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Linear spaces with many small lines*

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Abstract

In this paper some of the work in linear spaces in which most of the lines have few points is surveyed. This includes existence results, blocking sets and embeddings. Also, it is shown that any linear space of order v can be embedded in a linear space of order about $13v$ in which there are no lines of size 2.

1. Introduction

There have been many papers written on linear spaces, also known as pairwise-balanced designs (PBDs) with $\lambda = 1$. Many of the papers written on linear spaces involve highly structured objects. Yet a recent result of Grable [13] proves that almost all linear spaces are rigid (have a trivial automorphism group), so it makes sense to say something about such linear spaces; probably the most natural thing to say is when they exist! In this survey, results concerning the existence of linear spaces in which most lines have at most 4 points is considered. This type of problem goes back to the last century when Rev. Kirkman [20] considered the existence of linear spaces in which all lines have 3 points (Steiner triple systems). However, rather than giving an historical perspective, the point here will be to give the most versatile known constructions of these linear spaces, constructions that make it easy to solve the existence problem when additional properties are required of the linear space. Some such properties will then be added to the linear space (such as the existence of a blocking set) to demonstrate the use of the neat constructions.

In Section 3, some results concerning the embedding of linear spaces into linear spaces with no lines of size 2 are considered. Then finally some open, difficult problems are listed.

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The terminology used here is standard, so where definitions are not given, the reader is referred to [2].

2. Existence results

2.1. All lines have size 3

Of course, these linear spaces are the renowned Steiner triple systems. Over the years, various constructions have been given to show that there exist linear spaces of order v with all lines having size 3 if and only if $v=6x+1$ or $v=6x+3$.

Probably, the most versatile constructions for such spaces are the Bose and Skolem constructions, which use, respectively, as underlying structure an *idempotent* (cell (i,i) contains symbol i for $1 \leqslant i \leqslant 2x+1$) or *half-idempotent* (cells (i,i) and $(x+i,\ x+i)$ contain symbol i for $1 \leqslant i \leqslant x$) latin square, that is also symmetric (see Fig. 1).

Bose construction [3, 22]: Let L be a symmetric, idempotent latin square of order $2x+1$ on the symbols $1,2,\ldots,2x+1$. Define the set of lines B of a linear space of order $6x+3$ on the set of points $\{1,2,\ldots,2x+1\} \times \{1,2,3\}$ by

(1) for $1 \leqslant i \leqslant 2x+1$, $\{(i,1),(i,2),(i,3)\} \in B$, and

(2) for $1 \leqslant i < j \leqslant 2x+1$, $1 \leqslant k \leqslant 3$, $\{(i,k),(j,k),(i \cdot j, k+1)\} \in B$,

where $i \cdot j$ is the symbol in cell (i,j) of L, and the second coordinate is reduced modulo 3 (see Fig. 2).

Skolem construction [22, 37]: Let L be symmetric half-idempotent latin square of order $2x$ on the symbols $1,2,\ldots,2x$. Define the set of lines B of a linear space of order $6x+1$ on the set of points $(\{1,2,\ldots,2x\} \times \{1,2,3\}) \cup \{\infty\}$ by

(3) for $1 \leqslant i \leqslant x$, $\{(i,1),(i,2),(i,3)\} \in B$,

(4) for $1 \leqslant i \leqslant x$, $1 \leqslant k \leqslant 3$, $\{\infty,(x+i,k),(i,k+1)\} \in B$, and

(5) for $1 \leqslant i < j \leqslant 2x$, $1 \leqslant k \leqslant 3$, $\{(i,k),(j,k),(i \cdot j, k+1)\} \in B$,

where $i \cdot j$ is the symbol in cell (i,j) of L, and the second coordinate is reduced modulo 3.

By using the properties of the latin squares used, it is easy to check that these constructions produce Steiner triple systems. Of course, for the constructions to be of any use, one has to find the relevant latin squares, but this too is trivial: define cell (i,j) of L to contain $(i+j)/2$ or $(i+j+2x+1)/2$, whichever is an integer (reducing modulo $2x+1$) to produce a symmetric idempotent latin square of order $2x+1$; similarly, one can define a half-indempotent latin square of order $2x$.

