International Journal of Solids and Structures 46 (2009) 2477–2480

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

On vectorially parameterized natural strain measures of the non-linear Cosserat continuum

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ARTICLE INFO

Article history:
Received 12 November 2008
Received in revised form 20 January 2009
Available online 7 February 2009

Keywords:
Cosserat continuum
Micropolar continuum
Finite rotation vector
Strain measures
Wryness tensor

ABSTRACT

The natural Lagrangian stretch and wryness tensors of the non-linear Cosserat continuum are expressed in terms of the general finite rotation vector. These expressions are then specialized for seven particular definitions of the rotation vectors known in the literature. It is expected that some of the vectorially parameterized strain measures derived here may be more convenient than others in specific applications.

1. Introduction

In the recent paper by Pietraszkiewicz and Eremeyev (2009) we applied three different ways of defining the strain measures of the non-linear Cosserat continuum. We found in particular that the most natural definitions for the Lagrangian relative stretch $\mathbf{E}$ and wryness (or change of the microstructure curvature) $\boldsymbol{\Gamma}$ tensors are

$$\mathbf{E} = \mathbf{Q}^T(\mathbf{I} + \operatorname{Grad}\mathbf{u}) - \mathbf{I}, \quad \boldsymbol{\Gamma} = -\frac{1}{2}\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{Grad}\mathbf{Q}). \tag{1}$$

Here $\mathbf{u} \in E$ is the translation vector, $\mathbf{Q} \in SO(3)$ the proper orthogonal microrotation tensor, $\mathbf{I}$ the identity (metric) tensor in the undeformed configuration, $\boldsymbol{\epsilon} = -\mathbf{I} \times \mathbf{I}$ the third-order skew Ricci tensor, $\times$ the vector product, and the double dot product $:$ of two third-order tensors $\mathbf{A}, \mathbf{B}$ represented in the orthonormal base $\mathbf{h}_a$, $a = 1, 2, 3$, is defined as $\mathbf{A} : \mathbf{B} = A_{amn}B_{mnb}\mathbf{h}_a \otimes \mathbf{h}_b$.

The orthonormal vectors $\mathbf{h}_a$ were interpreted by Pietraszkiewicz and Eremeyev (2009) as the natural base vectors of three-orthogonal system of curvilinear arc-length coordinates s_a such that $\mathbf{h}_a = \partial\mathbf{x}/\partial s_a \equiv \mathbf{x}_{,a}$, where $\mathbf{x} \in E$ is the position vector of a material particle in the reference configuration of the body. Then gradients of the vector $\mathbf{v}(\mathbf{x}) \in E$ and second-order tensor $\mathbf{T}(\mathbf{x}) \in E \otimes E$ fields were defined by $\operatorname{Grad}\mathbf{v} = \mathbf{v}_{,a} \otimes \mathbf{h}_a$ and $\operatorname{Grad}\mathbf{T} = \mathbf{T}_{,a} \otimes \mathbf{h}_a$, respectively.

While three components of $\mathbf{u}$ in (1) are all independent, nine components of $\mathbf{Q}$ in (1) are subject to six constraints following from the orthogonality conditions $\mathbf{Q}^{-1} = \mathbf{Q}^T$, $\det\mathbf{Q} = +1$, so that only three rotational parameters of $\mathbf{Q}$ are independent.

In the literature, many techniques how to parameterize the rotation group $SO(3)$ were developed, see for example Rooney (1977), Guo (1981), Pietraszkiewicz and Badur (1983), Altman (1986), Atluri and Cazzani (1995), Borri et al. (2000), Geradin and Cardona (2001) and Chróścielewski et al. (2004). These parameterizations can roughly be classified as vectorial and non-vectorial ones. Various finite rotation vectors as well as the Cayley–Gibbs and exponential map parameters are examples of the vectorial parameterization, for they all have three independent scalar parameters as Cartesian components of a generalized vector in the 3D vector space E. The non-vectorial parameterizations are expressed either in terms of three scalar parameters that cannot be treated as vector components, such as Euler-type angles for example, or through more scalar parameters subject to additional constraints, such as unit quaternions, Cayley–Klein parameters, or direction cosines. Each of these parameterizations has some advantages and drawbacks widely discussed in the literature.

