ORIGINAL ARTICLE

On some first-order differential subordination

M. Nunokawa a, E. Yavuz Duman b, J. Sokół c,*, N.E. Cho d, S. Owa e

a University of Gunma, Hoshikuki-cho 798-8, Chuow-Ward, Chiba 260-0808, Japan
b Department of Mathematics and Computer Science, Istanbul Kültür University, İstanbul, Turkey
c Department of Mathematics, Rzeszów University of Technology, Al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland
d Department of Applied Mathematics, College of Natural Sciences, Pukyong National University, Pusan 608-737, Republic of Korea
e Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan

Received 2 July 2013; accepted 29 July 2013
Available online 15 September 2013

Abstract Let $A$ denote the class of functions $f$ that are analytic in the unit disc $D$ and normalized by $f(0) = f'(0) - 1 = 0$. In this paper, we investigate the class of functions such that $\Re\{f'(z) + zf''(z) - \beta\} > \alpha$ in $D$. We determine conditions for $\alpha$ and $\beta$ under which the function $f$ is univalent, close-to-convex, and convex. To obtain this, we first estimate $|\arg(f'(z))|$ which improves the earlier results.

KEYWORDS
Close-to-convex functions;
Convex functions;
Starlike functions;
Subordination

1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$, and denote by $\mathcal{A}$ the class of analytic functions in $D$ and usually normalized, i.e., $\mathcal{A} = \{f \in \mathcal{H} : f(0) = 0, f'(0) = 1\}$. We say that the $f \in \mathcal{H}$ is subordinate to $g \in \mathcal{H}$ in the unit disc $D$, written $f \prec g$ if and only if there exists an analytic function $w \in \mathcal{H}$ such that $|w(z)| \leq |z|$ and $f(z) = g[w(z)]$ for $z \in D$. Therefore, $f \prec g$ in $D$ implies $f(D) \subseteq g(D)$. In particular, if $g$ is univalent in $D$ then the Subordination Principle says that $f \prec g$ if and only if $f(0) = g(0)$ and $f([0,1]) \subseteq g([0,1])$, for all $r \in (0,1)$.

The class $S'_\alpha$ of starlike functions of order $\alpha < 1$ may be defined as
$$S'_\alpha := \{f \in \mathcal{A} : \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \ z \in D\}.$$ 

The class $S'_\alpha$ and the class $K_\alpha$ of convex functions of order $\alpha < 1$
$$K_\alpha := \{f \in \mathcal{A} : \Re \left( 1 + \frac{zf'(z)}{f(z)} \right) > \alpha, z \in D\}$$
were introduced by Robertson in [11]. If $\alpha \in [0,1)$, then a function in either of these sets is univalent. In particular, we denote $S'_\alpha = S', K_0 = K$, the classes of starlike and convex functions, respectively. Recall that $f \in A$ is said to be in the class $C(\beta)$,
of close-to-convex functions of order $\beta$ and type $z$, $0 \leq \beta < 1$, if and only if there exist $g \in K_z, \phi \in \mathbb{R}$, such that
\begin{equation}
\Re\left\{e^{i\phi}f(z)\right\} > \beta, \quad z \in \mathbb{D}.
\end{equation}
Functions defined by (1.1) with $\phi = 0$ were considered earlier by Ozaki [10], see also Umezawa [12,13]. Moreover, Lewandowski [6,9] defined the class of functions $f \in A$ for which the complement of $f(D)$ with respect to the complex plane is a linearly accessible domain in the large sense. The Lewandowski’s class is identical with the Kaplan’s class $C_0(0)$.

2. Main result

**Theorem 2.1.** Let $f(z) = z + \sum_{n=2}^{\infty}a_nz^n$ be analytic in the unit disc $\mathbb{D}$. If
\begin{equation}
f'(z) \neq 0, \quad f'(z) + zf'(z) \neq 0, \quad z \in \mathbb{D}
\end{equation}
and
\begin{equation}
\Re\{f'(z) + zf'(z)\} > \beta, \quad z \in \mathbb{D},
\end{equation}
then
\begin{equation}
|\arg\{f(z)\}| \leq \frac{\pi - \beta}{\beta} \log\{2(1 - \beta)\} \quad \beta \in (-\infty, 1/2) \cup (1/2, 1),
\end{equation}
and $f(z)$ is close-to-convex in $\mathbb{D}$ whenever $\beta > \beta_0$.

Moreover, $f$ is close-to-convex in $\mathbb{D}$ whenever $\beta > \beta_0$, where $-1.47 < \beta_0 < -1.46$ is the positive solution of the equation
\begin{equation}
\log\{2(1 - \beta)\} = 1 - 2\beta.
\end{equation}

**Proof.** Note that the assumptions (2.1) are necessary for $\beta < 0$ only. If $\beta \in (0, 1)$, then from (2.2) we have even more $\Re\{f'(z) + zf'(z)\} > 0$. Moreover, from (2.2) we have also that $\Re\{f(z)\} > 0$ so $z^f$ is univalent in $\mathbb{D}$ and $f(z) \neq 0$.

