Unsteady thin film flow of a fourth grade fluid over a vertical moving and oscillating belt

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Abstract This article studies the unsteady thin film flow of a fourth grade fluid over a moving and oscillating vertical belt. The problem is modeled in terms of non-linear partial differential equations with some physical conditions. Both problems of lift and drainage are studied. Two different techniques namely the adomian decomposition method (ADM) and the optimal homotopy asymptotic method (OHAM) are used for finding the analytical solutions. These solutions are compared and found in excellent agreement. For the physical analysis of the problem, graphical results are provided and discussed for various embedded flow parameters.

1. Introduction

Due to the diverse physical structures of non-Newtonian fluids, several constitutive models have been proposed in the literature to predict all of their salient features.

Generally, there are three non-Newtonian fluids models. They are known as (i) the differential type, (ii) the rate type, and (iii) the integral type. However, the most famous amongst them are the first two models. In this article, we will study the first model, the differential type fluid also known as grade n fluids and consider its subclass known as the fourth grade fluid. The simplest subclass of differential type is the second grade fluid for which one can reasonably hope to obtain an analytical solution. Although a second grade fluid model also known as grade
2 is able to predict the normal stress differences but it does not take into account the shear thinning and shear thickening phenomena that many fluids exhibit [1–4]. On the other hand, a third grade fluid model (grade 3 fluid) represents a more realistic description of the behavior of non-Newtonian fluids. This model is known to capture the non-Newtonian effects such as shear thinning or shear thickening as well as normal stresses [5–8]. There is another model for differential type fluids known as fourth grade fluid model (grade 4 fluid) which at one time captures most of the non-Newtonian flow properties. Means, Cauchy stress tensor appearing in this fluid model contains several parameters. Due to which the governing equation for fourth grade fluid becomes quite complicated. This model is also very capable to predict the effects of normal stresses that lead to phenomena like rod-climbing and die-swell [9]. Therefore, very few studies, which illustrates the effect of fourth grade fluids even on steady flow, have been reported in the literature. In general, it is not easy to study fourth grade fluid even for steady flow problems. However, under certain assumptions based on the flow situation, it is possible and some investigations are carried out in this direction [10–14]. On the other hand, the science of thin liquid films has developed rapidly in recent years. Thin film flow problems appear in many fields, varying from specific situations in the flow in human lunes to lubrication problems in engineering which is probably one of the largest subfield of thin film flow problems [15]. Few other applications are found in coating flows, biofluids, microfluidic engineering, and medicine [16,17]. Two well known examples from everyday life are slipping on a wet bathroom floor and aquaplaning or hydroplaning on a wet road. The study of thin film flow for practical applications is a challenging interplay between fluid mechanics, structural mechanics and rheology. Based on this motivation, recently several researchers are getting interested to study the thin film flows. Amongst them, we discuss here some important contributions of the following researchers.

Siddiqui et al. [18,19] discussed the analytical solution of thin film flow of third grade fluid and Oldroyd-8 constant fluid through vertical moving belt by applying adomian decomposition method (ADM) and variational iteration method (VIM). Aiyessi et al. [20] investigated the unsteady magnetohydrodynamic (MHD) thin film flow of a third grade fluid with heat transfer and no slip boundary condition down an inclined plane. Gul et al. [21,22] studied analytically the thin film flows of second grade and third grade fluids through a vertical belt for lift and drainage problems using optimal homotopy asymptotic method (OHAM) and ADM. Few other investigations in this direction are mentioned in Refs. [23–40].

The basic theme of this article is to venture further in the regime of thin film flows and to extend this idea to the fourth grade fluid executing the unsteady motion over a vertical oscillating belt using two analytical techniques namely the OHAM and ADM methods. OHAM on the other hand, is a new powerful technique which is a generalized form of homotopy perturbation method (HPM) and HAM. It is a straightforward technique and does not require the existence of any small or large parameter as does the traditional perturbation method. OHAM has successfully applied to a number of nonlinear problems arising in the science and engineering by various researchers. This proves the validity and acceptability of OHAM as a useful technique. For the correctness and verification of OHAM solutions, they are compared with those obtained by ADM. The rest of the paper is arranged as follows: The constitutive equations for fourth grade fluid are given in Section 2. Using these constitutive equations for unidirectional and one dimension flow, the continuity equation is satisfied identically and the momentum equation is derived in Section 3 with some physical conditions. In Section 4, the basic concepts of ADM and OHAM are discussed. These methods are used in Section 5 where the analytical solutions for both lifting and drainage problems are obtained. The graphical results are given in Section 6. This paper ends with some conclusions in Section 7.