These constructions are extremely versatile. For example, in the Bose construction, one can replace (2) with

(2′) for $1 \leqslant i \leqslant 2x+1$, $1 \leqslant k \leqslant 3$, $\{(i,k),(j,k),(i \cdot_k j, k+1)\} \in B$,

where $i \cdot_k j$ is the symbol in cell (i,j) of L_k, where L_k is any symmetric idempotent latin square of order $2x+1$. This property can be extensively used in solving the intersection problem for Steiner triple systems: how many lines can two such linear spaces of order v have in common [26]?

1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2
3	1	4	2	5

A symmetric idempotent latin square of order 5

1	4	2	5	3	6
4	2	5	3	6	1
2	5	3	6	1	4
5	3	6	1	4	2
3	6	1	4	2	5
6	1	4	2	5	3

A symmetric half-idempotent latin square of order 6

Fig. 1.

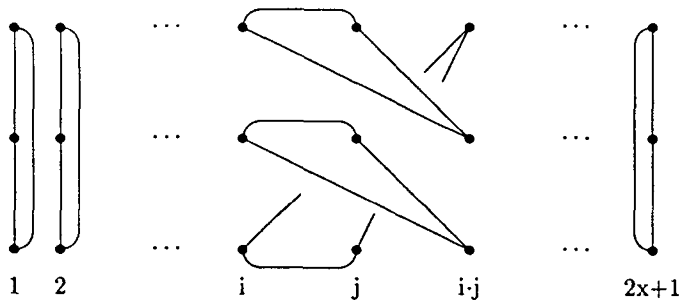

Fig. 2. The Bose construction.

		5	6	3	4
		6	5	4	3
5	6			1	2
6	5			2	1
3	4	1	2		
4	3	2	1		

Fig. 3. A symmetric latin square with holes of size 2 and of order 6.

Another very neat construction of these linear spaces is to use latin squares with holes; here we restrict the holes to size 2 for convenience. A *latin square with holes of size* 2 is a latin square of order $2x$ in which for $1 \leqslant i \leqslant x$ cells $h_i = \{2i-1, 2i\} \times \{2i-1, 2i\}$ contain a latin square of order 2 on the symbols $2i-1$ and $2i$; h_i is called a *hole*. (It is often the case that authors prefer to remove the symbols in the subsquares of order 2, leaving these $4x$ cells empty (see Fig. 3).)

Latin square with holes of size 2 *construction*: Let L be a symmetric latin square with holes of size 2 and of order $2x$. Define the set of lines B of the linear space of order $6x+1$ or $6x+3$ on the set of points $(\{1,2,\dots,2x\}\times\{1,2,3\})\cup\{\infty\}$ or $(\{1,2,\dots,2x\}\times\{1,2,3\})\cup\{\infty_1,\infty_2,\infty_3\}$, respectively, by

(6) for $1\leqslant i\leqslant x$ place the lines in a Steiner triple system of order 7 or 9 (of course, such systems exist) on the set of symbols $(\{2i-1,2i\}\times\{1,2,3\})\cup\{\infty\}$ or $(\{2i-1,2i\}\times\{1,2,3\})\cup\{\infty_1,\infty_2,\infty_3\}$ respectively, making sure that $\{\infty_1,\infty_2,\infty_3\}$ is a line in the latter case, and

(7) for $1\leqslant i<j\leqslant 2x$, $1\leqslant k\leqslant 3$, $\{(i,k),(j,k),(i\cdot j,k+1)\}\in B$.

Notice how similar this is to the Bose construction, but now the case where the linear space has $6x+1$ symbols is handled at the same time. Clearly, this linear space has many subspaces, all intersecting in the ∞ points, a property that can be useful (to keep the chromatic number high, for example).

Of course, we still have to construct the symmetric latin squares of order $2x$ with holes of size 2, but we delay that to Section 2.2. It suffices here to say that they do exist for all $x\neq 2$ [11].

Finally, we should not foreget Wilson's construction [43] for Steiner triple systems, which has been of great use in considering the existence of large sets of Steiner triple systems (i.e. a partition of all lines of size 3 of a v set, each element of which is the set of lines of a Steiner triple system). Let $v\in\{6x+1,6x+3\}$ and let G be the graph on the vertex set $\{1,\dots,v-3\}$ which joins i to j iff either $2i+j\equiv 0\pmod{v-2}$ or $j\equiv -i\pmod{v-2}$. Wilson shows that G has a 1-factorization with 1-factors F_0, F_{v-2} and F_{v-1}. Using this notation we have the following.

