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ORIGINAL ARTICLE

On some first-order differential subordination

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Abstract Let $\mathcal{A}$ denote the class of functions f that are analytic in the unit disc $\mathbb{D}$ and normalized by $f(0) = f'(0) - 1 = 0$. In this paper, we investigate the class of functions such that $\Re e\{f'(z) + zf''(z) - \beta\} > \alpha$ in $\mathbb{D}$. We determine conditions for α and β under which the function f is univalent, close-to-convex, and convex. To obtain this, we first estimate $|\mathrm{Arg}\{f'(z)\}|$ which improves the earlier results.

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1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and denote by $\mathcal{A}$ the class of analytic functions in $\mathbb{D}$ and usually normalized, i.e., $\mathcal{A} = \{f \in \mathcal{H} : f(0) = 0, f'(0) = 1\}$. We say that the $f \in \mathcal{H}$ is subordinate to $g \in \mathcal{H}$ in the unit disc $\mathbb{D}$, written $f \prec g$ if and only if there exists an analytic function $w \in \mathcal{H}$ such that $|w(z)| \leqslant |z|$ and $f(z) = g[w(z)]$ for $z \in \mathbb{D}$. Therefore, $f \prec g$ in $\mathbb{D}$ implies $f(\mathbb{D}) \subseteq g(\mathbb{D})$. In particular, if g is univalent in $\mathbb{D}$ then the Subordination Principle says that $f \prec g$ if and only if $f(0) = g(0)$ and $f(|z| < r) \subseteq g(|z| < r)$, for all $r \in (0,1)$.

The class $\mathcal{S}^*_\alpha$ of starlike functions of order $\alpha < 1$ may be defined as

$$\mathcal{S}^*_\alpha := \{f \in \mathcal{A} : \Re e \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{D}\}.$$

The class $\mathcal{S}^*_\alpha$ and the class $\mathcal{K}_\alpha$ of convex functions of order $\alpha < 1$

$$\mathcal{K}_\alpha := \{f \in \mathcal{A} : \Re e\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \mathbb{D}\}$$
$$= \{f \in \mathbb{D} : zf' \in \mathcal{S}^*_\alpha\}$$

were introduced by Robertson in [11]. If $\alpha \in [0;1)$, then a function in either of these sets is univalent. In particular, we denote $\mathcal{S}^*_0 = \mathcal{S}^*, \mathcal{K}_0 = \mathcal{K}$, the classes of starlike and convex functions, respectively. Recall that $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}_\alpha(\beta)$,

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[3], of close-to-convex functions of order β and type α, $0 \leqslant \beta < 1$, if and only if there exist $g \in \mathcal{K}_\alpha, \varphi \in \mathbb{R}$, such that

$$\Re\mathfrak{e}\left\{e^{i\varphi}\frac{f'(z)}{g'(z)}\right\} > \beta, \quad z \in \mathbb{D}. \tag{1.1}$$

Functions defined by (1.1) with $\varphi = 0$ were considered earlier by Ozaki [10], see also Umezawa [12,13]. Moreover, Lewandowski [5,6] defined the class of functions $f \in \mathcal{A}$ for which the complement of $f(\mathbb{D})$ with respect to the complex plane is a linearly accessible domain in the large sense. The Lewandowski's class is identical with the Kaplan's class $\mathcal{C}_0(0)$.

2. Main result

Theorem 2.1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc $\mathbb{D}$. If*

$$f'(z) \neq 0,\ f'(z) + zf''(z) \neq 0, \quad z \in \mathbb{D} \tag{2.1}$$

and

$$\Re\mathfrak{e}\{f'(z) + zf''(z)\} > \beta, \quad z \in \mathbb{D}, \tag{2.2}$$

then

$$|\mathrm{Arg}\{f'(z)\}| \leqslant \begin{cases} \frac{\pi(1-\beta)}{2(1-2\beta)} \log\{2(1-\beta)\} & \beta \in (-\infty, 1/2) \cup (1/2, 1), \\ \frac{\pi}{2} - 1 & \beta = 1/2. \end{cases} \tag{2.3}$$

Moreover, f is close-to-convex in $\mathbb{D}$ whenever

$\beta > \beta_0$,

where $-1.47 < \beta_0 < -1.46$ is the positive solution of the equation

$$\log\{2(1-\beta)\} = \frac{1-2\beta}{1-\beta}. \tag{2.4}$$

Proof. Note that the assumptions (2.1) are necessary for $\beta < 0$ only. If $\beta \in [0, 1)$, then from (2.2) we have even more $\Re\mathfrak{e}\{f'(z) + zf''(z)\} > 0$. Moreover, from (2.2) we have also that $\Re\mathfrak{e}\{(zf'(z))'\} > 0$ so zf' is univalent in $\mathbb{D}$ and $f'(z) \neq 0$.