From the hypothesis (2.2), we have
\begin{equation}
f'(z) = \frac{f'(z) + zf'(z) - \beta}{1 - \beta} \frac{1 + z}{1 - z}, \quad z \in \mathbb{D}
\end{equation}
and, so it follows that
\begin{equation}
f'(z) + zf'(z) < (1 - \beta)\frac{1 + z}{1 - z} + \beta, \quad z \in \mathbb{D}.
\end{equation}

From (2.5), we have
\begin{equation}
|\arg\{f(\rho e^{i\theta}) + \rho e^{i\theta}f'((\rho e^{i\theta}))\}| \leq \sin^{-1}\left\{1 + \frac{1 - (1 - \beta)^2}{2(1 - \beta)\rho^2}\right\} \quad \text{for all} \quad \rho \in [0, 1),
\end{equation}
\begin{equation}
\theta \in (-\pi, \pi).
\end{equation}
On the other hand, it follows that
\begin{equation}
f(z) = \frac{z f'(z)}{z}
= -\frac{1}{\pi} \int_0^\pi \left\{(t f'(t))' + (t f'(t))dt
= -\frac{1}{\pi} \int_0^\pi (z f'(t) +zf'(t))dt
= -\frac{1}{\pi} \int_0^\pi \int_0^{1/\beta} (f(\rho e^{i\theta}) + \rho e^{i\theta}f'((\rho e^{i\theta})))e^{i\theta}d\rho
\end{equation}
where $z = \rho e^{i\theta}$, $\rho \in [0, 1)$, $\theta \in (-\pi, \pi)$. It is known that
\begin{equation}
\sin^{-1}x \leq \frac{\pi}{2}x \quad \text{for} \quad x \in [0, 1],
\end{equation}
(2.8)

Then, applying the same idea of [9, pp. 1292–1293], Theorem 2.2, applying also (2.5), (2.7) and (2.8), we have
\begin{equation}
|\arg\{f(z)\}| \leq |\arg\left\{\int_0^\rho (\rho f(\rho e^{i\theta}) + \rho e^{i\theta}f'((\rho e^{i\theta})))d\rho\right\}|
\leq \int_0^\rho |\arg\{f(\rho e^{i\theta}) + \rho e^{i\theta}f'((\rho e^{i\theta}))\}|d\rho,
\end{equation}
\begin{equation}
\leq \int_0^\rho \sin^{-1}\left\{\frac{2(1 - \beta)\rho}{1 + (1 - 2\beta)\rho^2}\right\}d\rho,
\end{equation}
\begin{equation}
\beta \in (-\infty, 1/2) \cup (1/2, 1),
\end{equation}
\begin{equation}
\beta \leq 1/2,
\end{equation}
\begin{equation}
\beta \leq 1,
\end{equation}
\begin{equation}
\beta \leq 2/3.
\end{equation}

Letting $r \to 1^-$ we obtain
\begin{equation}
|\arg\{f(z)\}| \leq \frac{\pi - \beta}{\beta} \log\{2(1 - \beta)\} \quad \beta \in (-\infty, 1/2) \cup (1/2, 1),
\end{equation}
\begin{equation}
\beta \leq 1/2.
\end{equation}

It is easy to see that there exists $\beta_0 < 1.47 < \beta_0 < -1.46$, such that
\begin{equation}
\frac{\pi(1 - \beta_0)}{2(1 - 2\beta_0)} \log\{2(1 - \beta_0)\} = \frac{\pi}{2}
\end{equation}
and so for $\beta > \beta_0$, we have
\begin{equation}
\Re\{f(z)\} > 0, \quad z \in \mathbb{D}.
\end{equation}
This means that $f$ is a close-to-convex function with respect to $g(z) = z$, see (1.1). It completes the proof. □

Recall here the well known theorem due to Hallenbeck and Ruscheweyh [2].

**Theorem A** (see [2]). Let the function $h$ be analytic and convex univalent in $|z| < 1$ with $h(0) = a$. Let also $p(z) = a + b_0z^0 + b_1z^1 + \cdots$ be analytic in $\mathbb{D}$ If
\begin{equation}
p(z) + \frac{2p'(z)}{c} > h(z), \quad z \in \mathbb{D}
\end{equation}
for $\Re\{c\} > 0$, $c \neq 0$, then $p(z) < q_a(z) < h(z)$, $z \in \mathbb{D}$, where
\begin{equation}
q_a(z) = \frac{1}{\pi} \int_0^{2\pi} e^{i\theta - 1}(h(t))dt.
\end{equation}
Moreover, the function $q_a(z)$ is convex univalent and is the best dominant of $p < q_a$ in the sense that if $p < q$, then $q_a < q$.

