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The residue fields of a zero-dimensional ring

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Abstract

Gilmer and Heinzer have considered the question: For an indexed family of fields $\mathscr{K}=\{K_\alpha\}_{\alpha\in A}$, under what conditions does there exist a zero-dimensional ring R (always commutative with unity) such that $\mathscr{K}$ is up to isomorphism the family of residue fields $\{R/M_\alpha\}_{\alpha\in A}$ of R? If $\mathscr{K}$ is the family of residue fields of a zero-dimensional ring R, then the associated bijection from the index set A to the spectrum of R (with the Zariski topology) gives A the topology of a Boolean space. The present paper considers the following question: Given a field F, a Boolean space X and a family $\{K_x\}_{x\in X}$ of extension fields of F, under what conditions does there exist a zero-dimensional F-algebra R such that $\mathscr{K}$ is up to F-isomorphism the family of residue fields of R and the associated bijection from X to Spec(R) is a homeomorphism? A necessary condition is that given x in X and any finite extension E of F in K_x, there exist a neighborhood V of x and, for each y in V, an F-embedding of E into K_y. We prove several partial converses of this result, under hypotheses which allow the "straightening" of the F-embeddings to make them compatible. We give particular attention to the cases where X has only one accumulation point and where X is countable; and we provide several examples. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction and general results

In [4], Gilmer and Heinzer consider the following question: For an indexed family of fields $\mathscr{K}=\{K_a\}_{a\in A}$, under what conditions is $\mathscr{K}$ up to isomorphism the family of residue fields $\{R/M_a\}_{a\in A}$ of some zero-dimensional ring R (always commutative with unity)? Since the family of residue fields of a ring R is, up to isomorphism, the same as the family of residue fields of R modulo its nilradical, we may assume without loss

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of generality that R is reduced. If such a ring exists, Gilmer and Heinzer say that $\mathscr{K}$ is *realizable* and call the ring R a *realization* of $\mathscr{K}$. We will often consider the relative version of the realizability question, in which each K_a is assumed to be an extension of a common base field F: If there is a zero-dimensional F-algebra R whose residue fields are exactly the K_a up to F-isomorphism, we say that $\mathscr{K}$ is *F-realizable* and that R is an *F-realization* of $\mathscr{K}$.

If R is a zero-dimensional ring, then $X = \mathrm{Spec}(R)$ is a *Boolean space*, that is, X is compact, Hausdorff and totally disconnected. Therefore, in considering the family of residue fields of a zero-dimensional ring, it is natural to consider the following question, which brings in topological considerations:

Question 1.1. Given a Boolean space X, which assignments of fields to the points of X come from affine schemes?

In this connection we use the following definition.

Definition. Let X be a Boolean space and $\mathscr{K} = \{K_x\}_{x\in X}$ a family of fields indexed by X. We say that the pair $(X, \mathscr{K})$ is *realizable* if there exist a (necessarily zero-dimensional) reduced ring R and a homeomorphism $X \to \mathrm{Spec}(R) : x \mapsto P_x$ (where $\mathrm{Spec}(R)$ has the usual Zariski topology) such that, for each point x in X, we have $R/P_x \cong K_x$. In this case we say that R is a *realization of* $(X, \mathscr{K})$.

Let F be a field, let X be a Boolean space, and let $\mathscr{K} = \{K_x : x \in X\}$ be a family of extension fields of F indexed by X. We say that $(X, \mathscr{K})$ is *F-realizable* if there exist a reduced F-algebra R and a homeomorphism $X \to \mathrm{Spec}(R) : x \mapsto P_x$ such that $R/P_x \cong_F K_x$ for every x in X. In this case we say that R is an *F-realization of* $(X, \mathscr{K})$.

Thus, a family $\mathscr{K} = \{K_x\}_{x\in X}$ of fields is realizable if and only if there is a Boolean topology on X and a realization of the pair $(X, \mathscr{K})$. An example due to R. Wiegand [4, Example 6.4] shows that a pair $(X, \mathscr{K})$ can have two non-isomorphic realizations.

The main results of this section are Theorems 1.2 and 1.4, which give, respectively, a necessary and a sufficient condition for realizability. At first sight these two conditions appear very similar, in that both criteria require that certain of the fields be embeddable in others. However, there is the distinction between the two criteria that in the necessary condition there is no requirement that the embeddings be compatible, whereas in the sufficient condition the embeddings are in fact set-theoretic inclusions. Much of the paper is devoted to finding situations under which embeddings can be made compatible, in order to narrow the gap between the necessary condition of (1.2) and the sufficient condition of (1.4) for realizability.

In Section 2 we consider the question of F-realizability when the infinite Boolean space X has a single non-isolated point. Write $X = A \cup \{\infty\}$, where A is the set of isolated points, and suppose $\{K_a\}_{a\in A}$ is a family of extension fields of F. For which algebraic extensions L/F does the assignment $K_\infty = L$ give an F-realizable pair? Assuming L/F is countably generated, we show in (2.7) that a necessary and sufficient condition for F-realizability is that each finite extension of F contained in L has an F-embedding into all but finitely many of the K_x's.

In Section 3 we present several examples illustrating realizability for infinite Boolean spaces with one non-isolated point. In Section 4 we consider realizability for countable Boolean spaces. We give examples of families of fields realizable for certain Boolean spaces but not for others. We prove in Theorem 4.11 a sufficient condition for realizability and raise in (4.2) a question for further investigation.

If $X = \text{Spec}(R)$ we will sometimes write P_x instead of x when we are thinking of x as a prime ideal rather than a point in a topological space. Also, if $r \in R$, we let $\text{V}(r) = \{P \in \text{Spec}(R) : r \in P\}$ and $\text{D}(r) = \text{Spec}(R) - \text{V}(r)$. If R is zero-dimensional and reduced, these sets are clopen, since $\text{V}(r) = \text{V}(e)$, where e is the idempotent generating the same principal ideal as r. We will usually identify $\text{Spec}(R/eR)$ with $\text{V}(e)$.

Theorem 1.2. *Let F be a field and R a zero-dimensional reduced F-algebra. Let $X = \text{Spec}(R)$, and put $K_x = R/P_x$ for each $x \in X$. Fix $x \in X$, and let E' be any field such that $F \subseteq E' \subseteq K_x$ and $[E' : F] < \infty$. There exist a clopen neighborhood $V = \text{V}(e)$ of x and a field E such that $F \subseteq E \subseteq R/eR$ and E maps F-isomorphically onto E' under the canonical map $R/eR \twoheadrightarrow R/P_x$. In particular, for each $y \in V$ there exists an F-embedding of E' into K_y.*

Proof. Suppose first that E' is a simple extension of F, say, $E' = F(c')$. Let f be the monic minimal polynomial for c' over F. Choose an element c in R mapping to c' via the canonical map $R \twoheadrightarrow K_x$. Then $f(c) \in P_x$, so $\text{V}(f(c))$ is a clopen neighborhood of x. Let $\bar{c}$ be the image of c in $R/f(c)R$, and let $E = F[\bar{c}] \subseteq R/f(c)R$. The canonical map $R/f(c)R \twoheadrightarrow R/P_x$ carries E onto E', and $\dim_F(E) \leq \deg(f) = \dim_F(E')$, so it follows that the map from E onto E' is an F-isomorphism.