The aim of this note is to express the strain measures (1) in terms of seven different vectorial parameters proposed in the literature. Each of these expressions may appear to be more convenient than others when solving specific problems of the non-linear Cosserat continuum.

2. The vectorial parameterization

The microrotation tensor $\mathbf{Q}$ represents the isometric and orientation-preserving transformation of the 3D vector space E into

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0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2009.01.030

itself. By the Euler theorem such a transformation can be expressed in terms of the angle of rotation ϕ about the axis of rotation described by the eigenvector $\mathbf{e}$ corresponding to the real eigenvalue +1 of $\mathbf{Q}$ such that

$$\mathbf{Qe} = +\mathbf{e}, \quad \cos\phi = \frac{1}{2}(\mathrm{tr}\mathbf{Q} - 1), \quad \sin\phi\mathbf{e} = \frac{1}{2}\mathrm{ax}(\mathbf{Q} - \mathbf{Q}^T), \tag{2}$$

where tr $\mathbf{A}$ is the trace of the second-order tensor $\mathbf{A}$, and ax $\mathbf{W}$ is the axial vector $\mathbf{w}$ of the skew second-order tensor $\mathbf{W}$ such that $\mathbf{W} = \mathbf{w} \times \mathbf{I} = \mathbf{I} \times \mathbf{w}$.

In terms of $\mathbf{e}$ and ϕ the microrotation tensor $\mathbf{Q}$ can be expressed by the Gibbs (1901) formula, see for example Beatty (1977), Guo (1981) and Pietraszkiewicz and Badur (1983),

$$\mathbf{Q} = \cos\phi\mathbf{I} + (1 - \cos\phi)\mathbf{e} \otimes \mathbf{e} + \sin\phi\mathbf{e} \times \mathbf{I}. \tag{3}$$

In the vectorial parameterization of $\mathbf{Q}$ one introduces a scalar function $p(\phi)$ generating three components of the finite rotation vector $\mathbf{p}$ defined as, see for example Bauchau and Trainelli (2003),

$$\mathbf{p} = p(\phi)\mathbf{e}. \tag{4}$$

The generating function $p(\phi)$ in (4) has to be an odd function of ϕ with the limit behaviour $\lim_{\phi\to 0}\frac{p(\phi)}{\phi} = \kappa$, where κ is a positive real normalization factor (usually 1 or $\frac{1}{2}$), and $p(0) = 0$. In terms of (4) the tensor $\mathbf{Q}$ and its transpose can be represented as

$$\begin{aligned}\mathbf{Q} &= \cos\phi\mathbf{I} + \frac{1 - \cos\phi}{p^2}\mathbf{p} \otimes \mathbf{p} + \frac{\sin\phi}{p}\mathbf{p} \times \mathbf{I},\\ \mathbf{Q}^T &= \cos\phi\mathbf{I} + \frac{1 - \cos\phi}{p^2}\mathbf{p} \otimes \mathbf{p} - \frac{\sin\phi}{p}\mathbf{p} \times \mathbf{I}.\end{aligned} \tag{5}$$

The finite rotation vector (4) is the generalized vector. The composition of two successive rotations $\mathbf{Q}_3 = \mathbf{Q}_2\mathbf{Q}_1$, when expressed in terms of the corresponding vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ with angles of rotation ϕ_1, ϕ_2, ϕ_3, reads

$$\begin{aligned}\cos\frac{\phi_3}{2} &= \cos\frac{\phi_1}{2}\cos\frac{\phi_2}{2} - \frac{\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}}{p_1 p_2}\mathbf{p}_1 \cdot \mathbf{p}_2,\\ \frac{\sin\frac{\phi_3}{2}}{p_3}\mathbf{p}_3 &= \frac{\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}}{p_1 p_2}\left(\frac{p_2}{\cos\frac{\phi_2}{2}}\mathbf{p}_1 + \frac{p_1}{\cos\frac{\phi_1}{2}}\mathbf{p}_2 - \mathbf{p}_1 \times \mathbf{p}_2\right).\end{aligned} \tag{6}$$

Eq. $(6)_1$ is used to compute ϕ_3, which also gives $\sin\frac{\phi_3}{2}$ and $p_3 = p(\phi_3)$. Then $(6)_2$ allows one to establish the vector $\mathbf{p}_3$.