Wilson's construction [43]: Let $V=\{0,1,\dots,v-1\}$. Define the set of lines B of a linear space (V,B) with all lines of size 3 by

(8) if $a\neq b\neq c\neq a$, $\{a,b,c\}\subseteq\{1,\dots,v-3\}$, $\{y,2y\pmod{v-2}\}\nsubseteq\{a,b,c\}$ for $1\leqslant y\leqslant v-3$, and $a+b+c\equiv 0\pmod{v-2}$, then $\{a,b,c\}\in B$,

(9) if $i\in\{0,v-2,v-1\}$ and $\{j,k\}\in F_i$, then $\{i,j,k\}\in B$, and

(10) $\{0,v-1,v\}\in B$.

2.2. *One line has* 5 *points, the rest have* 2

The usual counting arguments prove that a necessary condition for the existence of linear spaces of order v with one line of size 5 and the rest of size 3 is that $v\equiv 5\pmod 6$. This can be seen to be sufficient by using a modification of the Bose construction.

The modified Bose construction [6, 24]: Let L be a symmetric idempotent latin square of order $2x+1$. Define the set of lines B of a linear space of order $6x+5$ with one line of size 5, the rest of size 3 on the set of points $(\{1,2,\dots,2x+1\}\times\{1,2,3\})\cup\{\infty_1,\infty_2\}$ by

(1″) $\{\infty_1,\infty_2,(2x+1,1),(2x+1,2),(2x+1,3)\}\in B$,

(2″) for $1\leqslant i<j\leqslant 2x+1$, $1\leqslant k\leqslant 3$, $\{(i,k),(j,k),(\alpha_k(i\cdot j),k+1)\}\in B$, and

(3″) for $1\leqslant i\leqslant x$, $\{\infty_1,(2i-1,1)\ (2i,2)\}$, $\{\infty_1,(2i,1),(2i,3)\}$, $\{\infty_1,(2i-1,2),(2i-1,3)\}$, $\{\infty_2,(2i-1,1),(2i-1,3)\}$, $\{\infty_2,(2i,2),(2i,3)\}$ and $\{\infty_2,(2i,1),(2i+1,2)\}$

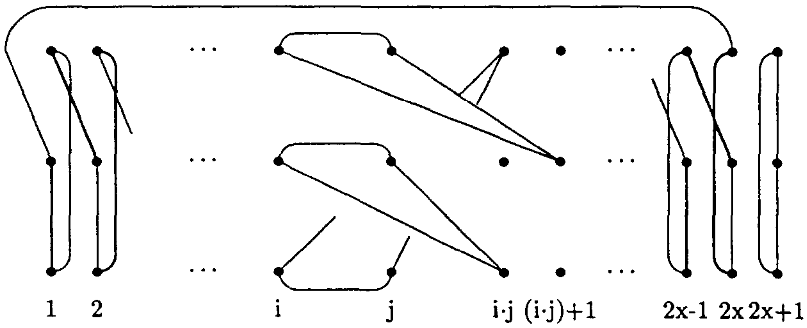

Fig. 4. The modified Bose construction. In the cycle of length $6x$, pairs joined by thick edges occur in lines with ∞_1, pairs joined by thin edge occur in lines with ∞_2.

are in B, reducing the first subscript modulo $2x$, and where α_2 and α_3 are identity permutations, α_1 is the permutation on $\{1, \ldots, 2x+1\}$ defined by

$$\alpha_1(l)=\begin{cases} l+1 & \text{if } 1 \leqslant l \leqslant 2x-1, \\ 1 & \text{if } l=2x, \\ 2x+1 & \text{if } l=2x+1. \end{cases}$$

The lines defined in (1″) and (2″) cover all pairs in $\{1, \ldots, 2x+1\} \times \{1, 2, 3\}$ except for $\{(i, 1), (i+1, 2)\}$, $\{(i, 2), (i, 3)\}$ and $\{(i, 1), (i, 3)\}$ for $1 \leqslant i \leqslant 2x$, reducing the first subscript modulo $2x$. Since these edges form a cycle of length $6x$ (which has even length), they can be joined with ∞_1 and ∞_2 as described in (3″) to form lines of size 3 (see Fig. 4).

This construction has also been useful in settling intersection problems [25] and support size problems in the area of pairwise-balanced designs [6].

We can also modify the 'latin squares with holes of size 2 construction' to form these linear spaces with one line of size 5, the rest of size 3. To do this replace (6) with:

(6′) for $1 \leqslant i \leqslant x$ place the lines in a linear space of order 11 with a subspace of order 5 (this is easy to construct) on the vertex set $(\{2i-1, 2i\} \times \{1,2,3\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$, the line of size 5 being $\{\infty_1, \ldots, \infty_5\}$.

