-1)^s l(c_1) \cdots \widehat{l(c_i)} \cdots l(c_{n+1-s}) \in k_nF$, $1 \leqslant i \leqslant n + 1 - s$; *and, in case* $s \geqslant 1$, *let* $\gamma = l(-1)^{s-1}l(c_1) \cdots l(c_{n+1-s}) \in k_nF$. *Then,*

(1) γ_i $(1 \leqslant i \leqslant n + 1 - s)$ *are* Z_2*-independent in* k_nF/H.

(2) *In case* $s \geqslant 1$, γ *and* γ_i $(1 \leqslant i \leqslant n + 1 - s)$ *are* Z_2*-independent in* k_nF/H.

Proof. (1) Suppose there exists a Z_2-linear combination $\sum_i \epsilon_i\gamma_i \in H$ where $\epsilon_i \in Z_2$ are not all zero. Without loss of generality, we may suppose that $\epsilon_1 = 1$. Multiplying by $l(c_1)$, we obtain

$$\epsilon_1 l(-1)^s\, l(c_1) \cdots l(c_{n+1-s}) + l(-1) \sum_{i \geqslant 2} \epsilon_i\gamma_i \in \sum_j l(a_j)\, k_nF.$$

But the first summand is z, and the second summand belongs to $l(-1)^{s+1}k_{n-s}F$, contradicting the given hypothesis.

(2) Suppose $s \geqslant 1$, and suppose there exist $\epsilon_i \in Z_2$ such that $\gamma + \sum_i \epsilon_i\gamma_i \in H$. Multiplying by $l(-1)$, we obtain

$$z + l(-1) \sum_i \epsilon_i\gamma_i \in \sum_j l(a_j)k_nF,$$

which again contradicts the given hypothesis.

COROLLARY 5.9. *If H above has codimension* $\leqslant n$ *in* k_nF, *then* $k_{n+1}F = \sum_j l(a_j)k_nF + l(-1)^2k_{n-1}F$. (*In particular, if* $\dim_{Z_2} k_nF \leqslant n$, *then* $k_{n+1}F = l(-1)k_nF$.)

Proof. First we claim that $k_{n+1}F = \sum_j l(a_j)k_nF + l(-1)k_nF$. In fact, if this fails to hold, there will exist a generator $z = l(c_1) \cdots l(c_{n+1})$ not belonging to $\sum_j l(a_j)k_nF + l(-1)k_nF$. According to the Lemma (in the case $s = 0$), k_nF/H will have at least $n + 1$ Z_2-independent elements, contradicting the hypothesis that H has codimension $\leqslant n$. To establish the corollary, it therefore suffices to show that

$$l(-1)\,k_nF \subset \sum_j l(a_j)\,k_nF + l(-1)^2\,k_{n-1}F.$$

If this inclusion does not hold, there will exist a generator

$$l(-1)l(d_1) \cdots l(d_n) \notin \textstyle\sum_j l(a_j)k_nF + l(-1)^2k_{n-1}F.$$

According to the Lemma (in the case $s = 1$), k_nF/H will have at least $n + 1$ Z_2-independent elements, again contradicting our hypothesis. The proof of Corollary 5.9 is now complete.

THEOREM 5.10. *If there exists an element* $a \in \dot{F}$ *such that* $H = l(a)k_{n-1}F$ *has codimension* $r \leqslant n$ *in* k_nF, *then* $\dim_{Z_2} k_{n+t}F \leqslant \dim_{Z_2} k_nF$ *for any* $t \geqslant 0$.