From the hypothesis (2.2), we have

$$\frac{f'(z) + f''(z) - \beta}{1-\beta} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}$$

and so, it follows that

$$f'(z) + zf''(z) \prec (1-\beta)\frac{1+z}{1-z} + \beta, \quad z \in \mathbb{D}. \tag{2.5}$$

From (2.5), we have

$$\begin{aligned} &|\mathrm{Arg}\{f'(\rho e^{i\theta}) + \rho e^{i\theta} f''(\rho e^{i\theta})\}| \\ &\quad \leqslant \sin^{-1}\left\{\frac{2(1-\beta)\rho}{1+(1-2\beta)\rho^2}\right\} \quad \text{for all} \quad \rho \in [0,1), \\ &\quad \theta \in (-\pi, \pi]. \end{aligned} \tag{2.6}$$

On the other hand, it follows that

$$\begin{aligned} f'(z) &= \frac{zf'(z)}{z} \\ &= \frac{1}{z}\int_0^z (tf'(t))' \mathrm{d}t \\ &= \frac{1}{z}\int_0^z (f'(t) + tf''(t)) \mathrm{d}t \\ &= \frac{1}{re^{i\theta}}\int_0^r (f'(\rho e^{i\theta}) + \rho e^{i\theta} f''(\rho e^{i\theta})) e^{i\theta} \mathrm{d}\rho \\ &= \frac{1}{r}\int_0^r (f'(\rho e^{i\theta}) + \rho e^{i\theta} f''(\rho e^{i\theta})) \mathrm{d}\rho, \end{aligned} \tag{2.7}$$

where $z = \rho e^{i\theta}$, $\rho \in [0, 1)$, $\theta \in (-\pi, \pi]$. It is known that

$$\sin^{-1} x \leqslant \frac{\pi}{2} x \quad \text{for} \quad x \in [0, 1]. \tag{2.8}$$

Then, applying the same idea of [9, pp. 1292–1293], Theorem 2.2, applying also (2.5), (2.7) and (2.8), we have

$$\begin{aligned} |\mathrm{Arg}\{f'(z)\}| &= \left|\mathrm{Arg}\left\{\frac{1}{r}\int_0^r (f'(\rho e^{i\theta}) + \rho e^{i\theta} f''(\rho e^{i\theta})) \mathrm{d}\rho, \right\}\right| \\ &\leqslant \int_0^r |\mathrm{Arg}\{f'(\rho e^{i\theta}) + \rho e^{i\theta} f''(\rho e^{i\theta})\}| \mathrm{d}\rho, \\ &\leqslant \int_0^r \sin^{-1}\{\frac{2(1-\beta)\rho}{1+(1-2\beta)\rho^2}\} \mathrm{d}\rho \\ &\leqslant \begin{cases} \frac{\pi}{2}\int_0^r \left\{\frac{2(1-\beta)\rho}{1+(1-2\beta)\rho^2}\right\} \mathrm{d}\rho & \beta \in (-\infty, 1/2) \cup (1/2, 1), \\ \int_0^r \sin^{-1}\rho \, \mathrm{d}\rho & \beta = 1/2, \end{cases} \\ &= \begin{cases} \frac{\pi(1-\beta)}{2(1-2\beta)}\int_0^r \left\{\frac{2(1-2\beta)\rho}{1+(1-2\beta)\rho^2}\right\} \mathrm{d}\rho & \beta \in (-\infty, 1/2) \cup (1/2, 1), \\ \int_0^r \sin^{-1}\rho \, \mathrm{d}\rho & \beta = 1/2, \end{cases} \\ &= \begin{cases} \frac{\pi(1-\beta)}{2(1-2\beta)} \{\log\{1+(1-2\beta)\rho^2\}\}\Big|_{\rho=0}^{\rho=r} & \beta \in (-\infty, 1/2) \cup (1/2, 1), \\ \left\{\rho\sin^{-1}\rho + \sqrt{1-\rho^2}\right\}\Big|_{\rho=0}^{\rho=r} & \beta = 1/2, \end{cases} \\ &= \begin{cases} \frac{\pi(1-\beta)}{2(1-2\beta)} \log\{1+(1-2\beta)r^2\} & \beta \in (-\infty, 1/2) \cup (1/2, 1), \\ r\sin^{-1} r + \sqrt{1-r^2} - 1 & \beta = 1/2. \end{cases} \end{aligned}$$