Perhaps, it is not surprising that Wilson's construction too can be modified to form these linear spaces. In fact, with his construction it is possible to produce such a linear space containing a parallel class (i.e. a group divisible design with one group of size 5, all other groups and all blocks of size 3). For details, see [23], but the construction is the same as the 'Wilson construction' except that now $v=6x+5$, and (10) is replaced with $\{0, 2x+1, 4x+2, v-2, v-1\} \in B$.

We can now return to the construction of symmetric latin squares of order $2x$ with holes of size 2. We have now used the Bose, Skolem and modified Bose constructions to form linear spaces with lines of size 3 and (perhaps) one line of size 5 of orders $6x+1$, $6x+3$ and $6x+5$. Let $(\{1, \ldots, v\}, B)$ be such a linear space of order $v \in \{6x+1, 6x+3, 6x+5\}$. This linear space can be used to construct a symmetric latin square with holes of size 2 and of order $v-1$, $v \neq 5$. Provided that $v \neq 5$, we can assume that for

$1 \leqslant i \leqslant (v-1)/2$, $\{2i-1, 2i, v\} \in B$, so if $v = 6x+5$ and $v \neq 5$ then the point v is not in the line of size 5. For all $v \neq 5$ define a symmetric latin square L with holes of size 2 by

(11) cells (i, j) and (j, i) of L contain symbol k if $\{i, j, k\} \in B$, $v \notin \{i, j, k\}$, and

(12) if $v = 6x+5$ and $\{a, b, c, d, e\} \in B$, then let L' be an idempotent symmetric latin square of order 5 with rows, columns and symbols in $\{a, b, c, d, e\}$ and for each $\{i, j\} \subseteq \{a, \ldots, e\}$ let cells (i, j) and (j, i) of L contain the symbol in cell (i, j) of L'.

Notice that since we assumed that the lines containing v are $\{2i-1, 2i, v\}$ for $1 \leqslant i \leqslant (v-1)/2$, the empty cells (holes) of L are $\{2i-1, 2i\} \times \{2i-1, 2i\}$. Therefore, we have the following result (as far as the author knows, this was first proved in [11]; the preceding proof is in [41]):

Theorem 2.1 (Fu [11]). *For all $x \neq 2$ there exists a symmetric latin square with holes of size 2 and of order $2x$.*

2.3. *One of the lines is very big*

In this section we consider linear spaces of order v in which one line is very big (i.e. of size cv where c is a constant), and obtain lower bounds on the number of lines in such spaces.

Suppose that we have a linear space with lines $L_1, \ldots, L_g$, $|L_i| = l_i$ where $l_g = k$ and $l_i \leqslant k$ for $1 \leqslant i \leqslant g-1$. Then Stinson [39] proved that

(13) $$g \geqslant 1 + (v-k)(2tk - (v-k-1))/(t^2+t),$$

where $t = \lfloor (v-1)/k \rfloor$. He proved this using the well-known method of variances as follows. Suppose that $L_i \cap L_g \neq \emptyset$ for $1 \leqslant i \leqslant g'$ and $L_i \cap L_g = \emptyset$ for $g'+1 \leqslant i \leqslant g-1$. For each $p \notin L_g$, p obviously occurs in k of $L_1, \ldots, L_g'$, so

(14) $$\sum_{i=1}^{g-1} (l_i - 1) \geqslant \sum_{i=1}^{g'} (l_i - 1) = k(v-k).$$

Also, counting pairs of points, neither being in L_g gives

(15) $$\sum_{i=1}^{g-1} \binom{l_i - 1}{2} \leqslant \sum_{i=1}^{g'} \binom{l_i - 1}{2} + \sum_{i=g'+1}^{g-1} \binom{l_i}{2} = \binom{v-k}{2}.$$