Proof. We may assume that $\dim_{Z_2} k_nF = m$ is finite, for otherwise there is nothing to prove. Let $\alpha_1, \ldots, \alpha_{m-r}$ be a set of generators with leading term $l(a)$ which form a basis for $H = l(a)k_{n-1}F$. Choose $\beta_1, \ldots, \beta_s$ ($s \leqslant r$) to be generators with leading term $l(-1)$ such that α_i, β_j form a basis for $H + l(-1)k_{n-1}F$. Pick generators δ_p ($1 \leqslant p \leqslant r - s$) such that α_i, β_j, and δ_p form a basis for k_nF. Then

$$l(a)k_nF = l(a) \cdot \langle \alpha_i, \beta_j, \delta_p \rangle = \langle l(-1)\alpha_i, l(a)\beta_j, l(a)\,\delta_p \rangle,$$
$$l(-1)^2k_{n-1}F \subset l(-1)\langle \alpha_i, \beta_j \rangle = \langle l(-1)\alpha_i, l(-1)\beta_j \rangle.$$

(Here, the symbol $\langle\ \rangle$ denotes taking the Z_2-span). By Corollary 5.9, we obtain

$$k_{n+1}F = \langle l(-1)\alpha_i, l(-1)\beta_j, l(a)\beta_j, l(a)\,\delta_p \rangle.$$

Write $\beta_j = l(-1)l(b_{j2}) \cdots l(b_{jn})$. We have

$$l(a)\beta_j = l(-1)l(a)l(b_{j2}) \cdots l(b_{jn}) \in \langle l(-1)\alpha_i \rangle.$$

Thus $k_{n+1}F$ is spanned by $l(-1)\alpha_i$ ($1 \leqslant i \leqslant m - r$), $l(-1)\beta_j$ ($1 \leqslant j \leqslant s$) and $l(a)\,\delta_p$ ($1 \leqslant p \leqslant r - s$). Its Z_2-dimension is therefore at most

$(m - r) + s + (r - s) = m = \dim_{Z_2} k_n F$, and we have established the conclusion of the theorem for $t = 1$. But the codimension of $l(a)k_n F$ in $k_{n+1}F$ is $\leqslant s \leqslant r \leqslant n < n + 1$, so we are done by induction.

COROLLARY 5.11. *Retain the hypothesis in Theorem* 5.10. *Then, either* $k_{n+1}F = l(-1)k_n F$, *or else* $\dim_{Z_2} k_{n+1}F < m$.

Proof. If the generators $l(-1)\alpha_i$ are Z_2-linearly dependent, then the argument in the above proof shows that $\dim_{Z_2} k_{n+1}F$ is less than m. We may now assume that $l(-1)\alpha_i$ are Z_2-linearly independent in $k_{n+1}F$. Write $\alpha_i = l(a)l(a_{i2}) \cdots l(a_{in})$, $1 \leqslant i \leqslant m - r$. Then $\alpha_i' = l(-1)l(a_{i2}) \cdots l(a_{in})$ must be also Z_2-linearly independent in $k_n F$. Therefore $H' = l(-1)k_{n-1}F$ has codimension $\leqslant r \leqslant n$. It follows from Corollary 5.9 that $k_{n+1}F = l(-1)k_n F$.

THEOREM 5.12. *For a given integer* $n > 1$, *suppose that* $\dim_{Z_2} k_n F = m \leqslant 6$. *Then* $\dim_{Z_2} k_{n+i}F \leqslant m$, *for all* $i \geqslant 0$.

Proof. It is clearly sufficient to prove this for $i = 1$. (1) Assume that $m \leqslant 4$. By Proposition 5.5, we may find $a \in \dot{F}$ such that $l(a)k_{n-1}F$ has codimension in $k_n F$ not greater than 2. In this case, the desired conclusion follows from Theorem 5.10.