2. Fundamental equation

The constitutive equations for fourth grade fluid model is

\[ T = -pI + \mu A_1 + \sum_{i=1}^{3} S_i, \]

where \( T \) is the Cauchy stress tensor, \( I \) is identity tensor, \( p \) is fluid pressure and \( S_i \) is the extra stress tensor

\[ S_1 = \alpha_1 A_2 + \alpha_2 A_1^3; \]

\[ S_2 = \beta_1 A_3 + \beta_2 (A_1 A_2 + A_2 A_1) + \beta_3 (tr A_2^2) A_1; \]

\[ S_3 = \gamma_1 A_4 + \gamma_2 (A_1 A_3 + A_3 A_1) + \gamma_3 A_2^3 + \left[ \gamma_4 (A_2 A_1^2 + A_1^2 A_2) + \gamma_5 (tr A_1) A_2 + \gamma_6 (tr A_2) A_1^2 + [\gamma_7 (tr A) + \gamma_8 (tr A_1)] A_1 \right]. \]

Here \( \mu \) is coefficient of viscosity, \( \alpha_i (i = 1, 2), \beta_j (j = 1, 2, 3) \) and \( \gamma_k (k = 1, 2, \ldots, 8) \) are material constants of second, third and fourth grades respectively. Rivlin–Ericksen tensors \( A_1, A_2, A_3 \) and \( A_4 \) are defined as:

\[ A_0 = I, \quad A_1 = (VV) + (VV)^T, \]

\[ A_n = \frac{DA_{n-1}}{Dt} + A_{n-1} (VV) + (VV)^T A_{n-1}, \quad n = 2, 3, 4. \]

The continuity and momentum equation for incompressible fluid are given by
\[ \text{div} \mathbf{V} = 0, \quad (5) \]
\[ \rho \frac{DV}{Dt} = \text{div} \mathbf{T} + \rho g, \quad (6) \]

where \( \mathbf{V} = (u, v, w) \) is the velocity vector, \( \rho \) is the fluid density, \( \frac{D}{Dt} \) is the material time derivative.

### 3. Formulation of lift problem

We consider an oscillating and moving wide flat belt placed in a large bath of a fourth grade fluid. The belt is moving in vertically upward direction with velocity \( U \) and pick up a thin film of a fourth grade fluid of constant thickness \( \delta \). The \( x \)- and \( y \)-axes are chosen in such a way that the belt is parallel to \( y \)-axis and perpendicular to \( x \)-axis as shown in the physical configuration (see Figures 1 and 2). The pressure gradient is not considered here and the flow is assumed as unsteady and laminar.

By taking velocity field of the form

\[ \mathbf{V} = (0, v(x, t), 0), \quad (7) \]

Subject to the following boundary conditions

\[ v(x, t) = U + U \cos \omega t, \quad \text{at} \quad x = 0, \quad (8) \]
\[ \frac{\partial v(x, t)}{\partial x} = 0, \quad \text{at} \quad x = \delta, \quad (9) \]

Here \( \omega \) is used as frequency of the oscillating belt.

Using the velocity field given in Eq. (7), the continuity Eq. (5) is identically satisfied and the momentum Eq. (6) reduces to the form

\[ \rho \frac{\partial v}{\partial t} = \rho g. \quad (10) \]

From Eq. (1) the component \( T_{xy} \) of Cauchy stress component is

\[ T_{xy} = \mu \frac{\partial v}{\partial x} + \alpha_1 \frac{\partial}{\partial t} \left( \frac{\partial v}{\partial x} \right) + \beta_1 \frac{\partial^2}{\partial x^2} \left( \frac{\partial v}{\partial x} \right) \]
\[ + 2(\beta_2 + \beta_3) \left( \frac{\partial v}{\partial x} \right)^3 + \gamma_1 \frac{\partial^3}{\partial x^3} \left( \frac{\partial v}{\partial x} \right) \]
\[ + 2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8) \frac{\partial}{\partial t} \left( \frac{\partial v}{\partial x} \right)^3. \quad (11) \]

Inserting Eq. (11) into Eq. (10), we get

\[ \rho \frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial x^2} + \alpha_1 \frac{\partial}{\partial t} \left( \frac{\partial^2 v}{\partial x^2} \right) + 6(\beta_2 + \beta_3) \left( \frac{\partial v}{\partial x} \right)^2 \left( \frac{\partial^2 v}{\partial x^2} \right) \]
\[ + \gamma_1 \frac{\partial^3}{\partial x^3} \left( \frac{\partial^2 v}{\partial x^2} \right) + 2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8) \frac{\partial}{\partial t} \left( \frac{\partial v}{\partial x} \right)^3 \]
\[ + 3\gamma_7 + \gamma_8 \frac{\partial}{\partial x} \left[ \left( \frac{\partial v}{\partial x} \right)^2 \frac{\partial}{\partial t} \left( \frac{\partial^2 v}{\partial x^2} \right) \right] - \rho g. \quad (12) \]

Introducing the following non-dimensional variables as mentioned in Refs. [13,21,22].
Here $\tilde{t}$ is a dimensionless time parameter, $\tilde{\alpha}$ is dimensionless second grade non-Newtonian parameter, $\tilde{\beta}, \tilde{\beta}_1$ are dimensionless third grade non-Newtonian parameters and $\tilde{\gamma}, \tilde{\gamma}_1$ are dimensionless fourth grade non-Newtonian parameters.