Letting $r \to 1^-$ we obtain

$$|\mathrm{Arg}\{f'(z)\}| \leqslant \begin{cases} \frac{\pi(1-\beta)}{2(1-2\beta)} \log\{2(1-\beta)\} & \beta \in (-\infty, 1/2) \cup (1/2, 1), \\ \frac{\pi}{2} - 1 & \beta = 1/2. \end{cases}$$

It is easy to see that there exists β_0, $-1.47 < \beta_0 < -1.46$, such that

$$\frac{\pi(1-\beta_0)}{2(1-2\beta_0)} \log\{2(1-\beta_0)\} = \frac{\pi}{2}$$

and so for $\beta > \beta_0$, we have

$\Re\mathfrak{e}\{f'(z)\} > 0, \ z \in \mathbb{D}$.

This means that f is a close-to-convex function with respect to $g(z) = z$, see (1.1). It completes the proof. □

Recall here the well known theorem due to Hallenbeck and Ruscheweyh [2].

Theorem A (see [2]). *Let the function h be analytic and convex univalent in $|z| < 1$ with $h(0) = a$. Let also $p(z) = a + b_n z^n + b_{n+1} z^{n+1} + \cdots$ be analytic in $\mathbb{D}$ If*

$$p(z) + \frac{zp'(z)}{c} \prec h(z), \quad z \in \mathbb{D}$$

for $\Re\mathfrak{e}\{c\} \geqslant 0, c \neq 0$, then

$$p(z) \prec q_n(z) \prec h(z), \quad z \in \mathbb{D},$$

where $q_n(z) = \frac{c}{nz^{c/n}}\int_0^z t^{c/n-1} h(t) \mathrm{d}t$. Moreover, the function $q_n(z)$ is convex univalent and is the best dominant of $p \prec q_n$ in the sense that if $p \prec q$, then $q_n \prec q$.

The condition (2.2) becomes

$$f'(z) + zf''(z) \prec h_\beta(z) = (1-\beta)\frac{1-z}{1+z} + \beta$$

where h_β is convex univalent and maps the unit disc onto the half-plane $\Re\mathfrak{e}\{w\} > \beta$. Using the above theorem with $n = 1$, $c = 1$, we immediately get

$$f'(z) \prec \frac{1}{z}\int_0^z h_\beta(t)\mathrm{d}t = q_\beta(z) = 2\beta - 1 + 2(1-\beta)\frac{\log(1+z)}{z},$$

where the best dominant q_β is convex univalent with real coefficients, so it is easy to find the bounds for real part, but it is much harder to find the bounds for the argument. Mocanu, Ripeanu, and Popovici [8] showed that for $\beta = 0$, we have the bound $|\mathrm{Arg}\{f'(z)\}| < 0.9110\ldots$, while (2.3) gives in this case $|\mathrm{Arg}\{f'(z)\}| < 1.08879\ldots$. However, as we have seen, applying the new idea of [9, pp. 1292–1293] we obtain in Theorem 2.1 this bound for all $\beta < 1$. Recall here also the interesting result of Mocanu [7], see also Ali [1], namely if

$$\Re e\{f'(z) + zf''(z)\} > \frac{6-\pi^2}{24-\pi^2} = -0.2739\ldots, \quad z \in \mathbb{D},$$

then $f \in \mathcal{S}^*$. Therefore, Theorem 2.1 extends the Ali's result in the sense that the condition (2.2) implies the univalence and close-to-convexity of f also for β such that

$$-1.46\ldots = \beta_0 < \beta < \frac{6-\pi^2}{24-\pi^2} = -0.2739\ldots$$

Theorem 2.2. *Under the assumptions of Theorem 2.1, we have*

$$|\mathrm{Arg}\{f'(z) - \beta\}| < \pi/2 - \log 2 = 0.877649\ldots. \tag{2.9}$$

Proof. Let us put

$$p(z) = \frac{1}{1-\beta}(f'(z) + zf''(z) - \beta), \quad z \in \mathbb{D}.$$

From the hypothesis (2.9), we have

$$\frac{1}{1-\beta}(f'(z) + f''(z) - \beta) \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}$$

or

$$f'(z) + zf''(z) - \beta \prec (1-\beta)\frac{1+z}{1-z}, \quad z \in \mathbb{D}. \tag{2.10}$$