(2) $m = 5$. If $n \geqslant 3$, the same argument as used in (1) applies, and we are done. We now handle the case when $k_2 F$ has $2^5 = 32$ elements. We may suppose that $\dim_{Z_2} l(z)k_1 F \leqslant 2$, for any $z \in \dot{F}$, because otherwise we could argue again as in (1). By Proposition 5.5, there exists a basis of $k_2 F$ consisting of $\alpha_1 = l(x)l(a_1)$, $\beta_1 = l(x)l(b_1)$, $\alpha_2 = l(y)l(a_2)$, $\beta_2 = l(y)l(b_2)$, and $\gamma = l(u)l(v)$. We have $H_1 = \langle \alpha_1, \beta_1 \rangle = l(x)k_1 F$ and $H_2 = \langle \alpha_2, \beta_2 \rangle = l(y)k_1 F$. In particular, $l(x)l(y) \in H_1 \cap H_2 = 0$. Using this fact, and Proposition 5.1, we obtain the following spanning set for $k_3 F$: $S = \{l(-1)\alpha_1, l(-1)\beta_1, l(-1)\alpha_2, l(-1)\beta_2, l(x)\gamma, l(y)\gamma, l(-1)\gamma\}$. Consider $s = \dim_{Z_2} l(-1)k_1 F \leqslant 2$. If $s = 0$, $k_3 F$ is spanned by two generators. If $s = 1$, then $l(-1)k_2 F$ has at most one nonzero element, so $k_3 F$ is spanned by three generators. We may thus assume that $s = 2$. If so, we may take x above to be -1. Since $l(-1)l(y) = 0$, the spanning set S constructed above simplifies to $\{l(-1)\alpha_1, l(-1)\beta_1, l(y)\gamma, l(-1)\gamma\}$ and so $\dim_{Z_2} k_3 F \leqslant 4$ in this case.

(3) $m = 6$. As before, we need handle only the cases when $k_2 F$ or $k_3 F$ has 64 elements. The necessary arguments are analogous to those used in (2), and will be omitted.

At this point, it seems natural to ask whether it is true in general, that $\dim_{Z_2} k_{n+1}F \leqslant \dim_{Z_2} k_n F$ $(n > 1)$, say, in case the latter is finite. At present, we are not able to answer this question, though we suspect that a counterexample might exist when $n = 2$.* We have proved, however, a "stable version" of this fact, with a very satisfactory bound on the "stable range". This is the content of Theorem 5.13 below, for which we need some notations.

For $a \in \dot{F}$, we shall write $P_r(a) = l(a)k_{r-1}F \subset k_rF$. The special subspace $P_r(-1)$ will be denoted simply by P_r. We shall write $\bar{P}_r(a) = (P_r(a) + P_r)/P_r$. For a nonnegative integer i, $\binom{r}{i}$ will denote the usual binomial coefficient, with the convention that $\binom{r}{i} = 0$ for $i > r$.

THEOREM 5.13. *Let $n > 1$ be a fixed integer.* (1) *For $a \in \dot{F}$, suppose $\dim_{Z_2} \bar{P}_n(a) = m$ is finite. Write $m = t(n-1) + s$, where $0 \leqslant s < n - 1$. Then, for any $j \geqslant 0$,*

$$\dim_{Z_2} \bar{P}_{n+j}(a) \leqslant (n-1) \cdot \binom{t}{j+1} + s \cdot \binom{t}{j}.$$

(2) *Suppose the set of integers $\{\dim_{Z_2} \bar{P}_n(a) \colon a \in \dot{F}\}$ has a maximum m_0. Write $m_0 = t_0(n-1) + s_0$, where $0 \leqslant s_0 < n - 1$. Then, for $i \geqslant 0$, we have $k_{n+t_0+i}F = l(-1)^i k_{n+t_0}F$. In particular, $\dim_{Z_2} k_{n+t_0+i}F \leqslant \dim_{Z_2} k_{n+t_0}F$ for all $i \geqslant 0$.*

Proof. (1) Suppose there exists a generator $l(a)l(x_{11}) \cdots l(x_{1n}) \notin P_{n+1}$. (If this does not exist, proceed directly to the last step of the present construction.) Let $\alpha_{1i} = l(a) \cdots \widehat{l(x_{1i})} \cdots l(x_{1n}) \in P_n(a)$. This notation is meaningful for $1 \leqslant i \leqslant n$, but we shall need only the set

$$S_1 = \{\alpha_{1i} \colon 2 \leqslant i \leqslant n\} \subset l(a)l(x_{11})k_{n-2}F.$$