Since $\mathbf{Q}^T\mathbf{Q}_{,a} = -(\mathbf{Q}^T\mathbf{Q}_{,a})^T$ is skew it can be expressed through the axial vector γ_a,

$$\begin{aligned}\mathbf{Q}^T\mathbf{Q}_{,a} = \gamma_a \times \mathbf{I}, \quad \gamma_a &= -\frac{1}{2}\epsilon : (\mathbf{Q}^T\mathbf{Q}_{,a})\\ &= \phi_{,a}\mathbf{e} + [\sin\phi\mathbf{I} - (1 - \cos\phi)\mathbf{e} \times \mathbf{I}]\mathbf{e}_{,a}.\end{aligned} \tag{7}$$

The vector γ_a describes the change of the reference microstructure curvature of the Cosserat continuum along the arc-length coordinate line s_a. It is analogous to the vector $\mathbf{k}_j$ of change of curvature of the curvilinear coordinate line θ^j in classical continuum mechanics defined as $\mathbf{R}^T\mathbf{R}_{,j} = \mathbf{k}_j \times \mathbf{I}$ by Pietraszkiewicz and Badur (1983), where $\mathbf{R}$ was the rotation tensor following from the polar decomposition $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$. But in the Cosserat continuum $\mathbf{Q}$ is the independent field not related to $\mathbf{u}$ and therefore $\mathbf{Q} \neq \mathbf{R}$, in general.

Differentiating the vector $\mathbf{p}$ in (4) along the coordinate line s_a we obtain the transformation relations

$$\phi_{,a} = \frac{1}{p'}p_{,a}, \quad \mathbf{e}_{,a} = -\frac{1}{p^2}p_{,a}\mathbf{p} + \frac{1}{p}\mathbf{p}_{,a}, \quad p' = \frac{dp}{d\phi}, \tag{8}$$

which introduced into (7) lead to

$$\gamma_a = \frac{1}{p}\left(\frac{1}{p'} - \frac{\sin\phi}{p}\right)p_{,a}\mathbf{p} + \frac{\sin\phi}{p}\mathbf{p}_{,a} - \frac{1 - \cos\phi}{p^2}\mathbf{p} \times \mathbf{p}_{,a}. \tag{9}$$

Taking into account that $\mathbf{p} \cdot \mathbf{p}_{,a} = pp_{,a}$, we have the identities

$$\mathbf{p} = \frac{1}{pp_{,a}}(\mathbf{p} \otimes \mathbf{p})\mathbf{p}_{,a}, \quad \mathbf{p}_{,a} = \mathbf{I}\mathbf{p}_{,a}, \quad \mathbf{p} \times \mathbf{p}_{,a} = (\mathbf{p} \times \mathbf{I})\mathbf{p}_{,a}, \tag{10}$$

and the relation (9) can be given in the equivalent form

$$\gamma_a = \mathbf{A}\mathbf{p}_{,a}, \quad \mathbf{A} = \frac{\sin\phi}{p}\mathbf{I} + \frac{1}{p^2}\left(\frac{1}{p'} - \frac{\sin\phi}{p}\right)\mathbf{p} \otimes \mathbf{p} - \frac{1 - \cos\phi}{p^2}\mathbf{p} \times \mathbf{I}. \tag{11}$$