If E' is *not* a simple extension of F, there is a field F_1 such that $F \subset F_1' \subset E'$ and $E' = F_1'(c')$ for some c' in E'. By induction on $[E' : F]$ there exist a clopen neighborhood $\text{V}(e_1)$ of x defined by an idempotent e_1 in P_x, and a field F_1 such that $F \subseteq F_1 \subseteq R_1 := R/e_1R$ and F_1 maps F-isomorphically onto F_1' under the canonical map $R_1 \twoheadrightarrow R/P_x$. Applying the case of a simple extension to F_1 and $\text{Spec}(R_1) = \text{V}(e_1)$, we get a clopen neighborhood $\text{V}(e)$ of x in $\text{V}(e_1)$ and a field E such that $F_1 \subset E \subseteq R_1/eR_1$ and E maps F_1-isomorphically (hence F-isomorphically) onto E' under the canonical map $R/eR = R_1/eR_1 \twoheadrightarrow R_1/(P_x/e_1R) = R/P_x$. □

Since every neighborhood of a non-isolated point in a Boolean space is infinite, Theorem 1.2 implies:

Corollary 1.3. *Let F be a field, let X be an infinite Boolean space, and let $\{K_x\}_{x \in X}$ be a family of extension fields of F. Assume $(X, \mathscr{K})$ is F-realizable. If z is a non-isolated point of X, then every subfield E of K_z of finite degree over F is F-isomorphic to a subfield of K_x for infinitely many points x in X. If X is countable, then these infinitely many points x can all be selected to be isolated.*

The last sentence follows from the fact that in a countable Boolean space, the set of isolated points is dense (cf. (4.4) below).

Example. For each positive integer i, let K_i be the field obtained by adjoining to the field of rational numbers $\mathbb{Q}$ the square roots of all prime integers except the first i primes. (So, e.g., $K_2 = \mathbb{Q}(\sqrt{5}, \sqrt{7}, \sqrt{11}, \ldots)$.) Note that

$$K_1 \supset K_2 \supset K_3 \supset \ldots \quad \text{and} \quad \bigcap_{i=1}^{\infty} K_i = \mathbb{Q}.$$

Corollary 1.3 implies that the family $\mathscr{K} = \{K_i\}_{i=1}^{\infty}$ is not realizable. For, suppose R is a realization of $\mathscr{K}$. Choose a non-isolated point z in the spectrum of R, and let L be the residue field of R at z. Then by Theorem 1.2 every finite extension of $\mathbb{Q}$ in L must be embeddable in infinitely many of the elements of $\mathscr{K}$. But if $L = K_n$ and p is the $(n+1)^{\text{st}}$ prime, then $\sqrt{p} \in L$, but $\sqrt{p}$ has no possible image in any K_m with $m > n$; so we have a contradiction.

The next result is a partial converse to (1.2). Other partial converses with different hypotheses appear in Theorems 2.7 and 4.11 below.

Theorem 1.4. *Let X be a Boolean space, F a field, Ω an extension field of F, and $\mathscr{K} = \{K_x\}_{x \in X}$ a family of subfields of Ω each of which contains F. Assume that for each x in X and each c in a set C_x of generators of K_x/F, there is a clopen neighborhood U of x such that for all y in U, $c \in K_y$. Then the pair $(X, \mathscr{K})$ is F-realizable.*

Proof. Give Ω the discrete topology, let T denote the ring of continuous functions from X into Ω, and let R be the subring of T consisting of those functions $f : X \to \Omega$ such that $f(x) \in K_x$ for each $x \in X$. It is easy to verify that each element of R generates the same R-ideal as an idempotent, so R is zero-dimensional reduced.

For each clopen subset U of X, let e_U be the element of R that is 1 at each element of U and 0 at each element of $X - U$. Let P_x be the kernel of the evaluation map $f \mapsto f(x)$. If x and y are distinct elements of X and U is a clopen neighborhood of x such that $y \notin U$, then $e_U \in P_y - P_x$; hence the P_x's are distinct. Compactness and clopen refinements imply every proper ideal is contained in some P_x. The map from X to Spec(R) is a continuous bijection and hence a homeomorphism.

To complete the proof it suffices to see that each c in C_x is in the image of $R/P_x \to K_x$: Take a clopen neighborhood U of x for which $c \in K_y$ for all $y \in U$. Then $ce_U \in R$ and the image of ce_U in R/P_x is c. □

If E, K are extension fields of a field F inside a common algebraic extension such that K/F is normal and E has an F-embedding into K, then $E \subseteq K$. Thus, combining (1.2) and (1.4), we have the following:

Corollary 1.5. *Let F be a field, Ω an algebraic closure of F, X a Boolean space, and $\mathscr{K} = \{K_x\}_{x \in X}$ is a family of subfields of Ω each of which contains F. Suppose $\mathscr{K}$ has the property that each K_x/F is normal. (This holds, for example, if F is algebraic* *over a finite field.) Then the following conditions are equivalent:*

(1) *The pair $(X, \mathscr{K})$ is F-realizable.*

(2) *For each $x \in X$ and each c in a set of generators for K_x/F, there is a clopen neighborhood U of x such that $c \in K_y$ for all $y \in U$.*

(3) *For each $x \in X$ and each finite extension E of F in K_x, there is a clopen neighborhood U of x such that, for all $y \in U$, E has an F-embedding into K_y.*

Another corollary of (1.2) is the following:

Corollary 1.6. *Let F be a field, Ω an algebraic closure of F, and $\mathscr{K} = \{K_a\}_{a \in A}$ an infinite family of subfields of L each of which contains F. Assume that $K_a \cap K_b = F$ if $a \neq b$. If $\mathscr{L}$ is a family of algebraic extensions of F that contains $\mathscr{K}$, and if $\mathscr{L}$ is F-realizable, then $F \in \mathscr{L}$.*

Proof. Suppose R is an F-realization of $\mathscr{L}$, and let $X = \mathrm{Spec}(R)$. Then the index set A is identified with an infinite subset of X, which has an accumulation point x in X. Let L be the residue field of R at x. If $L \neq F$, choose $\alpha \in L - F$. By Theorem 1.2, α has conjugates (over F) in infinitely many of the K_a's. But since $K_a \cap K_b = F$ for $a \neq b$, this would mean that α had infinitely many conjugates, a contradiction. □

Here are some examples to which (1.6) applies:

Example 1.7. (1) Let Ω be an algebraic closure of the prime field F, and let $\{p_n\}_{n=1}^{\infty}$ be a family of distinct prime integers. Suppose $\{K_n\}_{n=1}^{\infty}$ is a family of subfields of Ω, with $[K_n : F] = p_n$ for each n. Then the K_n's are pairwise linearly disjoint over F. By (1.6), if $\mathscr{L}$ is a realizable family of algebraic extensions of F that contains $\{K_n\}_{n=1}^{\infty}$, then $F \in \mathscr{L}$.

(2) Let F be a prime field, let Ω be an algebraic closure of F, and let $\mathscr{L}$ be the family of proper extensions of F in Ω. Then $\mathscr{L}$ is not realizable, for there exists an infinite subfamily $\{K_a\}_{a \in A}$ of $\mathscr{L}$ such that the K_a's are pairwise linearly disjoint over F.

The restriction in (1.6) that the fields in $\mathscr{L}$ be algebraic over F is crucial. If, for example, $\{p_n\}_{n=1}^{\infty}$ is a family of distinct prime integers and K_n is an extension of F of degree p_n for each n, it is possible to realize the family $\{K_n\}_{n=1}^{\infty}$ as the family of residue fields at maximal ideals of a localization of the polynomial ring $F[x]$. Therefore, as noted below in (2.1.1), the family of fields $\{K_n\}_{n=1}^{\infty} \cup \{F(x)\}$ is realizable.