By Lemma 5.8, the $(n-1)$ elements in S_1 are Z_2-independent in the quotient space $\bar{P}_n(a)$ (so in this case $t \geqslant 1$). Assume, inductively, that we have defined S_1 (as above), $S_2 = \{\alpha_{2i} \colon 2 \leqslant i \leqslant n\}, \ldots,$ and $S_h = \{\alpha_{hi} \colon 2 \leqslant i \leqslant n\}$. Suppose there exists a generator

$$l(a)l(x_{h+1,1}) \cdots l(x_{h+1,n}) \notin P_{n+1} + \sum_{j=1}^{h} l(a)l(x_{j1})k_{n-1}F.$$

(If no such generator exists, proceed to the last step.) Let $\alpha_{h+1,i} = l(a) \cdots \widehat{l(x_{h+1,i})} \cdots l(x_{h+1,n}) \in P_n(a)$. Then, again by (the proof of) Lemma 5.8, these generators are Z_2-independent modulo $\langle P_n, S_1, \ldots, S_h \rangle$. We then define $S_{h+1} = \{\alpha_{h+1,i} \colon 2 \leqslant i \leqslant n\} \subset l(a)l(x_{h+1,1})k_{n-2}F$. Since $\dim_{Z_2} \bar{P}_n(a) = t(n-1) + s$ is finite, the above construction cannot proceed indefinitely.

* *Added in proof.* The power series field $F = \mathbf{R}((t_1))\,((t_2))\,((t_3))$ provides a counterexample. In fact, $\dim_{Z_2} k_2F = 7$, and $\dim_{Z_2} k_3F = 8$. This shows also that Theorem 5.12 is best possible.

Therefore, there exists an integer $d \leqslant t$ such that, after we obtain S_d, we get

$$(*)\qquad P_{n+1}(a) \subset P_{n+1} + \sum_{j=1}^{d} l(a)\, l(x_{j1})\, k_{n-1}F = P_{n+1} + \sum_{j=1}^{d} l(x_{j1})\, P_n(a).$$

Now, expand $S_1 \cup \cdots \cup S_d$ to a basis for $\bar{P}_n(a)$ by adding arbitrary sets of generators $S_{d+1} = \{\alpha_{d+1,i}: 2 \leqslant i \leqslant n-1\},\ldots, S_t = \{\alpha_{ti}: 2 \leqslant i \leqslant n\}$, and $B = \{\beta_e: 1 \leqslant e \leqslant s\}$ (recall $m = t(n-1) + s$). By Proposition 5.1 and $(*)$ above, $\bar{P}_{n+1}(a)$ is spanned by

$$M_d^{(n+1)} = \{l(x_{h1})\, \alpha_{pi} : 2 \leqslant i \leqslant n,\ 1 \leqslant h < p \leqslant d\} \cup \{l(x_{h1})\, \beta_e : 1 \leqslant h \leqslant d,\ 1 \leqslant e \leqslant s\}.$$

Now we set $\alpha_{h2} = l(a)l(x_{h1})\ldots$, for any h such that $d < h \leqslant t$. This device is introduced only for notational purposes, in order to make the symbol x_{h1} meaningful for *all* h, $1 \leqslant h \leqslant t$. Now, $M_d^{(n+1)}$ is a subset of

$$M_t^{(n+1)} = \{l(x_{h1})\, \alpha_{pi} : 2 \leqslant i \leqslant n,\ 1 \leqslant h < p \leqslant t\} \cup \{l(x_{h1})\, \beta_e : 1 \leqslant h \leqslant t,\ 1 \leqslant e \leqslant s\},$$

which has cardinality $(n-1) \cdot \binom{t}{2} + st$. We have thus established (1) for $j = 1$. For the subsequent groups $\bar{P}_{n+j}(a)$, we claim that the following set