Each of (13) and (14) meets equality iff $g' = g-1$, i.e. iff every line intersects L_g. But then for any integer t (since each term is the product of consecutive integers),

$$0 \leqslant \sum_{i=1}^{g-1} (l_i - t - 1)(l_i - t - 2) = \sum_{i=1}^{g-1} (l_i - 1)(l_i - 2) - 2t \sum_{i=1}^{g-1} (l_i - 1) + (t^2 + t) \sum_{i=1}^{g-1} 1$$

which gives (13) after using (14) and (15). In this case, you get equality iff $l_i \in \{t+1, t+2\}$, so we have that: equality is achieved in (13) iff every line meets L_g and all lines other than L_g have size $t+1$ or $t+2$. The existence of linear spaces meeting equality in (13) has been settled in the case where $t = 1$ [29, 38] and essentially settled by Rees [32, 33] in the case where $t=2$. Notice that if $t=2$ then $2=t=\lfloor(v-1/k\rfloor$, so $v/3<k\leqslant(v-1)/2k$, so in such linear spaces L_g is very long, and all other lines have size 3 or 4, all of which intersect L_g.

It is worth mentioning that Rees [30] has another bound on g, independent of (13), for which a simple description of the properties required of a linear space to meet equality can also be given, one of which is that each line other than L_g has size t, $t+1$ or $t+2$. For a neat comparison of these two bounds, see [35].

2.4. *All lines have size* 4.

In 1961 Hanani proved [15] that linear spaces of order v in which all lines have size 4 exist for all $v\equiv 1$ or $4 \pmod{12}$. Recently, two very good constructions of such linear spaces have been found by making use of skew Room frames with holes of size 2, which are defined as follows.

For $1\leqslant i\leqslant s$ define $h_i=\{2i-1, 2i\}$ and let $H=\{h_i \mid 1\leqslant i\leqslant s\}$; h_i is called a *hole*. A *Room frame of order* $2s$ *with holes* H is a $2s\times 2s$ array S in which

(a) for $1\leqslant i\leqslant s$, the cells $(2i-1, 2i-1)$ and $(2i, 2i)$ each contain the symbols in h_i, and the cells $(2i-1, 2i)$ and $(2i, 2i-1)$ are empty.

(b) each 2-element subset of $\{1,\dots,2s\}$ that is not an element of H occurs in exactly one cell of S, each cell of S contains a pair of symbols or is empty, and each element of $\{1,\dots,2s\}$ occurs exactly once in each row and exactly once in each column.

(Sometimes the cells in (a) are defined to be empty; this is a cosmetic difference only.) A *skew* Room frame is a Room frame in which for $1\leqslant i<j\leqslant 2s$, $\{i,j\}\notin H$, exactly one of cells (i,j) and (j,i) is occupied.

It has been shown [24, 40] that with at most 14 exceptions, there exists a skew Room frame of order $2s$ with holes H.

The skew Room-frame construction. Let S be a skew Room frame with holes H of size 2 and of order $2s$ (see Fig. 5). Define a linear space with set of points $(\{1,\dots,2s\}\times\{1,\dots,6\})\cup\{\infty\}$ or $(\{1,2,\dots,2s\}\times\{1,\dots,6\})\cup\{\infty_1, \infty_2, \infty_3, \infty_4\}$, if $v\equiv 1$ or $4 \pmod{12}$, respectively, and with set of lines B by

(16) for $1\leqslant i\leqslant s$ let B contain the lines of a linear space with all lines of size 4 defined on the vertex set $(\{2i-1, 2i\}\times\{1,\dots,6\})\cup\{\infty\}$ or $(\{2i-1, 2i\}\times\{1,\dots,6\})\cup\{\infty_1, \infty_2, \infty_3, \infty_4\}$ if $v\equiv 1$ or $4 \pmod{12}$, respectively, (if $v\equiv 4 \pmod{12}$, then make sure that $\{\infty_1,\dots,\infty_4\}$ is a line), and

(17) for each $\{a,b\}\subseteq\{1,2,\dots,2s\}$, $\{a,b\}\neq h_i$ for $1\leqslant i\leqslant s$, and for $1\leqslant j\leqslant 6$ let $\{(a,j), (b,j), (r,j+1), (c,j+4)\}\in B$, where $\{a,b\}$ occurs in cell (r,c) or (c,r) of S, and reducing the second component modulo 6.