Inserting above non-dimensional variables into Eq. (12) and boundary conditions (8) and (9), we get:

$$\frac{\partial v}{\partial t} = -S_t + \tilde{\beta} \frac{\partial^2 v}{\partial x^2} + \tilde{\alpha} \frac{\partial}{\partial t} \left( \frac{\partial^2 v}{\partial x^2} \right) + \tilde{\beta}_1 \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 v}{\partial x^2} \right) + 2\tilde{\gamma} \frac{\partial}{\partial x} \left[ \frac{\partial^2 v}{\partial x^2} \right], \quad \tilde{\gamma} = (3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8) \frac{\gamma_1 U^2}{\mu \partial^2}, \quad (13)$$

4. Solution techniques

4.1. Basic concept of adomian decomposition method (ADM)

ADM is used to decompose the unknown function $u(x,t)$ into a sum of an infinite number of components defined by the decomposition series.

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \quad (17)$$

The decomposition method is used to find the components $u_0(x,t), u_1(x,t), u_2(x,t), \ldots$ separately. The determination of these components can be obtained through simple integrals. To give a clear overview of ADM, we consider the partial differential equation in an operator form as

$$L_x u(x,t) + L_y u(x,t) + R u(x,t) + N u(x,t) = g(x,t), \quad (18)$$

$$L_x u(x,t) = g(x,t) - L_y u(x,t) - R u(x,t) - N u(x,t), \quad (19)$$

where $L_x = \frac{\partial^2}{\partial x^2}$ and $L_y = \frac{\partial}{\partial y}$ are linear operators in the partial differential equation and are easily invertible, $g(x,t)$ is a source term, $R u(x,t)$ is a remaining linear term and $N u(x,t)$ is non-linear analytical term expandable in the adomian polynomials $A_n$.

After applying the inverse operator $L_x^{-1}$ to both sides of Eq. (19), we write

$$L_x^{-1} L_x u(x,t) = L_x^{-1} g(x,t) - L_x^{-1} L_y u(x,t) - L_x^{-1} N u(x,t), \quad (20)$$

$$u(x,t) = f(x,t) - L_x^{-1} L_y u(x,t) - L_x^{-1} R u(x,t) - L_x^{-1} N u(x,t). \quad (21)$$

Here the function $f(x,t)$ represents the terms arising from $L_x^{-1} g(x,t)$ after using the given conditions. $L_x^{-1} = \int (\bullet) dx$ is used as inverse operator for the second order partial differential equation.

In this method, the series solution $u(x,t)$ is defined as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (22)$$

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x,t) - L_x^{-1} L_y \sum_{n=0}^{\infty} u_n(x,t) - L_x^{-1} R \sum_{n=0}^{\infty} u_n(x,t) - L_x^{-1} N \sum_{n=0}^{\infty} u_n(x,t). \quad (23)$$

The non-linear term is expanded in adomian polynomials as:

$$N(u(x,t)) = \sum_{n=0}^{\infty} A_n, \quad (24)$$

where the components $u_0(x,t), u_1(x,t), u_2(x,t), \ldots$ are periodically derived as:

$$u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots = f(x,t) - L_x^{-1} R(u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots) - L_x^{-1}(A_0 + A_1 + A_2 + \ldots). \quad (25)$$

To determine the components of the series $u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots$, it is important to note that ADM suggests that the zeroth component $u_0(x,t)$ is usually defined by the function $f(x,t)$ described above.

The formal recursive relation is defined as:

$$u_0(x,t) = f(x,t), \quad u_1(x,t) = -L_x^{-1} L_x u_0(x,t) - L_x^{-1} R u_0(x,t) - L_x^{-1}(A_0), \quad (26)$$

$$u_2(x,t) = -L_x^{-1} L_x u_1(x,t) - L_x^{-1} R u_1(x,t) - L_x^{-1}(A_1), \quad$$

$$u_3(x,t) = -L_x^{-1} L_x u_2(x,t) - L_x^{-1} R u_2(x,t) - L_x^{-1}(A_2), \quad$$

and so on.