Remark 1.8. If $\mathscr{K} = \{K_a\}_{a \in A}$ is a family of fields, the ring $T = \prod_{a \in A} K_a$ is zero-dimensional, and each maximal ideal M_a in $\mathrm{Spec}(T)$ that is associated with a coordinate a in A (i.e., M_a is the kernel of the projection of T onto K_a) is principal (generated by the idempotent element that projects to 0 in K_a and to 1 in each K_b for all $b \neq a$). Therefore $\mathscr{K}$ is up to isomorphism a subset of the family of residue fields of T. If the set A is infinite, however, then the direct sum $I = \oplus_{a \in A} K_a$ is a proper ideal of

T and T has additional maximal ideals other than the M_a, namely the maximal ideals containing I. Indeed, if A is infinite, there can never be a bijection between A and Spec(T) since Spec(T) has cardinality $2^{2^{|A|}}$. ([6, 9F, p. 136] or [9, Item 4, p. 133] says this is the number of ultrafilters of subsets of A, and maximal ideals of T are in one-to-one correspondence with these ultrafilters.)

2. The case of a single limit point

It seems reasonable to consider Question 1.1 in the case of the one-point compactification A^* of an infinite discrete space A. (Recall that A^* is the Boolean space $A \cup \{\infty\}$, where the neighborhoods of ∞ are the complements of finite subsets of A.) If $\mathscr{K} = \{K_a\}_{a \in A^*}$ is a family of fields, and R is a realization of the pair $(A^*, \mathscr{K})$, then R has precisely one non-principal maximal ideal—the only non-isolated point in Spec(R). Thus the limit point is distinguished both algebraically and topologically.

Definition. Given a family $\mathscr{K} = \{K_a\}_{a \in A}$ of fields, a field L is said to be *admissible* for the family $\mathscr{K}$ if the pair $(A^*, \mathscr{K}^*)$ is realizable, where $\mathscr{K}^* := \{K_a\}_{a \in A^*}$, with $K_\infty = L$. If each K_a contains a base field F, the extension field L of F is said to be F-*admissible* for the family $\mathscr{K}$ provided the pair $(A^*, \mathscr{K}^*)$ is F-realizable.

Suppose L is admissible for the family of fields $\{K_a\}$, with a realization R having principal primes P_a and non-principal prime Q. Then since Q is the limit of the P_a's in Spec(R), it is tempting to believe that $L = R/Q$ is in some sense a limit of the fields $K_a = R/P_a$. It is difficult, however, to interpret this heuristic, as we illustrate using the following source of examples.

(2.1) Let R be an arbitrary commutative ring, and for P in Spec(R) let $\mathrm{k}(P)$ denote the field of fractions of the integral domain R/P. The family of fields $\mathscr{K} = \{\mathrm{k}(P) : P \in \mathrm{Spec}(R)\}$ is realizable [10, 11, 4, (2.5)] as the family of residue fields of a reduced zero-dimensional ring R^0 which is an extension ring of $R/\mathbf{n}$, where $\mathbf{n}$ is the nilradical of R. Moreover, if P is a maximal ideal of R that is the radical of a finitely generated ideal, then PR^0 is a principal maximal ideal. Therefore, if R is a one-dimensional integral domain with Noetherian spectrum, then R^0 is a reduced zero-dimensional ring with at most one maximal ideal that is not finitely generated.

(2.1.1) Let p be a prime number, and, for each positive integer i, let K_i be a field with $|K_i| = p^i$. Then the simple transcendental extension $K_1(x)$ is admissible for $\{K_i\}_{i=1}^{\infty}$. This follows from (2.1) if we take for R a localization of the polynomial ring $K_1[x]$ at the complement of the union of the ideals generated by a family of irreducible polynomials, one of each positive integral degree.

(2.1.2) Let F be an infinite field, and for each positive integer i, let $K_i = F$. Then the simple transcendental extension $F(x)$ is admissible for $\{K_i\}_{i=1}^{\infty}$. Again this follows from (2.1): Let R be a localization of the polynomial ring $F[x]$ at the complement of the union of the ideals generated by the polynomials $x-a$ as a varies over a countably infinite subset of F.

(2.1.3) Other interesting examples of fields L admissible for a family of fields $\mathscr{K}$ follow from results of Heitmann in [8]. In particular, let $\mathscr{K}$ be a countable family of fields with the property that for every prime number p there are only finitely many fields in $\mathscr{K}$ of characteristic p. Then there is a field L of characteristic zero admissible for $\mathscr{K}$.

In the next few results we interpret the results of Sect. 1 in the case of a one-point compactification of an infinite discrete space. Theorem 2.2 (an immediate consequence of Theorem 1.2) gives a general necessary condition for F-admissibility; the sufficiency of this condition, under additional hypotheses, is treated in (2.3), (2.4), (2.7) and (2.8).

Theorem 2.2. *Let $\{K_a\}_{a\in A}\cup\{L\}$ be a family of extension fields of a field F. If L is F-admissible for the family $\{K_a\}_{a\in A}$, then every subfield E of L of finite degree over F has F-embeddings into all but finitely many of the K_a.*

We thank Steve McAdam for the results (2.3) and (2.4) given below.

Applying Theorem 1.4 to the case of the one-point compactification of an infinite discrete space gives:

Theorem 2.3. *Let Ω be an extension field of a field F, let $\mathscr{K}=\{K_a\}_{a\in A}$ be an infinite family of subfields of Ω, each containing F, and let L be a subfield of Ω containing F. If every element of L is in K_a for all but finitely many a in A, then L is F-admissible for $\mathscr{K}$.*

Applying Theorem 1.4 and Corollary 1.5 to the case of the one-point compactification of an infinite discrete space gives:

Theorem 2.4. *Let Ω, F, K and L be as in Theorem 2.3. Assume in addition that Ω is algebraic over F and that each K_a is a normal extension of F. Then L is F-admissible for $\mathscr{K}$ if and only if each element of L is in K_a for all but finitely many a in A.*

Let $\mathscr{K}$ be a family of extension fields of a field F, and let L be an extension field of F. An interesting question is: If L is admissible for $\mathscr{K}$, is L necessarily F-admissible for $\mathscr{K}$? The proof of (1.2), concerning F-realizability, uses the fact that R, as an F-algebra, contains a copy of F. Suppose R is a zero-dimensional reduced ring that realizes a family of extension fields of F. It seems natural to ask whether R must then contain an isomorphic copy of F. However, this need not be: An example of Ray Heitmann [3, pp. 46–47] shows that for a simple transcendental extension field $\mathbb{Q}(t)$ of the field $\mathbb{Q}$ of rational numbers, there exists a reduced zero-dimensional ring R such that every residue field of R is isomorphic to $\mathbb{Q}(t)$, but $\mathbb{Q}(t)$ does not embed in R.

(Of course, the family of residue fields of Heitmann's example R *is* $\mathbb{Q}(t)$-realizable by a different ring.)

To prove other partial converses of Theorem 2.2, we use some preparatory results.