Then, it follows that

$$\begin{aligned} f'(z) - \beta &= \frac{zf'(z)}{z} - \beta \\ &= \frac{1}{z}\int_0^z (tf'(t))'\mathrm{d}t - \beta \\ &= \frac{1}{z}\int_0^z (f'(t) + tf''(t))\mathrm{d}t - \beta \\ &= \frac{1}{re^{i\theta}}\int_0^r (f'(\rho e^{i\theta}) \\ &\quad + \rho e^{i\theta} f''(\rho e^{i\theta}))e^{i\theta}\mathrm{d}\rho - \beta \\ &= \frac{1}{r}\int_0^r (f'(\rho e^{i\theta}) + \rho e^{i\theta}f''(\rho e^{i\theta}) - \beta)\mathrm{d}\rho, \end{aligned} \tag{2.11}$$

where $z = \rho e^{i\theta}$, $\rho \in [0,1)$, $\theta \in (-\pi,\pi]$. Applying the same idea of [9, pp. 1292–1293], and (2.11), it follows that

$$\begin{aligned} &|\mathrm{Arg}\{f'(z) - \beta\}| \\ &= \left|\mathrm{Arg}\{\tfrac{1}{r}\textstyle\int_0^r (f'(\rho e^{i\theta}) + \rho e^{i\theta}f''(\rho e^{i\theta}) - \beta)\mathrm{d}\rho,\}\right| \\ &\leqslant \textstyle\int_0^r |\mathrm{Arg}\{f'(\rho e^{i\theta}) + \rho e^{i\theta}f''(\rho e^{i\theta}) - \beta\}|\mathrm{d}\rho, \\ &= \textstyle\int_0^r \sin^{-1}\frac{2\rho}{1+\rho^2}\mathrm{d}\rho \\ &= \left\{\rho\sin^{-1}\tfrac{2\rho}{1+\rho^2} - \log(1+\rho^2)\right\}\Big|_{\rho=0}^{\rho=r} \\ &= \left\{r\sin^{-1}\tfrac{2r}{1+r^2} - \log(1+r^2)\right\}. \end{aligned}$$

Letting $r \to 1^-$ we obtain

$|\mathrm{Arg}\{f'(z) - \beta\}| \leqslant \pi/2 - \log 2.$ □

If we take $\beta = 0$ in (2.9), then we can see that it improves that one of Mocanu et al. [8] of the form $|\mathrm{Arg}\{f'(z)\}| < 0.9110\ldots$.

Theorem 2.3. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc $\mathbb{D}$. If*

$$|\mathrm{Arg}\{f'(z) + zf''(z) - \beta\}| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}, \tag{2.12}$$

where $0 < \alpha \leqslant 1$, $0 \leqslant \beta < 1$, then

$$|\mathrm{Arg}\{f'(z) - \beta\}| < \frac{\alpha\pi}{2}\left\{1 - \frac{\log 4}{\pi}\right\}, \quad z \in \mathbb{D}. \tag{2.13}$$

Proof. By (2.12), we have

$$\left\{\frac{f'(z) + f''(z) - \beta}{1-\beta}\right\}^{1/\alpha} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}.$$

A simple geometric observation yields to

$$\begin{aligned} &|\mathrm{Arg}\{(f'(\rho e^{i\theta}) + \rho e^{i\theta}f''(\rho e^{i\theta}))^{1/\alpha} - \beta\}| \\ &\quad \leqslant \sin^{-1}\frac{2\rho}{1+\rho^2} \quad \text{for all} \quad \rho \in [0,1), \quad \theta \in (-\pi,\pi]. \end{aligned} \tag{2.14}$$

From (2.12), we have

$$\Re e\{[z(f'(z) - \beta)/(1-\beta)]'\} > 0, \quad z \in \mathbb{D},$$

hence the function $z(f'(z) - \beta)$ is univalent in the unit disc and it vanishes at $z = 0$ only. Therefore, $f'(z) - \beta \neq 0$ and so Arg $\{f'(z) - \beta\}$ exists for all $z \in \mathbb{D}$. Therefore, for $z = r\ e^{i\theta}$, $r \in [0,1)$, $\theta \in (-\pi,\pi]$, we have from (2.14)

$$\begin{aligned} &|\mathrm{Arg}\{f'(z) - \beta\}| \\ &= \left|\mathrm{Arg}\{\tfrac{1}{r}\textstyle\int_0^r (f'(\rho e^{i\theta}) + \rho e^{i\theta}f''(\rho e^{i\theta}) - \beta)\mathrm{d}\rho,\}\right| \\ &\leqslant \textstyle\int_0^r |\mathrm{Arg}\{f'(\rho e^{i\theta}) + \rho e^{i\theta}f''(\rho e^{i\theta}) - \beta\}|\mathrm{d}\rho, \\ &= \alpha\textstyle\int_0^r \sin^{-1}\frac{2\rho}{1+\rho^2}\mathrm{d}\rho \\ &= \alpha\left\{\rho\sin^{-1}\tfrac{2\rho}{1+\rho^2} - \log(1+\rho^2)\right\}\Big|_{\rho=0}^{\rho=r} \\ &= \alpha\left\{r\sin^{-1}\tfrac{2r}{1+r^2} - \log(1+r^2)\right\}. \end{aligned}$$