Substituting $(5)_2$ and (11) into (1), the natural Lagrangian stretch $\mathbf{E}$ and wryness $\mathbf{\Gamma}$ tensors can now be represented in terms of the finite rotation vector $\mathbf{p}$ by the general relations

$$\mathbf{E} = \left(\cos\phi\mathbf{I} + \frac{1 - \cos\phi}{p^2}\mathbf{p} \otimes \mathbf{p} - \frac{\sin\phi}{p}\mathbf{p} \times \mathbf{I}\right)(\mathbf{I} + \mathrm{Grad}\,\mathbf{u}) - \mathbf{I}, \tag{12}$$

$$\mathbf{\Gamma} = \left[\frac{\sin\phi}{p}\mathbf{I} + \frac{1}{p^2}\left(\frac{1}{p'} - \frac{\sin\phi}{p}\right)\mathbf{p} \otimes \mathbf{p} - \frac{1 - \cos\phi}{p^2}\mathbf{p} \times \mathbf{I}\right]\mathrm{Grad}\,\mathbf{p}. \tag{13}$$

3. Particular finite rotation vectors

Among definitions of $\mathbf{p}$ used most often in the literature let us mention the finite rotation vectors defined as

$$\boldsymbol{\theta} = 2\tan\frac{\phi}{2}\mathbf{e}, \quad \boldsymbol{\phi} = \phi\mathbf{e}, \quad \boldsymbol{\varpi} = \sin\phi\mathbf{e}, \quad \boldsymbol{\rho} = \tan\frac{\phi}{2}\mathbf{e}, \tag{14}$$

where the generating functions are $\theta = 2\tan\frac{\phi}{2}$, $\phi, \varpi = \sin\phi$, and $\rho = \tan\frac{\phi}{2}$, respectively. Within the non-linear Cosserat continuum the Cayley–Gibbs vector $\boldsymbol{\theta}$ was used for example by Shkutin (1980), Badur and Pietraszkiewicz (1986), Zubov (1997) and Nikitin and Zubov (1998), while the linear vector $\boldsymbol{\phi}$ (called also the exponential map) by Kafadar and Eringen (1971), Nistor (2002) and Ramezani and Naghdabadi (2007). The vector $\boldsymbol{\theta}$ was used in the non-linear theory of plates, see for example Hodges et al. (1993), and in the non-linear theory of composite beams by Hodges (2006), where the extensive review of the literature was given. In the non-linear theory of Cosserat-type shells and the classical continuum mechanics the vector $\boldsymbol{\varpi}$ was found to be convenient in papers by Pietraszkiewicz (1979) and Pietraszkiewicz and Badur (1983), while the Rodrigues rotation vector $\boldsymbol{\rho}$ was willingly used in analytical mechanics of rigid-body motion, see for example Pars (1965).

Less popular in the literature till now is the Euler–Rodrigues vector $\boldsymbol{\sigma}$, the Wiener–Milenkovic vector $\boldsymbol{\mu}$, and the Bauchau–Trainelli vector $\boldsymbol{\beta}$ defined by

$$\boldsymbol{\sigma} = 2\sin\frac{\phi}{2}\mathbf{e}, \quad \boldsymbol{\mu} = 4\tan\frac{\phi}{4}\mathbf{e}, \quad \boldsymbol{\beta} = 4\sin\frac{\phi}{4}\mathbf{e}, \tag{15}$$

whose generating functions are $\sigma = 2\sin\frac{\phi}{2}$, $\mu = 4\tan\frac{\phi}{4}$, and $\beta = 4\sin\frac{\phi}{4}$, respectively.

Introducing (14) and (15) into (12) and (13) and using appropriate trigonometric identities, after complex but elementary transformations we obtain the formulae for $\mathbf{E}$ and $\mathbf{\Gamma}$ expressed in terms of the corresponding finite rotation vectors. These formulae are given in Tables 1 and 2.