Lemma 2.5. *Let F be a field, and let R be a reduced zero-dimensional F-algebra. Assume that* $\operatorname{Spec}(R)=\{P_a=e_aR\}_{a\in A}\cup\{P_\infty\}$, *where A is an infinite set, $e_a^2=e_a$ for each a in A, and the composite map $F\hookrightarrow R\to R/P_\infty$ is an isomorphism. Let $f(x)$ be an irreducible monic polynomial in $F[x]$; assume for each a in A that $f(x)$ has a root c_a' in $K_a:=R/P_a$, and let c_a in R be a preimage of c_a' (possibly varying with a). There exists a reduced zero-dimensional integral extension ring S of R such that:*

(0) *the composite injection $F\hookrightarrow R\hookrightarrow S$ extends to an injection of $E=F[x]/(f(x))$ into S, so that S has the structure of an E-algebra that extends its structure as an F-algebra,*

(1) $\operatorname{Spec}(S)=\{e_aS\}_{a\in A}\cup\{P_\infty S\}$ *(for the same idempotents e_a),*

(2) *for each a in A, $S/e_aS\cong_E K_a$, and the composite map $E\to S\to S/e_aS=K_a$ maps both the element $c=x+(f(x))$ of E and the element c_a of R to c_a' in K_a, and*

(3) *the composite map $E\to S\to S/P_\infty S$ is an isomorphism.*

Proof. Let I be the ideal of the polynomial ring $R[x]$ generated by $f(x)$ and the elements of $\{(1-e_a)(x-c_a)\}_{a\in A}$, and let $S=R[x]/I$. Then $I\cap F[x]=f(x)F[x]$, so $E=F[x]/(f(x))\hookrightarrow S$ extends $F\hookrightarrow S$ and gives S the structure of an E-algebra. Since the image of e_a in $R_{P_b}=R/P_b\cong_F K_b$ is 0 if $a=b$ and 1 if $a\neq b$, we have $IR_{P_a}[x]=(x-c_a')K_a[x]$ for each a in A. Therefore e_aS is a prime ideal and for each a, $S/e_aS\cong_E K_a$ and the composite map $E=F[c]\to S\to S/e_a=K_a$ maps $c\to c_a'$. On the other hand, the image of e_a in $R_{P_\infty}=R/P_\infty=F$ is 1 for each a. Therefore $IR_{P_\infty}[x]=f(x)F[x]$. Hence $P_\infty S$ is a prime ideal and the composite map $E\to S\to S/P_\infty S=F[x]/(f(x))$ is an isomorphism of E onto $S/P_\infty S$. □

Lemma 2.5 gives an extension S of the F-algebra R such that:

(i) as topological spaces $\operatorname{Spec}(R)$ and $\operatorname{Spec}(S)$ are homeomorphic,

(ii) the residue fields of R and S at corresponding isolated points are isomorphic,

(iii) the residue field E of S at its unique non-isolated point is a simple algebraic extension of the residue field F of R at its unique non-isolated point, and

(iv) S has an E-algebra structure extending its F-algebra structure.

We show in (2.7) that in certain cases the construction given in Lemma 2.5 can be iterated to prove admissibility with respect to a family $\{K_a\}_{a\in A}$ of certain infinite algebraic extensions of F. The obstruction to a direct extension of Lemma 2.5 is that we must be sure that the set of extension rings of the base ring from repeated used of (2.5) can be arranged to be a directed union. We use the following lemma in the proof of (2.7). The idea involved in the proof may be viewed as an application of the König Graph Theorem, or, alternatively, as the fact that an inverse limit of an inverse system of finite nonempty sets is nonempty.

Lemma 2.6. *Suppose a field extension L/F is the directed union of a family $\mathscr{E}$ of subfields of L finite algebraic over F, and let K/F be a field extension. If each E in $\mathscr{E}$ has an F-embedding into K, then there is an F-embedding of L into K.*

Proof. For each $E \in \mathscr{E}$, let M_E be the (finite) set of F-embeddings of E into K. Give to each M_E the discrete topology and let $X = \prod_{E\in\mathscr{E}} M_E$ with the product topology. For each finite subset S of $\mathscr{E}$, let X_S denote the subset of X consisting of the "$\mathscr{E}$-tuples" of functions $(f_E)_{E\in\mathscr{E}}$ such that if $E_1, E_2 \in S$ with $E_1 \subseteq E_2$, then $f_{E_2}|_{E_1} = f_{E_1}$. Since $\prod_{E\in S} M_E$ has the discrete topology (because S is finite), the sets X_S are closed subsets of X. Because $\mathscr{E}$ is directed, our hypothesis implies that every finite intersection of sets X_S is nonempty. Since X is compact by Tychonoff's Theorem, $\bigcap\{X_S : S \text{ is a finite subset of } \mathscr{E}\}$ is nonempty. An element of this intersection is a coherent $\mathscr{E}$-tuple of functions that determines an F-embedding of $\bigcup_{E\in\mathscr{E}} E = L$ into K. □

Theorem 2.7. *Let L/F be a countably generated algebraic field extension, and let $\mathscr{K} = \{K_a\}_{a\in A}$ be a family of extension fields of F. Write $L = \bigcup_{n=0}^{\infty} E_n$, where $F = E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots$, and E_{n+1} is a simple extension of E_n for each $n \geq 0$. The following are equivalent*:

(i) *L is F-admissible for $\mathscr{K}$.*
(ii) *Every subfield of L/F of finite degree over F has an F-embedding into all but finitely many of the K_a's.*
(iii) *Each E_n has an F-embedding into all but finitely many of the K_a's.*

Proof. Theorem 2.2 shows that (i) implies (ii), and (ii) implies (iii) trivially. Assuming (iii), we will prove (i). Let R_0 be the (unital) F-subalgebra of the direct product $\prod_{a\in A} K_a$ generated by the direct sum $\oplus_{a\in A} K_a$. Then, by [4, (2.7)] we have

(1) $\operatorname{Spec}(R_0)$ is homeomorphic to $A^* := A \cup \{\infty\}$, the one-point compactification with, say, P_a the prime ideal corresponding to the point $a \in A$ and with $P_{0\infty}$ the non-principal prime ideal;
(2) $R_0/P_a \cong_F K_a$ for all a in A; and
(3) the composite map $F \hookrightarrow R_0 \to R_0/P_{0\infty}$ is an F-isomorphism.

Choose finite subsets $B_1 \subseteq B_2 \subseteq \cdots$ of A such that for $a \notin B_n$ the field E_n is F-isomorphic to a subfield of K_a. For $a \in A - (\bigcup_{n=1}^{\infty} B_n)$ choose (using (2.6)) an F-embedding $\phi_a : L \to K_a$, and let $\phi_{an} : E_n \to K_a$ be the restriction of ϕ_a. For a in $B_{n+1} - B_n$, let $\phi_{an} : E_n \to K_a$ be an F-algebra embedding, and for each $j \leq n$, define $\phi_{aj} : E_j \to K_a$ to be the restriction of ϕ_{na}. Thus ϕ_{am} is defined for all pairs (m, a) such that $a \notin B_m$; and if $j < m$, then ϕ_{am} extends ϕ_{aj}, so ϕ_{am} is an E_j-algebra embedding of E_m into K_a.

For each $a \in A$ write $P_a = e_a R_0$ with $e_a^2 = e_a$. We will use Lemma 2.5 to build an ascending chain $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ of reduced zero-dimensional F-algebras, each integral over the one below, such that for each n:

(1′) $\operatorname{Spec}(R_n) = \{e_a R_n\}_{a\in A} \cup \{P_{n\infty}\}$; and for each a in A, $R_n/e_a R_n \cong_F K_a$;

(2′) for the idempotent element $w_n = \sum_{b \in B_n}(1-e_b)$ of R_0, R_n/w_nR_n has an E_n-algebra structure extending its F-algebra structure such that for all $a \notin B_n$, the composite F-algebra embedding $E_n \to R_n/w_nR_n \to R_n/e_aR_n = K_a$ is ϕ_{na}; and

(3′) the composite map $E_n \to R_n/w_nR_n \to R_n/P_{n\infty}$ is an F-isomorphism of E_n onto $R_n/P_{n\infty}$.

Assuming for the moment that we have our ascending chain of rings, we put $R_\infty = \bigcup_{n=1}^{\infty} R_n$ and $P_\infty = \bigcup_{n=1}^{\infty} P_{n\infty}$. It is easily checked that R_∞ is an F-realization of the pair $(A^*, \mathscr{K}^*)$, where $K_\infty = L$ and $\mathscr{K}^* = \{K_a\}_{a\in A^*}$. This proves (i).