$$M_d^{(n+j)} = \{l(x_{h_1,1}) \cdots l(x_{h_j,1})\, \alpha_{pi} : 2 \leqslant i \leqslant n,\ 1 \leqslant h_1 < \cdots < h_j < p \leqslant d\} \cup \{l(x_{h_1,1}) \cdots l(x_{h_j,1})\, \beta_e : 1 \leqslant h_1 < \cdots < h_j \leqslant d,\ 1 \leqslant e \leqslant s\}$$

spans $\bar{P}_{n+j}(a)$. If we show this, then the proof of (1) is complete, since

$$\dim_{Z_2} \bar{P}_{n+j}(a) \leqslant \operatorname{card} M_d^{(n+j)} \leqslant \operatorname{card} M_t^{(n+j)} = (n-1)\binom{t}{j+1} + s\binom{t}{j},$$

where $M_t^{(n+j)}$ is analogous to $M_d^{(n+j)}$, with t replacing d. We shall now prove our claim by induction on j. We may assume, inductively, that $M_d^{(n+j-1)}$ spans $\bar{P}_{n+j-1}(a)$. By Proposition 5.1 and $(*)$, we know that $\bar{P}_{n+j}(a)$ is spanned by the set

$$M = \{l(x_{h_1,1}) \cdots l(x_{h_j,1})\, \alpha_{pi} : 2 \leqslant i \leqslant n,\ 1 \leqslant h_1 < \cdots < h_j \leqslant d,\ 1 \leqslant p \leqslant d,\ p \text{ different from } h_1,\ldots, h_j\} \cup \{l(x_{h_1,1}) \cdots l(x_{h_j,1})\, \beta_e : 1 \leqslant h_1 < \cdots < h_j \leqslant d,\ 1 \leqslant e \leqslant s\}.$$

It suffices to show that any $\gamma = l(x_{h_1,1}) \cdots l(x_{h_j,1})\alpha_{pi}$ in M above is caught in $\langle M_d^{(n+j)}\rangle$ (the span of $M_d^{(n+j)}$ formed in $\bar{P}_{n+j}(a)$). Suppose this is false, Choose a counterexample γ with h_1 minimal. By inductive hypothesis,

$\sigma = l(x_{h_2,1}) \cdots l(x_{h_j,1})\alpha_{pi}$ can be expressed as a Z_2-combination of the generators in $M_d^{(n+j-1)}$. By the choice of γ, we know that $l(x_{h_1,1}) \cdot \sigma = \gamma \notin \langle M_d^{(n+j)} \rangle$. Hence, σ has a summand $\sigma_0 = l(x_{q_2,1}) \cdots l(x_{q_j,1})\alpha_{uv}$, $(1 \leqslant q_2 < \cdots < q_j < u \leqslant d, 2 \leqslant v \leqslant n)$, such that $l(x_{h_1,1}) \cdot \sigma_0 \notin \langle M_d^{(n+j)} \rangle$. Therefore, h_1 is automatically different from all of $q_2, \ldots, q_j$ and u. Also $q_2 < h_1$ (lest $l(x_{h_1,1}) \cdot \sigma_0 \in M_d^{(n+j)}$), but this contradicts the minimality of h_1. The claim is now established, and the proof of (1) is complete.

(2) For any $a \in \dot{F}$ as in (1), we have $m \leqslant m_0$, and so $t \leqslant t_0$. Therefore, by (1), $\bar{P}_{n+t_0+1}(a) = 0$, i.e., $P_{n+t_0+1}(a) \subset P_{n+t_0+1}$. Since this holds for all a, we conclude that $k_{n+t_0+1}F = l(-1)k_{n+t_0}F$, and all other statements in (2) follow immediately from this. Q.E.D.

Remark. By similar techniques, one may also obtain a bound on $\dim_{Z_2} k_{n+j}F$ in terms of certain invariants of k_nF, in case k_nF is finite. We suppress the gory details in the interest of sanity.

COROLLARY 5.14. *Retain the hypothesis and notations in* (2) *of the Theorem. Then* $I^{n+t_0+i}F = 2^i \cdot I^{n+t_0}F$ *for all* $i \geqslant 0$.