Letting $r \to 1^-$ we obtain

$$|\mathrm{Arg}\{f'(z) - \beta\}| \leqslant \alpha(\pi/2 - \log 2) = \frac{\alpha\pi}{2}\left\{1 - \frac{\log 4}{\pi}\right\}, \quad z \in \mathbb{D}.$$

It completes the proof. □

Theorem 2.4. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc $\mathbb{D}$. If*

$$|\mathrm{Arg}\{f'(z) + zf''(z) - \beta\}| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}, \tag{2.15}$$

where $0 \leqslant \beta < 1$ and

$$0 < \alpha \leqslant \frac{\pi}{2(\pi - \log 2)} = 0.6415\ldots, \tag{2.16}$$

then f is convex in $\mathbb{D}$.

Proof. From Theorem 2.3, we have

$$|\mathrm{Arg}\{f'(z)-\beta\}| \leqslant \alpha(\pi/2-\log 2), \quad z \in \mathbb{D}.$$

Then we also have

$$|\mathrm{Arg}\{f'(z)\}| \leqslant \alpha(\pi/2-\log 2), \quad z \in \mathbb{D},$$

since we supposed $0 < \beta < 1$ and by the same reason, we have

$$|\mathrm{Arg}\{f'(z)+zf''(z)\}| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}.$$

Then, we have

$$\left|\mathrm{Arg}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right| - |\mathrm{Arg}\{f'(z)\}| \leqslant |\mathrm{Arg}\{f'(z)+zf''(z)\}| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D},$$

and it follows that

$$\left|\mathrm{Arg}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\alpha\pi}{2} + \alpha(\pi/2-\log 2) = \alpha(\pi-\log 2), \quad z \in \mathbb{D}.$$

Therefore, if $\alpha(\pi-\log 2) \leqslant \pi/2$, then f is convex.

It completes the proof. □

Note that Krzyż [4] gave an example : if f satisfy the condition $\Re\mathfrak{e}\{f'(z)+zf''(z)\} > 0$ in $\mathbb{D}$, then f may not be convex in $\mathbb{D}$, but if f satisfy the conditions of Theorem 2.4, then f is convex in $\mathbb{D}$.

References

[1] R.M. Ali, On a subclass of starlike functions, Rocky Mountain J. Math. 24 (2) (1994) 447–451.

[2] D.I. Hallenbeck, St. Ruscheweyh, Subordination by convex functions, Proc. Am. Math. Soc. 52 (1975) 191–195.

[3] W. Kaplan, Close-to-convex schlicht functions, Michigan Math. J. 1 (1952) 169–185.

[4] J. Krzyż, A counter example concerning univalent functions, Mat. Fiz. Chem. 2 (1962) 57–58.

[5] Z. Lewandowski, Sur l'identité de certaines classes de fonctions univalentes I, Ann. Univ. Mariae Curie-Skodowska Sect. A 12 (1958) 131–146.

[6] Z. Lewandowski, Sur l'identité de certaines classes de fonctions univalentes II, Ann. Univ. Mariae Curie-Skodowska Sect. A 14 (1960) 19–46.

[7] P.T. Mocanu, On starlike images by Alexander integral, Babeş-Bolayi Univ., Fac. of Math. and Phys. Res. Sem., Itinerant Seminar Functional Equations, Approximation and Convexity, Preprint 6 (1987) 245–250.

[8] P.T. Mocanu, D. Ripeanu, M. Popovici, The best bound for the argumentof certain analytic funcions with positive real part, Babeş-Bolayi Univ., Fac. of Math. Res. Sem., Seminar of Geometric Function Theory, Preprint 5 (1986) 91–98.

[9] M. Nunokawa, S. Owa, E. Yavuz Duman, M. Aydoğan, Some properties of analytic functions relating to the Miller and Mocanu result, Comp. Math. Appl. 61 (2011) 1291–1295.

[10] S. Ozaki, On the theory of multivalent functions, Sci. Rep. Tokyo Bunrika Daig. A2 (1935) 167–188.

[11] M.S. Robertson, On the theory of univalent functions, Ann. Math. 37 (1936) 374–408.

[12] T. Umezawa, On the theory of univalent functions, Tohoku Math. J. 7 (1955) 212–228.

[13] T. Umezawa, Multivalently close-to-convex functions, Proc. Am. Math. Soc. 8 (1957) 869–874.