To build the chain of rings, assume inductively that we have constructed R_n. Write $E_{n+1} = E_n(c)$, and let $g(x)$ be the minimal polynomial of c over E_n. Let $c'_a := \phi_{a,n+1}(c)$ for each $a \in A - B_n$, and let $c_a \in T_n := R_n/w_nR_n$ be a preimage of c'_a. Observe that T_n satisfies the conditions of (2.5) with respect to the field E_n, the family of E_n-algebras $\{K_a\}_{a\in A-B_n}$, the polynomial $g(x)$, and the elements c, c'_a and c_a. Let S_n be the integral extension ring of T_n given by (2.5), and set $R_{n+1} := w_nR_n \times S_n$. Then R_{n+1} is an extension ring of R_n satisfying the conditions (1′),(2′),(3′). This completes the induction and the proof of (2.7). □

In the case where L is algebraic over its prime subfield, Theorem 2.7 gives the following:

Corollary 2.8. *Let L be a field algebraic over its prime field F, and let $\mathscr{K} = \{K_a\}_{a\in A}$ be a family of fields of the same characteristic as F. Then L is admissible for $\mathscr{K}$ if and only if each subfield of L finite over F embeds in all but finitely many K_a's.*

Remark 2.9. With $\{K_a\}_{a\in A}$, L and F as in (2.2), let K'_a denote the algebraic closure of F in K_a, and let L' be the algebraic closure of F in L. Then if L is F-admissible for $\{K_a\}_{a\in A}$, then L' is F-admissible for $\{K'_a\}_{a\in A}$. To see this, suppose R is a zero-dimensional reduced F-algebra such that $\mathrm{Spec}(R) = \{P_a\}_{a\in A} \cup \{Q\}$, where each P_a is principal with $R/P_a \cong_F K_a$ and $R/Q \cong_F L$. Then the integral closure R' of F in R contains all the idempotent elements of R; so the canonical map $\mathrm{Spec}(R) \to \mathrm{Spec}(R')$ is a bijective map of sets. Also, the fact that R' is integrally closed in R implies that $S^{-1}R'$ is integrally closed in $S^{-1}R$ for each multiplicatively closed subset S of R'; cf. [1, (5.12)]; hence if $P'_a = P_a \cap R'$ and $Q' = Q \cap R'$, then $\mathrm{Spec}(R') = \{P'_a\}_{a\in A} \cup \{Q'\}$, where $R'/P'_a = K'_a$ and $R'/Q' = L'$. Therefore L' is F-admissible for $\{K'_a\}_{a\in A}$. Moreover, in view of (2.7), if L'' is a countably generated subfield of L'/F, then L'' is F-admissible for both $\{K_a\}_{a\in A}$ and $\{K'_a\}_{a\in A}$.

3. More examples on admissibility

(3.1) Example (3.6) of [4] shows the existence of a zero-dimensional reduced ring R with the following properties: (1) $\mathrm{Spec}(R) = \{P_1, P_2, \ldots\} \cup \{Q\}$, where each P_i is principal, so $\mathrm{Spec}(R)$ is the one-point compactification of a countably infinite discrete space, and the field R/Q is admissible for the family $\{R/P_i\}_{i=1}^{\infty}$, (2) the fields R/P_i are pairwise incomparable finite fields of the same characteristic, while the field R/Q is an absolutely algebraic infinite field of the same characteristic, (3) none of the fields R/P_i can be embedded in R/Q.

(3.2) *Steinitz numbers*: Let F be a prime field of positive characteristic, and let Ω be an algebraic closure of F. Suppose $\{K_a\}_{a\in A}$ is a family of subfields of Ω. Corollary 2.8 gives precise conditions in order that a subfield L of Ω be admissible for $\{K_a\}_{a\in A}$. Each subfield K of Ω is uniquely determined by its Steinitz number defined as follows: let $\{p_i\}_{i=1}^{\infty}$ be an enumeration of the prime numbers. The *Steinitz number*, $\mathscr{S}(K)$, of K is defined to be the tuple $(s_1, s_2, \ldots)$, where each s_i is either a nonnegative integer or the symbol ∞, determined as follows: if K contains a field extension of F of degree p_i^m, but does not contain an extension of F of degree p_i^{m+1}, then $s_i = m$, while if K contains an extension of F of degree p_i^m for each nonnegative integer m, then $s_i = \infty$. Each tuple $(s_1, s_2, \ldots)$ is the Steinitz number of a subfield of Ω, and for subfields K and L of Ω with Steinitz numbers $\mathscr{S}(K) = (s_1, s_2, \ldots)$ and $\mathscr{S}(L) = (t_1, t_2, \ldots)$, we have $K \subseteq L$ if and only if $s_i \leq t_i$ for every positive integer i.

Suppose $\{K_a\}_{a\in A}$ is a family of subfields of Ω and $\mathscr{S}(K_a) = (s_{a1}, s_{a2}, \ldots)$. We associate with $\{K_a\}_{a\in A}$ the tuple $(e_1, e_2, \ldots)$, where we define e_i to be the nonnegative integer m if $s_{ai} \geq m$ for all but finitely many a and $s_{ai} = m$ for infinitely many a; while if $s_{ai} \geq m$ for all but finitely many a holds for each nonnegative integer m, we define $e_i = \infty$. From (2.8) it follows that a subfield L of Ω with Steinitz number $\mathscr{S}(L) = (t_1, t_2, \ldots)$ is admissible for the family $\{K_a\}_{a\in A}$ if and only if $t_i \leq e_i$ for every positive integer i. Thus, for example, we have: (i) there exists an infinite field L that is admissible for $\{K_a\}_{a\in A}$ if and only if either infinitely many of the e_i are nonzero or at least one of the $e_i = \infty$; (ii) the field Ω is admissible for $\{K_a\}_{a\in A}$ if and only if every $e_i = \infty$; and (iii) the only subfield of Ω admissible for $\{K_a\}_{a\in A}$ is F if and only if every $e_i = 0$. We illustrate with several examples.

(3.2.1) For each positive integer n, define K_n to be the subfield of Ω having Steinitz number $\mathscr{S}(K_n) = (n, n-1, \ldots, 2, 1, 0, 0, \ldots)$. Then each K_n is of finite degree over F, and the family $\{K_n\}_{n=1}^{\infty}$ has the property that each associated $e_i = \infty$. Therefore Ω and every subfield of Ω is admissible for $\{K_n\}_{n=1}^{\infty}$.

(3.2.2) For each positive integer n, define K_n to be the subfield of Ω having Steinitz number $\mathscr{S}(K_n) = (1, 1, \ldots, 1, 0, 0, \ldots)$, that is, 1's in the first n positions and then 0's. Then each K_n is of finite degree over F, and the family $\{K_n\}_{n=1}^{\infty}$ has the property that each associated e_i is equal to 1. Therefore a subfield L of Ω is admissible for $\{K_n\}_{n=1}^{\infty}$ if and only if the degree over F of every element of L is a product of distinct prime numbers.

(3.2.3) To obtain an example such as (3.6) of [4] mentioned in (3.1), let K_n have Steinitz number $\mathscr{S}(K_n) = (s_{n1}, s_{n2}, \ldots)$, where $s_{n1} = n, s_{nn} = 1$, and $s_{ni} = 0$, for i different from 1 and n.