The condition (2.2) becomes
\begin{equation}
f(z) + zf'(z) < h(\rho) = (1 - \beta)\frac{1 - z}{1 + z} + \beta
\end{equation}
where $h_0$ is convex univalent and maps the unit disc onto the half-plane $\Re\{w\} > \beta$. Using the above theorem with $n = 1$, $c = 1$, we immediately get

M. Nunokawa et al.
\[ f'(z) = \frac{1}{z} \int_0^z h(t) \, dt = q_\beta(z) = 2\beta - 1 + 2(1-\beta) \frac{\log(1+z)}{z}, \]

where the best dominant \( q_\beta \) is convex univalent with real coefficients, so it is easy to find the bounds for real part, but it is much harder to find the bounds for the argument. Mocanu, Ripeanu, and Popovici [8] showed that for \( \beta = 0 \), we have the bound \( |\arg(f'(z))| < 0.9110 \ldots \), while (2.3) gives in this case \( |\arg(f'(z))| < 1.08879 \ldots \). However, as we have seen, applying the new idea of [9, pp. 1292–1293] we obtain in Theorem 2.1 this bound for all \( \beta < 1 \). Recall here also the interesting result of Mocanu [7], see also Ali [1], namely if

\[ \Re\{f'(z) + zf''(z)\} > \frac{6 - \pi^2}{24 - \pi^2} = -0.2739 \ldots, \quad z \in \mathbb{D}, \]

then \( f \in S' \). Therefore, Theorem 2.1 extends the Ali’s result in the sense that the condition (2.2) implies the univalence and close-to-convexity of \( f \) also for \( \beta \) such that

\[-1.46 \ldots \beta_0 < \beta < \frac{6 - \pi^2}{24 - \pi^2} = -0.2739 \ldots \]

**Theorem 2.2.** Under the assumptions of Theorem 2.1, we have

\[ |\arg(f'(z) - \beta)| < \frac{\pi}{2} - \log 2 = 0.877649 \ldots \tag{2.9} \]

**Proof.** Let us put

\[ p(z) = \frac{1}{1-\beta} (f'(z) + zf''(z) - \beta), \quad z \in \mathbb{D}. \]

From the hypothesis (2.9), we have

\[ \frac{1}{1-\beta} (f'(z) + zf''(z) - \beta) < \frac{1+z}{1-z}, \quad z \in \mathbb{D}, \]

or

\[ f'(z) + zf''(z) - \beta < (1-\beta) \frac{1+z}{1-z}, \quad z \in \mathbb{D}. \tag{2.10} \]

Then, it follows that

\[ f'(z) - \beta = \frac{zf(z)}{z} - \beta \]

\[ = \frac{1}{z} \int_0^z (f'(t))' \, dt - \beta \]

\[ = \frac{1}{z} \int_0^z (f'(t) + zf''(t)) \, dt - \beta \]

\[ = \frac{1}{re^{\theta}} \int_0^r (f'(pe^{\theta})) + pe^{\theta}zf''(pe^{\theta})e^{\theta} \, d\rho - \beta \]

\[ = \frac{1}{r} \int_0^r (f'(pe^{\theta}) + pe^{\theta}zf''(pe^{\theta})) - \beta \, d\rho, \]

where \( z = pe^{\theta}, \rho \in [0,1], \theta \in (-\pi, \pi) \). Applying the same idea of [9, pp. 1292–1293], and (2.11), it follows that

\[ |\arg(f'(z) - \beta)| \]

\[ = |\arg\left\{ \frac{1}{r} \int_0^r (f'(pe^{\theta}) + pe^{\theta}zf''(pe^{\theta}) - \beta) \, d\rho \right\} | \]

\[ \leq \int_0^r |\arg(f'(pe^{\theta}) + pe^{\theta}zf''(pe^{\theta}) - \beta)| \, d\rho, \]

\[ = \int_0^r \sin^{-1} \frac{2\rho}{1+\rho^2} \, d\rho \]

\[ = \left\{ \rho \sin^{-1} \frac{2\rho}{1+\rho^2} - \log(1+\rho^2) \right\} \bigg|_{\rho=0}^{\rho=\rho} \]

\[ = \left\{ r \sin^{-1} \frac{2r}{1+r^2} - \log(1+r^2) \right\}. \]

Letting \( r \to 1^- \) we obtain

\[ |\arg(f'(z) - \beta)| < \pi/2 - \log 2. \quad \square \]

If we take \( \beta = 0 \) in (2.9), then we can see that it improves that one of Mocanu et al. [8] of the form \( |\arg(f'(z))| < 0.9110 \ldots \)