Note that for $\{r,c\}\notin H$, the skew property ensures that exactly one of (r,c) and (c,r) contain symbols. It is not hard to check that this defines the required linear space. Informally, the pairs of points occurring on the same level (i.e. have same second coordinate) are covered by a line of the linear space since each pair of symbols not in a hole occur together in exactly one cell of S, the pairs of points one level or two levels apart are covered by a line since each row or column, respectively, of S contains every

1 2			6 9		8 10		3 5	4 7	
	1 2	6 10		7 9		4 5			3 8
5 10		3 4			2 7		1 9	6 8	
	5 9		3 4	1 8		2 10			6 7
8 9		1 7		5 6			4 10	2 3	
	7 10		2 8		5 6	3 9			1 4
4 6		2 9		3 10		7 8		1 5	
	3 6		1 10		4 9		7 8		2 5
	4 8		5 7		1 3		2 6	9 10	
3 7		5 8		2 4		1 6			9 10

Fig. 5. A skew Room frame of order 10 with holes H.

symbol, and the pairs of points three levels apart are covered since the skew property of S guarantees that for every pair $\{r, c\} \neq H$, exactly one of (r, c) and (c, r) contains a pair of symbols (see Fig. 6).

This construction is fantastic for settling existence problems for linear spaces with all lines of size 4 with given weak chromatic number. The *weak chromatic number* of a linear space is the least number of colours required to colour the points in such a way that there are no monochromatic lines. Note then that settling the existence problem for linear spaces with weak chromatic number 2 is equivalent to settling the existence problem for linear spaces with blocking sets (the points receiving one colour form a blocking set when only two colours are being used). Clearly, if we colour the points with second coordinate i, $i+2$ or $i+4$ (for $1 \leqslant i \leqslant 2$) with colour i, then none of the lines defined in (17) are monochromatic, so it remains to find a linear space on 13 or 16 points with a blocking set of size 6 or 8, respectively, to be used in (16). This problem received much attention, and was finally settled by Hoffman et al. [18]. They used a result of Stinson on nesting group divisible designs, a result that can also be obtained by using skew Room frames.

In a recent survey by Colbourn and Rosa [7], one open problem they suggest is to settle the existence problem for weakly 3-chromatic linear spaces with all lines having size 4. Clearly, if we colour the vertices with second coordinate i or $i+3$ (for $1 \leqslant i \leqslant 3$) with colour i, then none of the lines defined in (17) are monochromatic.

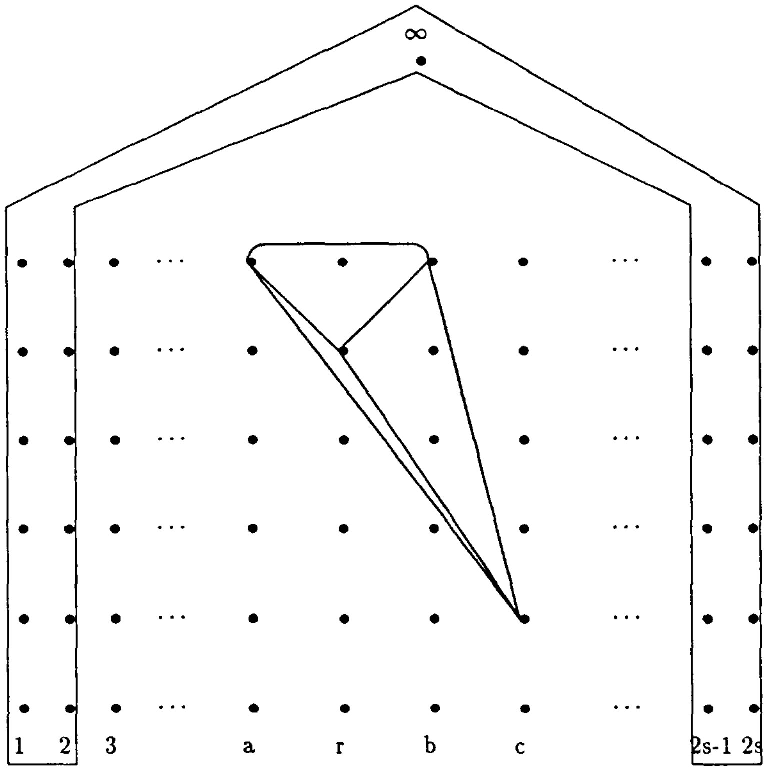

Fig. 6. A linear space with all lines of size 4. Cell (r, c) or (c, r) of the skew Room frame contains symbols a and b.

However, there are no weakly 3-chromatic linear spaces of order 13 or 16 with all lines of size 4. They do exist of orders 25 and 28, however, so the construction will do very well provided the skew Room frame has one hole $h_1 = \{1, 2, 3, 4\}$ of size 4. For then, (16) is modified in the obvious way to place a subspace of order 25 or 28 on the points $(h_1 \times \{1, \ldots, 6\}) \cup \{\infty\}$ or $(h_1 \times \{1, \ldots, 6\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ that is weakly 3-chromatic. The existence problem for such skew Room frames is yet to be settled, but hopefully, will be soon.