(3.3) Since the algebraic closure Ω of a countable field F is again countable, an easy way to get a family $\{K_n\}_{n=1}^{\infty}$ of finite algebraic extensions of F such that Ω is F-admissible for $\{K_n\}_{n=1}^{\infty}$ is to enumerate the elements of Ω, say $\Omega=\{c_n\}_{n=1}^{\infty}$, and let $K_n=F(c_1,\dots,c_n)$. Then every subfield E of Ω/F of finite degree over F is F-isomorphic to a subfield of K_n for all but finitely many of the K_n. Hence by (2.7), Ω is F-admissible for $\{K_n\}_{n=1}^{\infty}$.

Discussion 3.4. Given a family of fields $\mathscr{K}=\{K_i\colon i=0,1,2,\dots\}$, consider the following conditions:
(1) $K_0:=L$ is admissible for the family $\mathscr{G}=\{K_i\colon i=1,2,\dots\}$.
(2) There is a one-dimensional integral domain with fraction field L and with $\mathscr{G}$ the family of residue fields at the maximal ideals.
(3) There is a one-dimensional integral domain with Noetherian spectrum, with fraction field L and with $\mathscr{G}$ as the family of residue fields at the maximal ideals.
(4) There is a one-dimensional Noetherian domain with fraction field L and with $\mathscr{G}$ the family of residue fields at the maximal ideals.
(5) There is a principal ideal domain with fraction field L and with $\mathscr{G}$ the family of residue fields at maximal ideals.

Obviously $(5)\Rightarrow(4)\Rightarrow(3)\Rightarrow(2)$, and by (2.1), $(3)\Rightarrow(1)$. There are simple examples to show that, in general, (1) does not imply (2). For example, if L is a finite field, then L is not the fraction field of an integral domain of positive dimension, but L is admissible for the family $\mathscr{G}$ where each $K_j\cong L$.

Also, the fact that a ring S with Noetherian spectrum has for each positive integer m only finitely many prime ideals P such that $|S/P|\le m$ [5, Theorem 2.2(c)] implies the existence of examples showing that (2) does not imply (3). To get an example of a one-dimensional integral domain S having infinitely many maximal ideals P such that $|S/P|\le m$, one can start with the prime field F of characteristic 3 and realize S as an infinite integral extension of the polynomial ring $F[t]$. Making use of the argument given in [7, (1.2)], it is possible to obtain an infinite chain $F(t)=K_0\subset K_1\subset K_2\dots$ of algebraic field extensions of $F(t)$ such that $[K_{n+1}:K_n]=2$ for each positive integer n, and such that the integral closure S of $F[t]$ in the field $\bigcup_{n=1}^{\infty}K_n$ has infinitely many maximal ideals P lying over $tF[t]$ in $F[t]$, each of these maximal ideals P having the property that $S/P\cong F$.

4. Realizability for countable Boolean spaces

Remarks 4.1. (4.1.1) If X is a Boolean space with finitely many non-isolated points, then X is the disjoint union of finitely many one-point compactifications of discrete spaces. Thus the question of whether a family of fields $\{K_x\}_{x\in X}$ is realizable for X reduces to admissibility considerations.

(4.1.2) Let X be a set and $\mathscr{K}=\{K_x\}_{x\in X}$ a family of algebraic extension fields of a prime field F. Corollary 2.8 implies that $\mathscr{K}$ is realizable for the one-point compactification of a discrete space if and only if there exists L in $\mathscr{K}$ such that every subfield of L of finite degree over F embeds in all but finitely many of the K_x. More generally, suppose n is a positive integer. The family $\mathscr{K}$ is realizable for a Boolean space with n non-isolated points if and only if there exist $L_1,\dots,L_n \in \mathscr{K}$ such that $\mathscr{K}$ can be expressed as the disjoint union of n infinite subfamilies, say $\mathscr{K}_1,\dots,\mathscr{K}_n$, such that for $1 \le i \le n$ every subfield of L_i of finite degree over F embeds in all but finitely many of the fields in $\mathscr{K}_i$.

(4.1.3) In view of (4.1.2), it is easy to give examples of families of fields that are realizable for a countable Boolean space with n non-isolated points, but are not realizable for a Boolean space with fewer than n non-isolated points. For example, let F be a finite prime field, let Ω be an algebraic closure of F, and let $p_1,\dots,p_n$ be distinct prime numbers. For $1 \le i \le n$ and $j \ge 1$, let K_{ij} be the unique subfield of Ω of degree p_i^j over F. The family of fields $\mathscr{K} = \bigcup_{i=1}^n \mathscr{K}_i$, where $\mathscr{K}_i = \{K_{ij}\}_{j=1}^\infty$, is realizable for a Boolean space with n non-isolated points, but Theorem 2.2 implies that $\mathscr{K}$ is not realizable for a Boolean space with fewer than n non-isolated points.

(4.1.4) There also exist families of fields that are realizable for a countable Boolean space with one non-isolated point, but not for a Boolean space with more than one non-isolated point. For example, let F be a prime field, let $\{p_j\}_{j=1}^\infty$ be an infinite set of distinct prime numbers, and let K_j be an extension field of F of degree p_j in a fixed algebraic closure of F. The family $\mathscr{K} = \{F\} \cup \{K_j\}_{j=1}^\infty$ is realizable for a Boolean space that is the one-point compactification of a countable discrete space, but is not realizable for a Boolean space with more than one non-isolated point.

(4.1.5) Let F be a prime field, and suppose $\mathscr{K} = \{K_x\}_{x\in X}$ is a family of field extensions of F each of which is of finite degree over F. If $\mathscr{K}$ is realizable, then Theorem 3.1 of [4] implies there exists a finite subset $\{K_i\}_{i=1}^n$ of $\mathscr{K}$ such that each K_x contains an isomorphic copy of K_i for some i. Hence the realizability of a family $\mathscr{K}$ of extension fields of F of finite degree implies the realizability of this family $\mathscr{K}$ for a Boolean space with finitely many non-isolated points.

(4.1.6) There exists a family of algebraic extensions of a finite prime field F that is realizable for a Boolean space X having infinitely many non-isolated points, but is not realizable for a Boolean space with finitely many non-isolated points. To obtain such an example one can modify [4, (3.6)], which is described in (3.1) above, as follows: (1) to each $R/P_i = K_i$ associate a countably infinite number of copies K_{ij}, $j = 1,2\dots,$ of K_i, (2) let $T_i = \prod_{j=1}^\infty K_{ij}$, and let R_i be the K_i-subalgebra of T_i generated by the direct sum ideal of T_i (thus, Spec(R_i) is a Boolean space with one non-isolated point, and R_i is a realization of the family of fields $\{K_{ij}\}_{j=1}^\infty \cup \{K_i\}$), (3) let $T = \prod_{i=1}^\infty R_i$, and let S be the (unitary) subring of T generated by the direct sum ideal $I = \oplus_{i=1}^\infty R_i$ of T. Then I is a prime ideal of S, and $S/I \cong R/Q = L$. The family of residue fields of S is the disjoint union of the family of residue fields of the R_i together with $\{L\}$. To see that the family of residue fields of S is not realizable for a Boolean space with finitely many non-isolated points we make use of the incomparability of the K_i

and L. Assume that S' is a reduced zero-dimensional ring that has the same family of residue fields as S. Let P'_{ij} be the prime of S' at which the residue field is K_{ij}. Then the sets $\{P'_{i,j}\}_{j=1}^{\infty}$, $i=1,2,\ldots$, are infinite pairwise disjoint sets in Spec(S'), and in view of (1.2) each one is closed, so each must contain a different non-isolated point.

Several of the previous remarks concern the decomposition of a realizable family into several subfamilies, each realizable in its own right. The following question concerns the patching together of realizable families:

Question 4.2. Let X be a countably infinite Boolean space and $\{K_x\colon x\in X\}=\mathscr{K}$ be a family of fields. Assume that for every closed subset Y of X such that Y is homeomorphic to the one-point compactification of a countable discrete space the family $\{K_y : y\in Y\}$ is realizable for Y. Does it follow that $\mathscr{K}$ is realizable for X?