With all the vectorial parameterizations the singularities occur for some values of ϕ following from singularities of the generating functions $p(\phi)$, when $p \to \infty$, from singularities of the inverse relations $\mathbf{p} = \mathbf{p}(\mathbf{Q})$, as well as from singularities of $\mathbf{A}$ and $\mathbf{A}^{-1}$, see Bauchau and Trainelli (2003). Hence, we also indicate in Tables 1 and 2 the ranges of validity of ϕ for the analysis to be singular-free while using these strain measures in problems of the non-linear Cosserat continuum. When in applications there appear arbitrary values of the rotation angle ϕ, one needs at least five independent scalar parameters to parameterize the rotation group $SO(3)$ in the globally one-to-one and singular-free manner, see for example Hopf (1940), Stuelpnagel (1964) and Perelyaev (2006). For the finite rotation vectors $\boldsymbol{\mu}$ and $\boldsymbol{\beta}$, Bauchau and Trainelli (2003)

Table 1
The natural Lagrangian stretch tensor for different finite rotation vectors.

$\mathbf{p}$	$\phi \in$	$\mathbf{E}$
$\boldsymbol{\theta} \equiv 2\tan\frac{\phi}{2}\mathbf{e}$	$(-\pi, \pi)$	$\frac{1}{1+\frac{\theta^2}{4}}\left[\left(1-\frac{\theta^2}{4}\right)\mathbf{I}+\frac{1}{2}\boldsymbol{\theta}\otimes\boldsymbol{\theta}-\boldsymbol{\theta}\times\mathbf{I}\right](\mathbf{I}+\mathrm{Grad}\,\mathbf{u})-\mathbf{I}$
$\boldsymbol{\phi} \equiv \phi\mathbf{e}$	$(-2\pi, 2\pi)$	$\left(\cos\phi\mathbf{I}+\frac{1-\cos\phi}{\phi^2}\boldsymbol{\phi}\otimes\boldsymbol{\phi}-\frac{\sin\phi}{\phi}\boldsymbol{\phi}\times\mathbf{I}\right)(\mathbf{I}+\mathrm{Grad}\,\mathbf{u})-\mathbf{I}$
$\boldsymbol{\varpi} \equiv \sin\phi\mathbf{e}$	$(-\pi, \pi)$	$\left(\cos\phi\mathbf{I}+\frac{1-\cos\phi}{\varpi^2}\boldsymbol{\varpi}\otimes\boldsymbol{\varpi}-\boldsymbol{\varpi}\times\mathbf{I}\right)(\mathbf{I}+\mathrm{Grad}\,\mathbf{u})-\mathbf{I}$
$\boldsymbol{\rho} \equiv \tan\frac{\phi}{2}\mathbf{e}$	$(-\pi, \pi)$	$\frac{1}{1+\rho^2}[(1-\rho^2)\mathbf{I}+2\boldsymbol{\rho}\otimes\boldsymbol{\rho}-2\boldsymbol{\rho}\times\mathbf{I}](\mathbf{I}+\mathrm{Grad}\,\mathbf{u})-\mathbf{I}$
$\boldsymbol{\sigma} \equiv 2\sin\frac{\phi}{2}\mathbf{e}$	$(-\pi, \pi)$	$\left[\left(1-\frac{1}{2}\sigma^2\right)\mathbf{I}+\frac{1}{2}\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}-\cos\frac{\phi}{2}\boldsymbol{\sigma}\times\mathbf{I}\right](\mathbf{I}+\mathrm{Grad}\,\mathbf{u})-\mathbf{I}$
$\boldsymbol{\mu} \equiv 4\tan\frac{\phi}{4}\mathbf{e}$	$(-2\pi, 2\pi)$	$\frac{1}{\left(1+\frac{\mu^2}{16}\right)^2}\left\{\left[1-\frac{\mu^2}{16}\left(\frac{3}{8}-\frac{\mu^2}{16}\right)\right]\mathbf{I}+\frac{1}{2}\boldsymbol{\mu}\otimes\boldsymbol{\mu}-(1-\frac{\mu^2}{16})\boldsymbol{\mu}\otimes\mathbf{I}\right\}(\mathbf{I}+\mathrm{Grad}\,\mathbf{u})-\mathbf{I}$
$\boldsymbol{\beta} \equiv 4\sin\frac{\phi}{4}\mathbf{e}$	$(-2\pi, 2\pi)$	$\left\{\left[1-\frac{\beta^2}{2}\left(1-\frac{\beta^2}{16}\right)\right]\mathbf{I}+\frac{1}{2}\left(1-\frac{\beta^2}{8}\right)\boldsymbol{\beta}\otimes\boldsymbol{\beta}-\sqrt{1-\frac{\beta^2}{16}}\left(1-\frac{\beta^2}{8}\right)\boldsymbol{\beta}\times\mathbf{I}\right\}(\mathbf{I}+\mathrm{Grad}\,\mathbf{u})-\mathbf{I}$