2.5. *Other results*

In this section, we just briefly present three more results related to this survey.

Having settled the existence problem for all linear spaces having lines of just size 3 or just size 4, it is reasonable to allow both line sizes, and indeed to specify there be exactly α lines of size 3, β of size 4. This problem has been solved for all orders $v \geqslant 96$ [8]. It is also worth pointing out a result of Rees [31], related to his work described in Section 2.3, which settles the existence problem of finding a linear space with lines of size 2 and 3 which can be partitioned into α parallel classes of lines of size 3 and β parallel classes of lines of size 2: they exist

(a) for all $v \equiv 0 \pmod 6$, when $0 < \alpha < v/2$ (so $\beta = v - 1 - 2\alpha$), except if $\alpha = (v/2) - 1$ and $v \in \{6, 12\}$, in which cases they do not exist,

(b) for all $v \equiv 0 \pmod 2$ when $\alpha = 0$ (this is a 1-factorization of K_n), and

(c) for all $v \equiv 3 \pmod 6$ when $\alpha = (v-1)/2$ (this is a Kirkman triple system).

The existence of linear spaces of order $v = n + s$ that contain one line of size s, the rest of size at least 3 has been completely solved for n even, and has been solved for n odd, $n \geqslant 91$ by Hartman and Heinrich [16], following work on this problem by Mendelsohn and Rees [28] and by Jungnickel and Lenz [19].

The problem of finding group divisible designs with one group of size u, the rest of the groups having size t and all other lines having size 3 has been completely solved by Colbourn et al. [5].

Finally, Greig [14] has settled the existence problem for linear spaces in which

(a) each line has size 7 (with 40 possible exceptions),

(b) each line has size 8 (with 78 possible exceptions),

(c) each line has size 9 (with 157 possible exceptions),

(d) each line has size 8 that is resolvable (with 95 possible exceptions).

3. Embedding linear spaces

A linear space (V, B) is *embedded* in a linear space (W, B') if $V \subseteq W$ and there exists an injection $\phi: B \to B'$ such that for each $l \in B$, l is the restriction of $\phi(l)$ to V. Obviously, it is easy to embed (V, B) into a linear space (W, B') of any order $t > |V|$ if you allow B' to contain lines of size 2, so normally the embeddings considered preclude this possibility.

One of the earliest results was a proof by Doyen and Wilson [9] that any Steiner triple system (all lines have size 3) of order v can be embedded in a Steiner triple system of order t, for all $t \geqslant 2v+1$, $t \equiv 1$ or $3 \pmod 6$. This result is the best possible. More recently, Wei and Zhu [42] and Rees and Stinson [34] proved the corresponding result for linear spaces with all lines of size 4, the embedding being possible for all $t \geqslant 3v+1$, $t \equiv 1$ or $4 \pmod{12}$, and again this result is the best possible. Rees and Stinson also proved [36] a best possible result that for all $t \geqslant 3v$, $t \equiv 3 \pmod 6$ any resolvable linear space with all lines of size 3 and of order v can be embedded in a resolvable linear space with all lines of size 3 and of order t (these are Kirkman triple systems).

Corresponding to these problems are the more difficult problems of starting with a linear space in which all lines have size at most k (instead of exactly k), or even those in which all lines have size either 2 or k, and embedding it in a linear space in which all lines have size k. For example, Lindner [21] proved that any linear space of order v with each line having size at most 3 can be embedded in a Steiner triple system of order $t = 6v+3$. This was subsequently improved to any $t \geqslant 4v+1$, $t \equiv 1$ or $3 \pmod 6$ by

Andersen et al. [1], but the widely conjectured best possible result of $t \geqslant 2v+1$, $t \equiv 1$ or $3 \pmod 6$ is still an open problem. The situation is even worse for the embedding of a linear space with all lines of size at most 4 into one with lines having size 4: no embedding where t is even just a polynomial function of v is known! Almost certainly the right result is that if all lines have size 2 or 4 then such an embedding is possible for all $t \geqslant 3v+1$, $t \equiv 1$ or $4 \pmod{12}$. Clearly, much work remains to be done.