(4.3) Suppose R is a zero-dimensional reduced ring. If K is a finite field such that $R/P\cong K$ for each P in an infinite subset Y of Spec(R), then there exists a non-isolated point Q in Spec(R) such that R/Q embeds in K. (To see this, choose $f\in\mathbb{Z}[x]$ such that $f(\alpha)=0$ for every $\alpha\in K$. Now $R'=R/(\bigcap_{P\in Y}P)$ embeds in $\prod_{P\in Y}(R/P)$, and every element of the product is also a root of f. Thus, for each maximal ideal Q' of R', the residue field R'/Q' consists of roots of f. Choosing Q' to be an accumulation point of Y in Spec(R') and letting Q be the inverse image of Q' in R yields the result.) But as noted in (2.1.1), if K is an infinite field, then the transcendental extension field $K(x)$ is admissible for the family consisting of infinitely many copies of K.

Discussion 4.4. A well-known topological result states that an infinite Boolean space with no isolated points must be uncountable. (An argument for this is as follows: Suppose X is an infinite Boolean space having no isolated points. We can then write $X=U_0\cup U_1$, where U_0 and U_1 are nonempty disjoint clopen subsets of X, and then write $U_i=U_{i0}\cup U_{i1}$, where U_{i0} and U_{i1} are nonempty disjoint clopen subsets of $U_i, i=0$ or 1. The assumption that X has no isolated points implies that this process can be continued to obtain for each sequence of 0's and 1's a descending sequence of nonempty clopen subsets of X. Since X is compact, these descending sequences of clopen subsets of X have nonempty intersections, which implies that X is uncountable.) Therefore if X is a countably infinite Boolean space, then X has an isolated point. Applying this to clopen subsets of X implies that the isolated points of X are dense in X. It follows that, in a countably infinite Boolean space, a non-isolated point is the limit of a sequence of isolated points. Therefore we have:

Corollary 4.5. *Suppose that X is a countable Boolean space and p is a non-isolated point of X. There exists a closed subspace Y of X containing p such that p is non-isolated in Y and Y is the one-point compactification of a denumerable set of isolated points of X.*

In algebraic terms, Corollary 4.5 implies:

Corollary 4.6. *Suppose R is a reduced zero-dimensional ring such that* Spec(R) *is countably infinite. Let Q be a maximal ideal of R that is not finitely generated. There exists an ideal $I \subset Q$ such that Q/I is the unique non-principal maximal ideal of R/I.*

The following corollary generalizes the example following (1.3).

Corollary 4.7. *Let Ω be an algebraic closure of the field F, and let $\mathscr{L} = \{L_n\}_{n=1}^{\infty}$ be a countably infinite family of fields with $F \subset L_n \subseteq \Omega$ for each n. Assume $L := \bigcap_{n=1}^{\infty} L_n$ is not in $\mathscr{L}$, and that for every infinite set W of natural numbers, $\bigcap_{n\in W} L_n = L$. Then $\mathscr{L}$ is not a realizable family.*

Proof. Suppose $\mathscr{L}$ is the family of residue fields of the zero-dimensional ring R. By (4.6) there is an infinite subfamily $\mathscr{K} = \{K_i\}_{i=1}^{\infty}$ of $\mathscr{L}$ that is the family of residue fields of a homomorphic image S of R, where S has precisely one maximal ideal Q that is not finitely generated. Then $S/Q = K \in \mathscr{L}$. Since $L \notin \mathscr{L}$, $L \subsetneq K$. Take c in $K - L$, let $c = c_1, \ldots, c_s$ be the conjugates of c over F in K, and set $E = F(c_1, \ldots, c_s)$. By (2.2), for all but finitely many i, there exists an embedding of E into K_i. Since there are only finitely many different F-embeddings of E into Ω, it follows that the same F-isomorphic copy of E is contained in infinitely many of the K_i, so it is contained in L. But the only F-isomorphic copy of E in K is E itself. This contradicts the fact that $c \notin L$. □

(4.8) If one does not assume the fields L_n in (4.6) are algebraic over a common field, then it may happen that the fields are all isomorphic and hence the family is realizable. For example, if $\{x_i\}_{i=1}^{\infty}$ is a set of indeterminates over a field F and $L_n = F[\{x_i\}_{i>n}]$, then $\{L_n\}_{n=1}^{\infty}$ is realizable.

We close by proving a fairly general theorem (4.9) on F-realizability for countable Boolean spaces. We use a method somewhat different from that in Section 2.

Recall the Cantor–Bendixson derivative of a topological space X: $X^{(0)} = X$, for the ordinal α, $X^{(\alpha+1)} = X^{(\alpha)} - \{\text{isolated points of } X^{(\alpha)}\}$, and $X^{(\beta)} = \bigcap_{\alpha<\beta} X^{(\alpha)}$ if β is a limit ordinal. If X is a denumerable Boolean space, we know from (4.4) that $X^{(\alpha)} - X^{(\alpha+1)}$ is a dense subset of $X^{(\alpha)}$, so there is an ordinal γ such that $X^{(\gamma)} = \emptyset$. By compactness, the least such ordinal γ is a successor; we let $\lambda(X)$ be the smallest ordinal λ such that $X^{(\lambda+1)} = \emptyset$. For an element $x \in X$, we let $\lambda(x)$ be the least ordinal α such that $x \notin X^{(\alpha+1)}$.

The following theorem should be compared with (1.4). Note that here we impose no compatibility assumptions on the embeddings. The proof consists of "straightening" the embeddings to make them compatible.

Theorem 4.9. *Let X be a denumerable Boolean space, let F be a field, and let $\mathscr{K} = \{K_x\colon\ x \in X\}$ be a family of extension fields of F indexed by X. Assume* *either that $\lambda(X)$ is finite or that each K_x is a finite algebraic extension of F. Suppose each point x in X has a neighborhood U_x such that K_x has an F-embedding into K_y for every y in U_x. Then the pair $(X,\mathscr{K})$ is F-realizable.*

Before embarking on the proof we note the following corollary of (4.9) and (1.2).

Corollary 4.10. *Let X be a denumerable Boolean space, let F be a field, and let $\mathscr{K}=\{K_x: x\in X\}$ be a family of extension fields of F indexed by X. Assume $[K_x:F]<\infty$ for each x in X. Then the pair $(X,\mathscr{K})$ is F-realizable if and only if each point x in X has a neighborhood U_x such that K_x has an F-embedding into K_y for every $y\in U_x$.*

It is interesting to compare (4.10) with (3.1) and (3.1A) of [4] where realizability is considered on the family $\mathscr{K}=\{K_x: x\in X\}$ without regard to the Boolean topology on X.

Theorem 4.9 is a special case of the following:

Theorem 4.11. *Let X be a denumerable Boolean space, let F be a field, and let $\mathscr{K}=\{K_x: x\in X\}$ be a family of extension fields of F indexed by X. Assume that for every x in X there exist a neighborhood U_x of x and an F-embedding $f_{yx}:K_x\to K_y$ for every y in U_x. Assume further that the following condition is satisfied:*

(†) *There is no infinite sequence $x_1,x_2,x_3,\dots$ in X such that for all $i\geq 1$,*

(i) $x_i\in U_{x_{i+1}}$,

(ii) $\lambda(x_i)<\lambda(x_{i+1})$, *and*

(iii) K_{i+1} *is F-isomorphic to a proper subfield of K_i (where $K_j=K_{x_j}$).*

Then the pair $(X,\mathscr{K})$ is F-realizable.