Table 2
The natural Lagrangian wryness tensor for different finite rotation vectors.

$\mathbf{p}$	$\phi \in$	$\boldsymbol{\Gamma}$
$\boldsymbol{\theta} \equiv 2\tan\frac{\phi}{2}\mathbf{e}$	$(-\pi, \pi)$	$\frac{1}{1+\frac{\theta^2}{4}}\left(\mathbf{I}-\frac{1}{2}\boldsymbol{\theta}\times\mathbf{I}\right)\mathrm{Grad}\,\boldsymbol{\theta}$
$\boldsymbol{\phi} \equiv \phi\mathbf{e}$	$(-2\pi, 2\pi)$	$\left(\frac{\sin\phi}{\phi}\mathbf{I}+\frac{\phi-\sin\phi}{\phi^3}\boldsymbol{\phi}\otimes\boldsymbol{\phi}-\frac{1-\cos\phi}{\phi^2}\boldsymbol{\phi}\times\mathbf{I}\right)\mathrm{Grad}\,\boldsymbol{\phi}$
$\boldsymbol{\varpi} \equiv \sin\phi\mathbf{e}$	$(-\pi, \pi)$	$\left[\mathbf{I}+\frac{1}{\varpi^2}\left(\frac{1}{\cos\phi}-1\right)\boldsymbol{\varpi}\otimes\boldsymbol{\varpi}-\frac{1-\cos\phi}{\varpi^2}\boldsymbol{\varpi}\times\mathbf{I}\right]\mathrm{Grad}\,\boldsymbol{\varpi}$
$\boldsymbol{\rho} \equiv \tan\frac{\phi}{2}\mathbf{e}$	$(-\pi, \pi)$	$\frac{2}{1+\rho^2}(\mathbf{I}-\boldsymbol{\rho}\times\mathbf{I})\mathrm{Grad}\,\boldsymbol{\rho}$
$\boldsymbol{\sigma} \equiv 2\sin\frac{\phi}{2}\mathbf{e}$	$(-\pi, \pi)$	$\left(\cos\frac{\phi}{2}\mathbf{I}-\frac{1}{4\cos\frac{\phi}{2}}\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}-\frac{1}{2}\boldsymbol{\sigma}\times\mathbf{I}\right)\mathrm{Grad}\,\boldsymbol{\sigma}$
$\boldsymbol{\mu} \equiv 4\tan\frac{\phi}{4}\mathbf{e}$	$(-2\pi, 2\pi)$	$\frac{1}{(1+\frac{\mu^2}{16})^2}\left[\left(1-\frac{\mu^2}{16}\right)\mathbf{I}+\frac{1}{8}\boldsymbol{\mu}\otimes\boldsymbol{\mu}-\frac{1}{2}\boldsymbol{\mu}\times\mathbf{I}\right]\mathrm{Grad}\,\boldsymbol{\mu}$
$\boldsymbol{\beta} \equiv 4\sin\frac{\phi}{4}\mathbf{e}$	$(-2\pi, 2\pi)$	$\left[\sqrt{1-\frac{\beta^2}{16}}(1-\frac{\beta^2}{8})\mathbf{I}+\frac{1-\left(1-\frac{\beta^2}{8}\right)\left(1-\frac{\beta^2}{16}\right)}{\beta^2\sqrt{1-\frac{\beta^2}{16}}}\boldsymbol{\beta}\otimes\boldsymbol{\beta}-\frac{1}{2}(1-\frac{\beta^2}{16})\boldsymbol{\beta}\times\mathbf{I}\right]\mathrm{Grad}\,\boldsymbol{\beta}$

described procedures how to handle arbitrary rotations by combining appropriate update and rescaling operations.