It is worth pointing out that the $t=6v+3$ embedding and the $t \geqslant 4v+1$ embedding follow quite different paths. The $t=6v+3$ embedding relies heavily on the lines of size 3 in the given linear space, whereas the $t \geqslant 4v+1$ embedding essentially ignores them, focussing on the lines of size 2 that are to eventually occur in lines of size 3. In fact, a close look at the proof of the Andersen et al. [1] result shows that it actually proves the following proposition.

Proposition 3.1. *Let (V, B) be a linear space in which*

(a) *the number of lines of size 2 is divisible by 3, and*

(b) *each point is contained in an even number of lines of size 2.*

Then (V, B) can be embedded in a linear space of order t, for any $t \geqslant 4|V|+1$, $t \equiv 1$ or $3 \pmod 6$ in which there are no lines of size 2.

Using this observation, one can obtain the following polynomial embedding of any linear space.

Theorem 3.2. *Let L be any linear space of order v. Then L can be embedded in a linear space of order t and with no lines of size 2, for any $t \geqslant 4x+1$, $t \equiv 1$ or $3 \pmod 6$ where x is at most $v+19+9\lfloor (v+6)/4 \rfloor$.*

Proof. We can assume that v is odd. We shall first embed $L=(V, B)$ in a linear space (W, B') that satisfies the conditions of Proposition 3.1. The result will then follow from Proposition 3.1.

To define (W, B') we shall define all lines of length greater than 2. Let P be the set of points contained in an odd number of lines of size 2 in B. Clearly, $|P|$ is even. If $|P| \equiv 2 \pmod 4$ then add 3 lines $\{1,2,7,8\}$, $\{3,4,7,9\}$ and $\{5,6,8,9\}$ to B, where $\{1,\dots,9\} \cap V = \emptyset$. Partition the points in P or in $P \cup \{1,\dots,6\}$, if $|P| \equiv 0$ or $2 \pmod 4$, respectively, into x sets of size 4, say $s_1,\dots,s_x$ (so $x \in \{|P|/4, (|P|+6)/4\}$). Then for $1 \leqslant i \leqslant x$, take a projective plane of order 3, with one line being s_i and the other 9 points being 9 further new points, and add all the lines in this plane except for s_i to B. This adds a total of $9x \leqslant 9\lfloor (v+6)/4 \rfloor$ new points.

Finally, add 9 or 10 more new points to make the total number of points (new and old) odd. Then on these 9 or 10 points, if the current number of lines of size 2 (defined on all points) is congruent to $i \pmod 3$, form i lines of size 5 and add these to B, thus making the number of lines of size 2 divisible by 3. For each point p (old or new), the number of pairs $\{p, q\}$ contained in lines of size greater than 2 is even (so since the number of points is odd, the number of lines of size 2 containing p is even, so (b) of Proposition 3.1 is satisfied).

The result is a linear space (W, B') that satisfies (a) and (b) of Proposition 3.1. Also, we have that

$$|W| \leqslant v+9+9\lfloor (v+6)/4 \rfloor +10$$

and so the theorem follows. □

Loosely speaking, Theorem 3.2 embeds any linear space of order v into a linear space with no lines of size 2 and of order around $13v$.

4. Open problems

Here are some open problems, none of which are likely to be easy to solve!

(1) Prove that any linear space of order v in which all lines have size at most 3 can be embedded in a linear space of order at most $2v+1$ in which all lines have size 3. (This best possible embedding has been proved for partial triple systems of index divisible by 4 by Hilton and Rodger [17].)

(2) Find a small embedding (i.e. polynomial in v) of a linear space of order v in which all lines have size at most 4 into a linear space in which all lines have size 4.

(3) Solve the existence problem for resolvable linear spaces with all lines of size 3 or 4 (see Section 2.3). Or, solve the existence problem for linear spaces that can be resolved into α parallel classes of lines of size 3 and β parallel classes of lines of size 4.

(4) Show that for any $k \geqslant 4$ and any $\chi \geqslant 2$ there exists a linear space, all lines having size k, with weak chromatic number χ. Even doing this for $k=4$ would be good! (One can combine results of Erdös and Hajnal [10], Lovász [27] and Ganter [12] to obtain linear spaces with all lines of size k and weak chromatic number at least χ. Also, de Brandes et al. [4] have proved that for all sufficiently large orders v and for any χ there exists a Steiner triple system of order v and weak chromatic number $\chi \geqslant 2$.)

(5) Find a versatile construction for linear spaces with all lines of size 5.

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