4.2. Optimal homotopy asymptotic method (OHAM)

For the analysis of OHAM, we consider the boundary value problem as considered in Refs. [12,13]:

$$\tilde{L}(\tilde{u}(x)) + \tilde{N}(\tilde{u}(x)) + \tilde{G}(x) = 0, \quad \tilde{B} \left( \tilde{u}, \frac{\partial \tilde{u}}{\partial x} \right) = 0, \quad (27)$$
where $L$ a linear operator in the differential equation is, $\tilde{N}$ is a non-linear term, $x \in R$ is an independent variable, $\tilde{B}$ is a boundry operator and $\tilde{G}$ is a source term. According to Refs. [12,13] we construct a set of equation for OHAM:

$$\left[1 - p\right] \left[L(\tilde{u}(x)) + \tilde{G}(x)\right] = H(p, c_i) \left[L\tilde{\phi}(x, p) + \tilde{G}(x) + \tilde{R}\tilde{\phi}(x, p)\right],$$

$\tilde{B}\left[\tilde{\phi}(x, p), \frac{\partial \tilde{\phi}}{\partial x}\right] = 0. \quad (28)$

Here $p \in [0, 1]$ is an embedding parameter, $H(p, c_i)$ is a non-zero auxiliary function for $p \neq 0$ and $H(0, c_i) = 0$, $\tilde{\phi}(x, p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$, it holds that:

$$\tilde{\phi}(x, 0) = \tilde{u}_0(x), \quad \tilde{\phi}(x, 1) = \tilde{u}(x), \quad (29)$$

Note that, when $p$ varies from 0 to 1 then $\tilde{\phi}(x, p)$ also varies from $\tilde{u}_0(x)$ to $\tilde{u}(x)$.

The zeroth component solution $\tilde{u}_0(x)$ is obtained from Eq. (28) when $p = 0$ i.e.

$$\tilde{L}(\tilde{u}_0(x)) + \tilde{G}(x) = 0, \quad \tilde{B}\left(\tilde{u}_0(x), \frac{\partial \tilde{u}_0(x)}{\partial x}\right) = 0, \quad (30)$$

auxiliary function $H(p, c_i)$ is chosen as:

$$H(p, c_i) = pc_1 + p^2c_2 + \cdots \quad (31)$$

where $c_1, c_2$ are auxiliary constants.

Marina [12,13] uses a special procedure to expand $\tilde{\phi}(x, p)$ with respect to $p$ by using Taylor series.

$$\tilde{\phi}(x, p, c_i) = \tilde{u}_0(x) + \sum_{k \geq 1} u_k(x, c_i)p^k, \quad i = 1, 2, \ldots. \quad (32)$$

Inserting Eq. (32) into Eq. (28), collecting the same powers of $p$, and equating each coefficient of $p$, the zero order problem is given in Eq. (30) and the first and second order are given in Eqs. (33) and (34).

$$\tilde{L}(\tilde{u}_1(x)) + \tilde{G}(x) = c_1\tilde{N}_0(\tilde{u}_0(x)), \quad (33)$$

$$\tilde{L}(\tilde{u}_2(x)) - \tilde{L}(\tilde{u}_1(x)) = c_2\tilde{N}_0(\tilde{u}_0(x)) + c_1\left[\tilde{L}(\tilde{u}_1(x))\tilde{N}_0(\tilde{u}_0(x), \tilde{u}_1(x))\right],$$

$$\tilde{B}\left(\tilde{u}_2(x), \frac{\partial \tilde{u}_2(x)}{\partial x}\right) = 0. \quad (34)$$

The general governing equations for $\tilde{u}_k(x)$ are given by

$$\tilde{L}(\tilde{u}_k(x)) - \tilde{L}(\tilde{u}_{k-1}(x)) = c_k\tilde{N}_0(\tilde{u}_0(x)) + \sum_{i=0}^{k-1} c_i \left[\tilde{L}(\tilde{u}_{k-1}(x))\right], \quad k = 2, 3 \quad (35)$$

$$\tilde{B}\left(\tilde{u}_k(x), \frac{\partial \tilde{u}_k(x)}{\partial x}\right) = 0.$$}

Here $\tilde{N}_m(\tilde{u}_0(x), \tilde{u}_1(x), \ldots, \tilde{u}_{m-1}(x))$ is the coefficient of $p^m$, in the expansion of $\tilde{N}(\tilde{\phi}(x, p))$.

$$\tilde{N}(\tilde{\phi}(x, p, c_i)) = \tilde{N}_0(\tilde{u}_0(x)) + \sum_{m=1}^{\infty} \tilde{N}_{k-1}(u_0(x), \tilde{u}_1(x), \ldots, \tilde{u}_{m-1}(x))p^m, \quad (36)$$

the convergence of the series in Eq. (32) depend upon the convergence control parameters $c_i, c_2, \ldots$.