With the vectors $\boldsymbol{\theta}, \boldsymbol{\rho}, \boldsymbol{\mu}$, or $\boldsymbol{\beta}$ the formulae for $\mathbf{E}, \boldsymbol{\Gamma}$ in Tables 1 and 2 do not contain any trigonometric expressions of ϕ. This might suggest some convenience in further purely algebraic transformations. With the vectors $\boldsymbol{\phi}, \boldsymbol{\mu}$, or $\boldsymbol{\beta}$ the formulae for $\mathbf{E}, \boldsymbol{\Gamma}$ have broader range of singular-free behaviour. When $|\phi| < \pi$ the values of $\mu(\phi)$ and $\beta(\phi)$ are not much different from ϕ, that is $\mu(\phi) \approx \phi \approx \beta(\phi)$. In the limit the sin-type generating functions ϖ, σ, β converge to ϕ from below, while the tan-type ones θ, ρ, μ, from above.

When the values of $\mathbf{u}$ and ϕ as well as their spatial gradients are infinitesimal

$$\|\mathbf{u}\| \ll 1, \quad \|\mathrm{Grad}\,\mathbf{u}\| \ll 1, \quad |\phi| \ll 1, \quad \|\mathrm{Grad}\,\phi\| \ll 1,$$

we also have $\sin\phi \approx \phi$, $\cos\phi \approx 1$, and $p(\phi) \approx \kappa\phi$. Then from (3), (15) and (14) it follows that

$$\mathbf{p} \approx \kappa\boldsymbol{\vartheta}, \quad \mathbf{Q} \approx \mathbf{I} + \boldsymbol{\vartheta} \times \mathbf{I},$$

where $\boldsymbol{\vartheta} = \phi\mathbf{e}$ is now the infinitesimal rotation vector. Then from (12) and (13) we obtain

$$\mathbf{E} \approx \boldsymbol{\varepsilon} \equiv \mathrm{Grad}\,\mathbf{u} - \boldsymbol{\vartheta} \times \mathbf{I}, \quad \boldsymbol{\Gamma} \approx \boldsymbol{\gamma} \equiv \mathrm{Grad}\,\boldsymbol{\vartheta}. \tag{16}$$

The infinitesimal strain measures $\boldsymbol{\varepsilon}$, $\boldsymbol{\gamma}$ or their transpose were used in many papers and books on the linear theory of the Cosserat continuum. Let us mention here the books by Kröner (1968), Nowacki (1986), Eringen (1999) and Dyszlewicz (2004), where many references to other papers can be found.

4. Conclusions

Within the non-linear Cosserat continuum, introduction of the finite rotation vector gives the possibility to formulate the boundary-value problem in terms of displacement and finite rotation vectors as the primary unknown variables. In this note the natural Lagrangian stretch and wryness tensors derived by Pietraszkiewicz and Eremeyev (2009) have been expressed in terms of the general finite rotation vector. These expressions have then been specialized for seven different definitions of the rotation vectors known in the literature. Each of the particular forms of the strain measures has some advantages and drawbacks, and each of them may be more convenient than others in specific applications.

References

Altman, S.L., 1986. Rotations, Quaternions, and Double Groups. Clarendon Press, Oxford.

Atluri, S.N., Cazzani, A., 1995. Rotations in computational solid mechanics. Achieves of Computational Mechanics and Engineering 2 (1), 49–138.

Badur, J., Pietraszkiewicz, W., 1986. On geometrically non-linear theory of elastic shells derived from pseudo-Cosserat continuum with constrained micro-rotations. In: Pietraszkiewicz, W. (Ed.), Finite Rotations in Structural Mechanics. Springer, Wien, pp. 19–32.

Bauchau, O.A., Trainelli, L., 2003. The vectorial parameterization of rotation. Nonlinear Dynamics 32 (1), 71–92.