Proof. This is an immediate consequence of the Theorem just proved, and the Main Theorem 3.2.

6. APPLICATIONS TO MILNOR'S CONJECTURE

We are now in a position to apply the results of the preceding sections to study Milnor's epimorphism (Theorem 1.7) $s_n: k_nF \to I^nF/I^{n+1}F$. In various special cases, we will be able to show that s_n are isomorphisms. This section consists of Theorem 6.1, and five of its corollaries.

Let $B(F)$ denote the Brauer group of F, and $GB(F)$ denote the graded Brauer group of F (see p. 102 in [9]). Let $f: W(F) \to GB(F)$ be the group homomorphism given by what Scharlau calls [9] the Clifford–Minkowski–Hasse–Witt–Wall–Delzant–Stieffel–Whitney-invariant. Since $f(I^3F) = 0$, f factors through $W(F)/I^3F \to GB(F)$, which we denote by $\bar{f}$. In [8], Pfister has studied the question

*Q*1. Is $\bar{f}: W(F)/I^3F \to GB(F)$ a monomorphism? (It is known that $\ker \bar{f} \subset I^2F/I^3F$.)

On the other hand, by the universal property of k_2F, it is easily seen that the rule $l(x)l(y) \mapsto (x, y/F) \in B(F)$ defines a group homomorphism $g: k_2F \to B(F)$ [5, 189]. In [4] and [5], Milnor has considered the question

*Q*2. Is $g: k_2F \to B(F)$ a monomorphism?

Now, by an easy calculation, it can be shown that the following diagram is commutative:

$$\begin{array}{ccc} k_2F & \xrightarrow{g} & B(F) \\ {\scriptstyle s_2}\downarrow & & \downarrow{\scriptstyle i} \\ I^2F/I^3F & \xrightarrow{\bar{f}} & GB(F) \end{array} \qquad (i = \text{inclusion})$$

Since s_2 is an isomorphism by Theorem 1.7, it follows immediately that $\bar{f}$ is a monomorphism if and only if g is a monomorphism. In other words, $Q1$ and $Q2$ are equivalent. We shall now give a sufficient condition for $\bar{f}, g$ to be monomorphisms, and for s_n to be an isomorphism.

THEOREM 6.1. (1) *For a given integer n, suppose that every element of k_nF can be expressed as the sum of at most three generators. Then s_n is an isomorphism.*

(2) *If every element of k_2F can be expressed as a sum of three generators, then $\bar{f}, g$ are monomorphisms. (In particular, if a form σ belongs to I^2F, then the Clifford algebra of σ splits over F if and only if σ actually belongs to I^3F. Compare Satz* 14 *and its Zusatz in* [8].)

Proof. (1) Suppose $s_n(z) = 0 \in I^nF/I^{n+1}F$, where $z \in k_nF$. By hypothesis, we may express

$$z = \sum_{i=1}^{3} l(-a_{i1}) \cdots l(-a_{in}).$$

Let φ_i denote the n-fold Pfister form $\langle\langle a_{i1}, \ldots, a_{in}\rangle\rangle$. Then we have

$$s_n(z) = \varphi_1 + \varphi_2 + \varphi_3 \equiv 0 \ (\mathrm{mod}\, I^{n+1}F).$$

By Theorem 4.8, there exists an $(n-1)$-linkage $\sigma = \langle\langle b_2, \ldots, b_n\rangle\rangle$ such that $\varphi_1 \cong \sigma\langle 1, -xy\rangle$, $\varphi_2 \cong \sigma\langle 1, x\rangle$ and $\varphi_3 \cong \sigma\langle 1, y\rangle$, for suitable elements $x, y \in \dot{F}$. According to the Main Theorem 3.2, these isometries imply that

$$\begin{aligned} l(-a_{11}) \cdots l(-a_{1n}) &= l(xy)l(-b_2) \cdots l(-b_n), \\ l(-a_{21}) \cdots l(-a_{2n}) &= l(-x)l(-b_2) \cdots l(-b_n), \\ \text{and} \quad l(-a_{31}) \cdots l(-a_{3n}) &= l(-y)l(-b_2) \cdots l(-b_n) \text{ in } k_nF. \end{aligned}$$

Adding these equations together, we obtain $z = 0 \in k_nF$, since $l(-x) + l(-y) = l(xy) = -l(xy)$ in k_1F. This proves that s_n is an isomorphism.