If it converges at $p = 1$ then the $m$th order approximation $\tilde{u}$ is

$$\tilde{u}(x, c_1, c_2, c_3, \ldots, c_m) = \tilde{u}_0(x) + \sum_{i=1}^{m} \tilde{u}_i(x, c_1, c_2, c_3, \ldots, c_i), \quad (37)$$

inserting Eq. (36) into Eq. (27), the residual is obtained as:

$$\tilde{R}(x, c_i) = \tilde{L}(\tilde{u}(x, c_i)) + \tilde{N}(\tilde{u}(x, c_i)) + \tilde{G}(x), \quad (38)$$

Numerous methods like Ritz method, method of least squares, Galerkin method and collocation method are used to find the optimal values of $c_i, i = 1, 2, 3, \ldots$.
We apply the method of least squares in our problem as given below:

\[ J(c_1, c_2, c_3, \ldots, c_m) = \int_a^b R^2(x, c_1, c_2, c_3, \ldots, c_m) dx. \]  

(39)

where \( a \) and \( b \) are the constant values taking from domain of the problem.

Convergence control parameters \( (c_1, c_2, c_3, \ldots, c_m) \) can be identified from:

\[ \frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = \frac{\partial J}{\partial c_3} = \ldots = 0. \]  

(40)

Finally, from these convergence control parameters, the approximate solution is well determined.

4.3. The ADM solution of lift problem

Rewrite Eq. (14) of the form of Eq. (18), we get

\[ L_v v(x, t) = S_t + \frac{\partial v}{\partial t} - \alpha \frac{\partial^2 v}{\partial t^2} \left( \frac{\partial^2 v}{\partial x^2} \right) - \beta_1 \frac{\partial^2 v}{\partial x^2} - 6 \beta_2 \frac{\partial^2 v}{\partial x^2} \]  

\[ \left( \frac{\partial^2 v}{\partial x^2} \right) - \gamma_1 \frac{\partial^2 v}{\partial x^2} \left( \frac{\partial^2 v}{\partial x^2} \right) - 2 \gamma_2 \left( \frac{\partial v}{\partial x} \right) \left( \frac{\partial^2 v}{\partial x^2} \right) + \left( \frac{\partial v}{\partial x} \right) \left( \frac{\partial^2 v}{\partial x^2} \right), \]  

(41)

Using the inverse operator \( L_x^{-1} \) on Eq. (33), we get

\[ L_x^{-1} L_v v(x, t) = L_x^{-1} \left( S_t + \frac{\partial v}{\partial t} - aL_x^{-1} \frac{\partial^2 v}{\partial x^2} \right) \]  

\[ -\beta_1 L_x^{-1} \left( \frac{\partial^2 v}{\partial x^2} \right) - 6 \beta_2 L_x^{-1} \left( \frac{\partial v}{\partial x} \right) \left( \frac{\partial^2 v}{\partial x^2} \right) \]  

\[ -\gamma_1 L_x^{-1} \left( \frac{\partial^2 v}{\partial x^2} \right) - 2 \gamma_2 L_x^{-1} \left( \frac{\partial v}{\partial x} \right) \left( \frac{\partial^2 v}{\partial x^2} \right), \]  

(42)

the components of velocity profile are obtained by comparing both side of Eq. (45).
Zeroth component problem:
\[ v_0(x, t) = b_0 + b_1 x + S_\gamma x^2, \]  
(46)

Solution of zeroth component problem using boundary conditions given in Eqs. (15) and (16) is:
\[ v_0(x, t) = 1 + \cos [\omega t] - \left(1 + \cos [\omega t] + \frac{S_\gamma}{2}\right)x + \left(\frac{S_\gamma}{2}\right)x^2, \]  
(47)

First component problem:
\[ v_1(x, t) = S_\gamma - aL_x^{-1}[A_0] - \beta_1 L_x^{-1}[B_0] - 6\beta_0 L_x^{-1}[C_0] - \gamma_1 L_x^{-1}[D_0] - 2\gamma L_x^{-1}[E_0], \]  
(48)

Solution of first component problem using boundary conditions given in Eqs. (15) and (16) is:
\[ v_1(x, t) = S_\gamma \left[8\gamma \cos \left[\frac{\omega t}{2}\right]^3 \sin \left[\frac{\omega t}{2}\right] - \frac{1}{35\gamma} \omega \sin [\omega t] \right] \]
\[ - 2\gamma \cos \left[\frac{\omega t}{2}\right]^4 - 2S_\gamma \cos \left[\frac{\omega t}{2}\right]^2 \]
\[ + \frac{2\gamma \omega}{3} \cos \left[\frac{\omega t}{2}\right] \sin \left[\frac{\omega t}{2}\right] - \beta S_\gamma^2 \right) \]
\[ + \left[\frac{1}{2} \omega \sin [\omega t] + 12\beta S_\gamma \right] \]
\[ - 8\gamma \omega S_\gamma \cos \left[\frac{\omega t}{2}\right]^3 \sin \left[\frac{\omega t}{2}\right] + 6\beta S_\gamma^2 \cos \left[\frac{\omega t}{2}\right]^2 \]
\[ - 2\gamma \omega S_\gamma^2 \cos \left[\frac{\omega t}{2}\right] \sin \left[\frac{\omega t}{2}\right] + \frac{3\beta S_\gamma^2}{4} \right)x^2 \]
\[ - \frac{1}{6} \omega \sin [\omega t] + 4\beta S_\gamma^2 \cos \left[\frac{\omega t}{2}\right]^2 \]
\[ - \frac{4}{3} \gamma \omega S_\gamma^2 \cos \left[\frac{\omega t}{2}\right] \sin \left[\frac{\omega t}{2}\right] + \beta S_\gamma^2 \right)x^3 \]
\[ + \left[\frac{\beta S_\gamma^3}{2}\right] x^4, \]  
(49)