Beatty, M.F., 1977. Vector analysis of finite rigid rotations. Journal of Applied Mechanics, Transactions of ASME E44 (3), 501–502.

Borri, M., Trainelli, L., Bottasso, C.L., 2000. On representations and parameterizations of motion. Multibody System Dynamics 4 (2–3), 129–193.

Chróścielewski, J., Makowski, J., Pietraszkiewicz, W., 2004. Statyka i dynamika powłok wielopłatowych. Nieliniowa teoria i metoda elementów skończonych. Wydawnictwo IPPT PAN, Warszawa.

Dyszlewicz, J., 2004. Micropolar Theory of Elasticity. Springer, Berlin.

Eringen, A.C., 1999. Microcontinuum Field Theory. I. Foundations and Solids. Springer, New York.

Geradin, M., Cardona, A., 2001. Flexible Multibody Dynamics: A Finite Element Approach. Wiley, Chichester.

Gibbs, J.W., 1901. Vector Analysis. Yale University Press, New Haven.

Guo, Z.-H., 1981. Representations of orthogonal tensors. Solid Mechanics Archives 6 (4), 451–466.

Hodges, D.H., 2006. Nonlinear Composite Beam Theory. AIAA, Washington, DC.

Hodges, D.H., Atilgan, A.R., Danielson, D.A., 1993. A geometrically nonlinear theory of elastic plates. Journal of Applied Mechanics 60 (1), 109–116.

Hopf, H., 1940. Systeme symmetrischer Bilinearformen und euklidische Modelle projektiven Räume. Vierteljahresschrift Naturforschung Gesellschaft Zürich 85 (32), 165–177.

Kafadar, C.B., Eringen, A.C., 1971. Micropolar media – I. The classical theory. International Journal of Engineering Science 9 (3), 271–305.

Kröner, E., 1968. Mechanics of Generalized Continua. Springer, Berlin.

Nikitin, E., Zubov, L.M., 1998. Conservation laws and conjugate solutions in the elasticity of simple materials and materials with couple stress. Journal of Elasticity 51 (1), 1–22.

Nistor, I., 2002. Variational principles for Cosserat bodies. International Journal of Non-Linear Mechanics 37 (3), 565–569.

Nowacki, W., 1986. Theory of Asymmetric Elasticity. Pergamon Press, Oxford.

Pars, L.A., 1965. A Treatise on Analytical Dynamics. Heinemann, London.

Perelyaev, S.E., 2006. On the global parameterizations of a group of 3D rotations (in Russian). Izvestiya RAN, Mekhanika Tverdogo Tela 41 (3), 30–44. (English Transl.: Mechanics of Solids 41 (3), 23–33 by Allerton Press, Inc.).

Pietraszkiewicz, W., 1979. Finite Rotations and Lagrangian Description in the Non-linear Theory of Shells. Polish Scienctific Publishers, Warszawa-Poznań.

Pietraszkiewicz, W., Badur, J., 1983. Finite rotations in the description of continuum deformation. International Journal of Engineering Science 21 (9), 1097–1115.

Pietraszkiewicz, W., Eremeyev, V.A., 2009. On natural strain measures of the non-linear micropolar continuum. International Journal of Solids and Structures 46 (3–4), 774–787.

Ramezani, S., Naghdabadi, R., 2007. Energy pairs in the micropolar continuum. International Journal of Solids and Structures 44 (14–15), 4810–4818.

Rooney, J., 1977. A survey of representations of spatial rotation about a fixed point. Environment and Planning B 4 (2), 185–210.

Shkutin, L.I., 1980. Nonlinear models of deformable momental continua. Journal of Applied Mechanics and Technical Physics 21 (6), 111–117 (in Russian).

Stuelpnagel, J., 1964. On the parametrization of the three-dimensional rotation group. SIAM Reviews 6 (4), 422–429.

Zubov, L.M., 1997. Nonlinear Theory of Dislocations and Disclinations in Elastic Bodies. Springer, Berlin.