(2) For $z \in k_2F$, suppose $g(z) = 1$ (the identity element of $B(F)$, in multiplicative notation). As in (1) above, we express $z = \sum_{i=1}^{3} l(-a_{i1})l(-a_{i2})$, and set $\varphi_i = \langle\langle a_{i1}, a_{i2}\rangle\rangle$. We repeat once more the ascent construction in the

proof of Theorem 1.4, and pass to the field $L = K(\sqrt{-\varphi_1'(X')})$ (here, $X' = (x_2, x_3, x_4)$, and $K = F(x_2, x_3, x_4)$). Since $L \otimes_F \varphi_1$ is hyperbolic over L, we have $(-a_{11}, -a_{12}/L) = 1 \in B(L)$, and hence $(-a_{21}, -a_{22}/L) \cong (-a_{31}, -a_{32}/L)$. By Theorem 1.8, we conclude that $L \otimes_F \varphi_2 \cong L \otimes_F \varphi_3$. Let 2^r be the Witt index of the form $q = \varphi_2 \perp \langle -1 \rangle \varphi_3$ over F. We claim that $r \neq 0$. Indeed if $r = 0$, the anisotropic part q_a of q is six dimensional, whereas $L \otimes_F (q_a) = 0 \in W(L)$ implies (by the argument of Theorem 1.4) that $\dim q_a$ is a multiple of $\dim \varphi_1 = 4$. Hence $r \geqslant 1$, and by Proposition 4.4, φ_2 and φ_3 are linked over F. Using Theorem 1.8 we may write $l(-a_{21})l(-a_{22}) = l(b)l(x)$, and $l(-a_{31})l(-a_{32}) = l(b)l(y)$, where $b, x, y \in \dot{F}$. Now

$$z = l(b)l(xy) + l(-a_{11})l(-a_{12}),$$

and $g(z) = 1 \in B(F)$ implies that $z = 0 \in k_2F$ by Theorem 1.8. Q.E.D.

COROLLARY 6.2. *For a given integer $n > 1$, suppose that* $\dim_{Z_2} k_nF = m \leqslant 6$. *Then s_{n+i} is an isomorphism for all $i \geqslant 0$. If $n = 2$, then f and g are monomorphisms.*

Proof. By Corollary 5.7, each element in k_nF can be expressed as the sum of three generators. It follows from the above Theorem that s_n is an isomorphism. The corresponding statement for s_{n+i} $(i > 0)$ now follows from Theorem 5.12. Q.E.D.

Remark. If F has at most 16 square classes, then $\dim_{Z_2} k_2F \leqslant 7$, and it can be shown that Theorem 6.1(2) applies. In general, if $\dim_{Z_2} k_2F \leqslant 7$, it can also be shown that s_m are all isomorphisms.

We shall now investigate some special cases of Theorem 6.1.

COROLLARY 6.3. *For a given integer $n > 1$, suppose that every element of k_nF is equal to a generator. Then every element of k_rF $(r \geqslant n)$ is also equal to a generator, and every pair of r-fold Pfister forms $(r \geqslant n)$ are linked. In particular, s_r is an isomorphism for all $r \geqslant n$.*

Proof. It suffices to show that the sum of any two generators in $k_{n+1}F$ is equal to a generator, for then everything else will follow by induction. Let α, β be two generators in $k_{n+1}F$. By Lemma 5.4, there exists $a \in \dot{F}$ such that α, β both belong to $l(a)k_nF$. Thus, $\alpha + \beta$ is clearly a generator in $k_{n+1}F$.