Second component problem:
\[ v_2(x, t) = -L_x^{-1}\left(\frac{\partial v_1}{\partial t}\right) - aL_x^{-1}[A_1] - \beta_1 L_x^{-1}[B_1] - 6\beta_0 L_x^{-1}[C_1] - \gamma_1 L_x^{-1}[D_1] - 2\gamma L_x^{-1}[E_1], \]  
(50)

due to massive calculation the analytical result have been declared up to first order while graphical solutions are given up to second order.

4.4. OHAM solution for lift problem

Apply the standard form of OHAM to Eq. (14), we found zero, first and second components of velocity problem
\[ p^0 : \frac{\partial^2 v_0}{\partial x^2} = S_\gamma, \]  
(51)

\[ p^1 : \frac{\partial^2 v_0}{\partial x^2} = -S_\gamma + c_1 \frac{\partial v_0}{\partial x} \left(1 + c_1 + 6\beta_0 \left(\frac{\partial v_0}{\partial x}\right)^2 \right) \]
\[ + 4c_1 \gamma \left(\frac{\partial v_0}{\partial x}\right)^2 \]
\[ + c_1 \left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 \alpha + 2\gamma \left(\frac{\partial v_0}{\partial x}\right)^2 \]
\[ + c_1 \beta_1 \left(\frac{\partial^2 v_0}{\partial x^2}\right) \gamma \left(\frac{\partial^2 v_0}{\partial x^2}\right)^2. \]  
(52)

Solution of zero and first component problem using boundary conditions given in Eqs. (15) and (16) is:
\[ v_0(x, t) = 1 + \cos [\omega t] - \left(1 + \cos [\omega t] + \frac{S_\gamma}{2}\right)x + \left(\frac{S_\gamma}{2}\right)x^2, \]  
(53)

\[ v_1(x, t) = c_1 \left[ aS_\gamma \sin \left[\frac{\omega t}{2}\right] \left(8\gamma \cos \left[\frac{\omega t}{2}\right]^3 + \frac{2\gamma S_\gamma}{3} \cos \left[\frac{\omega t}{2}\right] - \frac{1}{3}\right) \right] \]
\[ - 12\beta S_\gamma \cos \left[\frac{\omega t}{2}\right]^4 - 2\beta S_\gamma^2 \cos \left[\frac{\omega t}{2}\right]^2 - \frac{3\beta S_\gamma^3}{4} \right)x \]
\[ + \frac{1}{2} \omega \sin [\omega t] \left(1 - 4\gamma S_\gamma \cos \left[\frac{\omega t}{2}\right] - 4\gamma S_\gamma^2 \cos \left[\frac{\omega t}{2}\right] \right) \]
\[ + 12\beta S_\gamma \cos \left[\frac{\omega t}{2}\right]^4 + 6\beta S_\gamma^2 \cos \left[\frac{\omega t}{2}\right]^2 + \frac{3\beta S_\gamma^3}{4} \right)x^2 \]
\[ + \frac{1}{2} \omega \sin [\omega t] \left(4\gamma S_\gamma^2 \cos \left[\frac{\omega t}{2}\right] - \frac{1}{2}\right) \]
\[ - 4\beta S_\gamma^2 \cos \left[\frac{\omega t}{2}\right]^2 - \beta S_\gamma^3 \right)x^3 + \left[\frac{\beta S_\gamma^3}{2}\right] x^4. \]  
(54)

Due to massive calculation, the analytical result have been given up to first term while graphical solutions are given up to the second order term.