**Theorem 2.3.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic in the unit disc \( \mathbb{D} \). If

\[ |\arg(f'(z) + zf''(z) - \beta)| \leq \frac{2\pi}{3}, \quad z \in \mathbb{D}, \tag{2.12} \]

where \( 0 < \alpha \leq 1, 0 \leq \beta < 1 \), then

\[ |\arg(f'(z) - \beta)| < \frac{2\pi}{3} \left( 1 - \log \frac{4}{\pi} \right), \quad z \in \mathbb{D}. \tag{2.13} \]

**Proof.** By (2.12), we have

\[ \left\{ f'(z) + zf''(z) - \beta \right\}^{1/3} < \frac{1+z}{1-z}, \quad z \in \mathbb{D}. \]

A simple geometric observation yields to

\[ |\arg(f'(pe^{\theta}) + pe^{\theta}zf''(pe^{\theta}) - \beta)| \]

\[ \leq \sin^{-1} \frac{2\rho}{1+\rho^2} \quad \text{for all} \quad \rho \in [0,1], \quad \theta \in (-\pi, \pi]. \tag{2.14} \]

From (2.12), we have

\[ \Re\{z(f'(z) - \beta)/(1-\beta)^\alpha\} > 0, \quad z \in \mathbb{D}, \]

hence the function \( z(f'(z) - \beta) \) is univalent in the unit disc and it vanishes at \( z = 0 \) only. Therefore, \( f'(z) - \beta \neq 0 \) and so \( \arg(f'(z) - \beta) \) exists for all \( z \in \mathbb{D} \). Therefore, for \( z = re^{i\theta}, \quad r \in [0,1], \quad \theta \in (-\pi, \pi] \), we have from (2.14)

\[ |\arg(f'(z) - \beta)| \]

\[ = \left| \arg\left\{ f'(pe^{\theta}) + pe^{\theta}zf''(pe^{\theta}) - \beta \right\} \right| \]

\[ \leq \int_0^r |\arg(f'(pe^{\theta}) + pe^{\theta}zf''(pe^{\theta}) - \beta)| \, d\rho, \]

\[ = z \left( \rho \sin^{-1} \frac{2\rho}{1+\rho^2} - \log(1+\rho^2) \bigg|_{\rho=0}^{\rho=}\right) \]

\[ = z \left( r \sin^{-1} \frac{2r}{1+r^2} - \log(1+r^2) \right). \]

Letting \( r \to 1^- \) we obtain

\[ |\arg(f'(z) - \beta)| \leq \pi(\pi/2 - \log 2) = \frac{\pi}{2} \left( 1 - \log \frac{4}{\pi} \right), \quad z \in \mathbb{D}. \]

It completes the proof. \( \square \)

**Theorem 2.4.** Let \( f(z) = z + \sum_{n=3}^{\infty} a_n z^n \) be analytic in the unit disc \( \mathbb{D} \). If

\[ |\arg(f'(z) + zf''(z) - \beta)| \leq \frac{2\pi}{3}, \quad z \in \mathbb{D}, \tag{2.15} \]

where \( 0 \leq \beta < 1 \) and

\[ 0 < \alpha \leq \frac{\pi}{2(\pi - \log 2)} = 0.6415 \ldots, \tag{2.16} \]

then \( f \) is convex in \( \mathbb{D} \).
Proof. From Theorem 2.3, we have

$$|\text{Arg}(f(z) - \beta)| \leq \alpha(\pi/2 - \log 2), \quad z \in \mathbb{D}. $$

Then we also have

$$|\text{Arg}(f(z))| \leq \alpha(\pi/2 - \log 2), \quad z \in \mathbb{D},$$

since we supposed $0 < \beta < 1$ and by the same reason, we have

$$|\text{Arg}(f(z) + zf''(z))| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}. $$

Then, we have

$$\left|\text{Arg}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| - |\text{Arg}(f(z))| \leq |\text{Arg}(f'(z) + zf''(z))|$$

$$< \frac{\alpha\pi}{2}, \quad z \in \mathbb{D},$$

and it follows that

$$\left|\text{Arg}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\alpha\pi}{2} + \alpha(\pi/2 - \log 2) = \alpha(\pi - \log 2), \quad z \in \mathbb{D}. $$

Therefore, if $\alpha(\pi - \log 2) \leq \pi/2$, then $f$ is convex. It completes the proof. □

Note that Krzyż [4] gave an example: if $f$ satisfy the condition $\Re\{f'(z) + zf''(z)\} > 0$ in $\mathbb{D}$, then $f$ may not be convex in $\mathbb{D}$, but if $f$ satisfy the conditions of Theorem 2.4, then $f$ is convex in $\mathbb{D}$.

References