EXAMPLE 6.4. The hypothesis of Corollary 6.3 holds in the following two cases. (1) F is a global field, and $n \geqslant 2$. (See [9], p. 135). (2) $\dim_{Z_2} k_nF \leqslant 2$. (See Proposition 5.5).

COROLLARY 6.5. *Suppose every 2^n-dimensional form over F is universal. Then*

(1) $I^{n+1}F = 0$ *and hence* $k_rF = 0$ *for all* $r \geqslant n+1$.

(2) s_n *is an isomorphism.*

(3) *Every element of* I^nF *is represented by an n-fold Pfister form, and every pair of n-fold Pfister forms are linked.*

Proof. From the hypothesis, it follows that every $(2^n + 1)$-dimensional form is isotropic, so every $(n+1)$-fold Pfister form is hyperbolic. This fact, together with Corollary 3.3, imply (1) above. Let φ, γ be two n-fold Pfister forms. Since the anisotropic part of $q = \varphi \perp \langle -1 \rangle \gamma$ is at most 2^n-dimensional, the Witt index of q must be $\geqslant 2^{n-1}$. It follows from Proposition 4.4 that φ and γ are linked, establishing the last statement of (3). The first part of (3) can be deduced easily from Korollar 3 of [1]. But here is an ad hoc proof. Observe that for φ above, and any $a \in \dot{F}$, we have $\langle a \rangle \varphi \cong \varphi$ by Corollary 2.4. It is thus sufficient to prove that, for given φ and γ as above, $\varphi + \gamma$ is equal to another n-fold Pfister form, in $W(F)$. But we have shown that φ, γ are linked, so $\varphi \cong \sigma\langle 1, x \rangle$, $\gamma \cong \sigma\langle 1, y \rangle$. Now $\varphi + \gamma = \varphi - \gamma = \sigma(\langle x \rangle - \langle y \rangle) = \sigma\langle 1, -xy \rangle \in W(F)$, proving the first statement of (3). Finally, from (3) and Lemma 5.4, it follows that every element of k_nF is equal to a generator. Therefore, Corollary 6.3 implies (2). Q.E.D.

COROLLARY 6.6. *If F is a C_3-field and $K = F(t)$ is purely transcendental of degree one over F, then s_r is an isomorphism for the fields F and K, for all r. (Example: $F = \mathbf{C}(X_1, X_2, X_3)$).*

Proof. By hypothesis, every 8-dimensional quadratic form is universal, so Corollary 6.5 applies for $n = 3$. Recalling Theorem 1.7, we conclude that s_r is an isomorphism for F, and for all r. Now any finite extension of F is again a C_3-field. It follows from Corollary 5.8 of [4] that s_r is also an isomorphism for K, for all r. Q.E.D.

REFERENCES

1. J. K. ARASON AND A. PFISTER, Beweis des Krullschen Durchschnittsatzes für den Wittring, to appear.
2. R. ELMAN AND T. Y. LAM, Determination of k_n ($n \geqslant 3$) for global fields, *Proc. Amer. Math. Soc.*, to appear.
3. D. HARRISON, "University of Kentucky Lectures," 1970.
4. J. MILNOR, Algebraic K-theory and quadratic forms, *Invent. Math.* **9** (1970), 318–344.
5. J. MILNOR, Algebraic K-theory, mimeographed notes, M.I.T., 1970.

6. O. T. O'MEARA, "Introduction to Quadratic Forms," Springer-Verlag, Berlin, 1963.
7. A. PFISTER, Multiplikative quadratische Formen, *Arch. Math.* **16** (1965), 363–370.
8. A. PFISTER, Quadratische Formen in beliebigen Körpern, *Invent. Math.* **1** (1966), 116–132.
9. W. SCHARLAU, "Quadratic Forms," Queen's Papers in Pure and Applied Mathematics, No. 22, Queen's University, Kingston, Ontario, 1969.
10. E. WITT, Theorie der quadratischen Formen in beliebigen Körpern, *J. Reine Angew. Math.* **176** (1937), 31–44.