The value of \( c_1 \) for the lift velocity components are
\[ c_1 = -0.0517334034, \quad c_2 = -0.0272742812. \]

5. Formulation of drainage problem

Now we consider the fourth grade fluid, falling on the stationary oscillating vertical belt. The fluid flow is due to gravity in downward direction. All the remaining assumptions are similar of the lift problem but Stock number is taken positive. Using value of \( T_{\gamma s} \) the governing equation of drainage problem become
\[ \rho \frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial x^2} + \alpha_1 \frac{\partial}{\partial x} \left(\frac{\partial^2 v}{\partial x^2}\right) + 6(\beta_2 + \beta_3) \left(\frac{\partial v}{\partial x}\right)^2 \left(\frac{\partial^2 v}{\partial x^2}\right) \]
\[ + \gamma_1 \frac{\partial^3}{\partial t^3} \left(\frac{\partial^2 v}{\partial x^2}\right) + 2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8) \]
\[ \frac{\partial}{\partial x} \left[\left(\frac{\partial v}{\partial x}\right)^2 \frac{\partial \left(\frac{\partial v}{\partial x}\right)}{\partial x^2}\right] + \rho g. \]  
(55)
Along with drainage boundary conditions
\[ v(x, t) = U \cos \omega t, \quad \text{at} \quad x = 0, \quad (56) \]
\[ \frac{\partial v(x, t)}{\partial x} = 0, \quad \text{at} \quad x = \delta. \quad (57) \]

The dimensionless form of Eq. (55) is given as:
\[
\frac{\partial v}{\partial t} = S_t + \frac{\partial^2 v}{\partial x^2} + \alpha \frac{\partial}{\partial t} \left( \frac{\partial^2 v}{\partial x^2} \right) + \beta_1 \frac{\partial^3}{\partial t^3} \left( \frac{\partial^2 v}{\partial x^2} \right)
+ 6\beta \left( \frac{\partial}{\partial x} \right)^2 \left( \frac{\partial^2 v}{\partial x^2} \right) + \gamma_1 \frac{\partial^3}{\partial t^3} \left( \frac{\partial^2 v}{\partial x^2} \right)
+ 2\gamma \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right)^2 \left( \frac{\partial^2 v}{\partial x^2} \right),
\]
with boundary conditions regarding in Eqs. (56) and (57)
\[ v(x, t) = \cos \omega t, \quad \text{at} \quad x = 0, \quad (59) \]
\[ \frac{\partial v(x, t)}{\partial x} = 0, \quad \text{at} \quad x = 1. \quad (60) \]

5.1. The ADM solution of drainage problem

From Eq. (58), we are applying the modified adomian decomposition method (MADM). For both lifting and drainage problems the modified adomian polynomial are same. After simplification zeroth, first and second velocity components problem are.

Zeroth component problem:
\[ v_0(x, t) = b_0 + b_1 x - S_t \frac{x^2}{2}. \quad (61) \]

Solution of zeroth component problem using boundary conditions in Eqs. (52) and (53) is:
\[ v_0(x, t) = \cos [\omega t] - \left( 1 + \cos [\omega t] + S_t \frac{S_t}{2} \right) x - \left( S_t \frac{S_t}{2} \right) x^2. \quad (62) \]

5.2. OHAM solution of drainage problem

Apply the standard form of OHAM on Eq. (58), we found zero, first and second components of velocity problem.

Zero component problem:
\[ p^0: \frac{\partial^2 v_0}{\partial x^2} = -S_t, \quad (65) \]
First component problem:

\[ p^1 : \frac{\partial^2 v_1}{\partial x^2} = S_i + c_1 S_i - c_1 \frac{\partial v_0}{\partial t} + \frac{\partial^2 v_0}{\partial x^2} - 1 + 6 \beta c_1 \left( \frac{\partial v_0}{\partial x} \right)^2 + 4 c_1 \frac{\partial}{\partial t} \left( \frac{\partial^2 v_0}{\partial x^2} \right) + \alpha + 2 \gamma \left( \frac{\partial v_0}{\partial x} \right)^2 + c_1 \beta_1 \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 v_0}{\partial x^2} \right) + \gamma \frac{\partial^3}{\partial x^3} \left( \frac{\partial^2 v_0}{\partial x^2} \right). \quad (66) \]

Solution of zeroth and first component problem using boundary conditions given in Eqs. (59) and (60):

\[ v_0(x,t) = \cos [\omega t] - \left( 1 + \cos [\omega t] \frac{S_i}{2} \right) x - \left( \frac{S_i}{2} \right)^2. \quad (67) \]

\[ v_1(x,t) = c_1 \left[ \frac{1}{3} \alpha \frac{\omega}{\sin [\omega t]} \left( \gamma S_i^2 - 1 - 6 \gamma S_i \cos [\omega t] \right) + 3 \beta S_i \cos [\omega t]^2 - 3 \beta S_i^2 \cos [\omega t] + \frac{3 \beta S_i^3}{4} \right] x + \left[ \frac{\omega}{\sin [\omega t]} \left( \frac{1}{2} - 2 \gamma S_i^2 + 2 \gamma S_i \cos [\omega t] \right) - 3 \beta S_i \cos [\omega t]^2 + 3 \beta S_i^2 \cos [\omega t] + \frac{3 \beta S_i^3}{4} \right] x^2 + \left[ \frac{1}{3} \frac{\omega}{\sin [\omega t]} \left( 2 \gamma S_i^2 - \frac{1}{2} - 2 \gamma S_i \cos [\omega t] \right) + \beta S_i^3 \right] x^3. \quad (68) \]

Due to massive calculation the analytical result have been given up to first term while graphical solutions are given up to second order.

The value of \( c_i \) for the drainage velocity components are:

\[ c_1 = -0.2661918234, \quad c_2 = -0.2382524332. \]
6. Results and discussion

Analytical solutions for unsteady thin film flow of a fourth grade fluid through a moving and oscillating vertical belt are obtained. The physical configuration of the problem is shown in Figures 1 and 2. Two different techniques namely ADM and OHAM are used for the solutions of lift and drainage problems. The results of velocity obtained from both methods for lift and drainage problems are compared numerically in Tables 1 and 2 and graphically in Figures 3 and 4 for different values of embedded parameters. Whereas, the effects of other physical parameters \( t, S_i, \beta, \gamma, \beta_1 \) and \( \gamma_1 \) on velocity profile for both lift and drainage problems are studied in Figures 5–18. All results for the thin film of fourth grade fluid near the belt are illustrated in the \( x \)-coordinate only for a selected domain \( x \in [0,1] \). In Figures 5–8, the velocity profiles are studied versus time for different values of independent variable \( x \). Due to the no-slip condition, the fluid near the belt oscillates

![Figure 9](image-url)  
Figure 9  Effect of non-Newtonian parameter \( \beta_1 \) on lift velocity profile when \( \omega = 5, \ \alpha = 0.2, \ \beta = 0.4, \ \gamma = 0.2, \ \gamma_1 = 0.2, \ S_i = 0.1 \).

![Figure 10](image-url)  
Figure 10  Effect of \( \beta_1 \) on drainage velocity when \( \omega = 5, \ \alpha = 0.2, \ \beta = 0.4, \ \gamma = 0.2, \ \gamma_1 = 0.2, \ S_i = 0.1 \).

![Figure 11](image-url)  
Figure 11  Effect of third grade parameter \( \beta \) on lift velocity when \( \omega = 5, \ \alpha = 0.2, \ \beta_1 = 0.7, \ \gamma = 2, \ \gamma_1 = 0.2, \ S_i = 0.2 \).

![Figure 12](image-url)  
Figure 12  Effect of \( \beta \) on drainage velocity when \( \omega = 5, \ \alpha = 0.2, \ \beta_1 = 0.7, \ \gamma = 2, \ \gamma_1 = 0.2, \ S_i = 0.2 \).

![Figure 13](image-url)  
Figure 13  Effect of fourth grade parameter \( \gamma \) on lift velocity when \( \omega = 5, \ \alpha = 0.2, \ \beta_1 = 0.7, \ t = 3, \ \gamma_1 = 0.2, \ S_i = 0.2 \).
jointly with the belt in the same period and oscillation reduces gradually towards the free surface shown in Figures 5 and 6. Increasing domain from (0,1) velocity amplitude raises gradually towards the surface of the fluid layer shown in Figures 7 and 8. It is found that velocity accepts and oscillating behavior. Generally, due to larger apparent viscosity, non-Newtonian fluid exhibits much thicker boundary layer than that of Newtonian fluid. With the increase in fourth grade parameters the apparent viscosity of the fluid become more bulky (viscous forces dominate), then the flow simultaneously adjust to the present driving force and oscillate closely with the similar phase in the whole flow domain.

Therefore, increasing non-Newtonian parameters α, β, γ, β₁ and γ₁ of second, third and fourth grade fluids causes further thickening of the boundary layer. So, increase in these higher order parameter increases velocity profiles of both lift and drainage problems shown in Figures 9–16.

Figures 17 and 18 display the effects of St on lift and drainage velocities. From Figure 17, we observed that by increasing the value of St the lift velocity decreases whereas Figure 18 shows that drainage velocity increases by increasing the St. The reason is that the friction force cause the effect of gravity and it seems to be smaller near the belt. While it increasing gradually towards the free surface.

7. Conclusion

In this article, we consider the unsteady flow of fourth grade fluid on moving and oscillating vertical belt. The analytical solution of lift and drainage velocity field achieved by applying two techniques namely ADM and OHAM. The result obtained from both techniques compared numerically
and graphically. The result of ADM and OHAM close associated to each other. The effects of model parameters are presented graphically on lift and drainage velocity field.

1. If the apparent viscosity of the fluid is more bulky (viscous forces dominate), then the flow simultaneously adjust to the present driving force and oscillate closely with the similar phase in the whole flow domain.

2. Due to oscillation of the belt, we observed that the fluid motion is maximum at the surface of the belt and minimum at the surface of the fluid.

3. The higher order parameters are more efficient in the increase of fluid motion as one go from belt towards free surface.

References


Unsteady thin film flow of a fourth grade fluid over a vertical moving and oscillating belt