Proof. Our first task is to shrink the open sets U_x in order to eliminate potential incompatibility problems among the field embeddings. We claim that one can find, for every x in X, a clopen set V_x such that the following are true:

(1) $x\in V_x\subseteq U_x$ for every x in X.

(2) If $y\in V_x$ and $y\neq x$, then $\lambda(y)<\lambda(x)$.

(3) If $V_x\cap V_y\neq\emptyset$ then either $y\in V_x$ or $x\in V_y$.

To see this, note that $D_\alpha:=X^{(\alpha)}-X^{(\alpha+1)}$ is discrete in the relative topology for every $\alpha\leq\lambda(X)$. Thus for each $x\in X$ there is a neighborhood W_x of x such that $W_x\cap D_{\lambda(x)}=\{x\}$. If we now choose, for every $x\in X$, a clopen set V_x such that $x\in V_x\subseteq U_x\cap W_x$, we will satisfy requirements (1) and (2). To accomplish (3) we may need to shrink the sets V_x. We do this by enumerating X, say, $X=\{x_1,x_2,x_3,\dots\}$, and by replacing V_{x_n} by $V_{x_n}-\bigcup\{V_{x_i}: i<n \text{ and } x_n\notin V_{x_i}\}$.

Using our clopen sets V_x, we will introduce a useful partial ordering on X. We first define a relation # on X by declaring that

$$x\,\#\,y\Leftrightarrow x\neq y \quad\text{and}\quad x\in V_y.$$

The relation # can be used to define a partial order $\preceq$ on X: $x \preceq y$ if and only if there is a sequence $x = z_0 \# z_1 \# \cdots \# z_m = y$ for some $m \geq 0$. (The relation $\preceq$ is anti-symmetric by (2): If $x \# y$, then $\lambda(x) < \lambda(y)$.) We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$.

For future reference we note

(4) $\lambda(x) < \lambda(y)$ if $x \prec y$.

The following observation tells us that our partially ordered set is actually a tree:

Claim. For each $x \in X$, the set $C_x := \{y \in X: x \preceq y\}$ is a chain.

To see this, let $p, q \in C_x$, say, $x = z_0 \# z_1 \# \cdots \# z_m = p$ and $x = w_0 \# w_1 \# \cdots \# w_n = q$. We may assume that $m > 0$ and $n > 0$ and proceed by induction on $m + n$. Since $x \in V_{z_1} \cap V_{w_1}$, we may assume, using (3) and symmetry, that either $z_1 = w_1$ or $z_1 \# w_1$. Then $p, q \in C_{z_1}$, and by induction (since $m + n$ has dropped by either 2 or 1) we have either $p \preceq q$ or $q \preceq p$. This completes the proof of the claim.

Now consider the chains C_x for x in $X - X^{(1)}$. Suppose for the moment that we have defined compatible F-embeddings $\sigma_{yz} = \sigma_{yz}^x : K_z \to K_y$ whenever $y, z \in C_x$ and $y \preceq z$. The compatibility conditions are

(a) σ_{yy} is the identity map on K_y for every $y \in C_x$, and

(b) $\sigma_{yz}\sigma_{zw} = \sigma_{yw}$ if $y, z, w \in C_x$ and $y \preceq z \preceq w$.

(The temporary superscript x reflects the possibility that the σ_{yz} may depend on x.)

Still assuming that we have defined compatible embeddings on each chain C_x, we now wish to define σ_{yz} for *every* pair y, z of elements of X with $y \preceq z$. Note first that U_y contains a point of $X - X^{(1)}$ (since $X - X^{(1)}$ is dense in X) and thus $y, z \in C_x$ for some x in $X - X^{(1)}$. The only problem is to define the embeddings consistently on the union of the various chains. Enumerate $X - X^{(1)}$, say, $X - X^{(1)} = \{x_1, x_2, \ldots\}$, and suppose we have defined an F-embedding $\sigma_{yz} : K_z \to K_y$ whenever $y \leq z$ and $y, z \in T_n := C_{x_1} \cup \cdots \cup C_{x_n}$ in such a way that

(c) σ_{yy} is the identity map on K_y for every $y \in T_n$, and

(d) $\sigma_{yz}\sigma_{zw} = \sigma_{yw}$ if $y, z, w \in T_n$ and $y \preceq z \preceq w$.

If $C_{x_{n+1}} \cap T_n = \emptyset$, we extend our definition to T_{n+1} by letting σ be $\sigma^{x_{n+1}}$ on $C_{x_{n+1}}$. Otherwise, we let t be the smallest element of $C_{x_{n+1}} \cap T_n$. (Note that $C_{x_{n+1}}$ is well-ordered, by (3) and (4).) Now $C_{x_{n+1}}$ is the union of two pieces meeting only at t, namely $A := \{y \in C_{x_{n+1}}: y \preceq t\}$ and $B := \{y \in C_{x_{n+1}}: t \preceq y\}$. Since $B \subseteq T_n$, we just need to know how to define σ_{yz} when either y or z is in A. If both y and z are in A we use $\sigma^{x_{n+1}}$. If $y \in A$ and $z \in T_n$ we cannot have $z \prec y$, so suppose $y \preceq z$. Then $t \preceq z$ by the Claim, and we simply let $\sigma_{yz} = \sigma_{yt}^{x_{n+1}}\sigma_{tz}$.

Still assuming that we have defined compatible embeddings on each chain C_x for x in $X - X^{(1)}$, we now have, for each pair of elements $y \preceq z$ in X, an F-embedding $\sigma_{yz} : K_z \to K_y$. Moreover, since any three elements with $y \preceq z \preceq w$ all belong to some chain C_x, our embeddings satisfy the compatibility conditions:

(e) σ_{yy} is the identity map on K_y for every $y \in X$, and

(f) $\sigma_{yz}\sigma_{zw} = \sigma_{yw}$ if $y, z, w \in X$ and $y \preceq z \preceq w$.

Once we have these maps, the proof of the theorem is rather standard, but we will put in the details. Let $\bigsqcup\mathscr{K}$ denote the disjoint union of the fields K_x, and let $\pi : \bigsqcup\mathscr{K} \to X$ be the map sending each α in K_x to x. We topologize $\bigsqcup\mathscr{K}$ as follows: Given α in K_x, a basic open neighborhood of α is a set of the form $W(\alpha) := \{\sigma_{yx}(\alpha)\colon y \in W\}$, where W is an open neighborhood of x in V_x. With this topology π maps $W(\alpha)$ homeomorphically onto W and thus is open and locally homeomorphic. Thus we have a sheaf. (The compatibility of the maps is needed here: If $\beta = \sigma_{yx}(\alpha) \in K_y$ with $y \in W$ we need to know that β has a basic neighborhood that is contained in $W(\alpha)$. But this is easy: If $W'(\beta)$ is any basic neighborhood of β, then $(W \cap W')(\beta)$ is contained in $W(\alpha)$ by the compatibility conditions.)

Clearly $R := \Gamma(X, \mathscr{K})$ is a reduced F-algebra. For each x in X let P_x be the kernel of the map $R \to K_x$ taking ρ to $\rho(x)$. We want this map to be surjective. Given any α in K_x, define $\rho(y) = 0$ outside the clopen set V_x, and for z in V_x we put $\rho(z) = \sigma_{zx}(\alpha)$. Then ρ is a global section whose value at x is α, as desired. So now we know that $R/P_x \cong_F K_x$.

Next we show that R is von Neumann regular. Given $\rho \in R$, let C be the support of ρ (a closed set for any sheaf). Then C is also open. (For, let $x \in C$, say, $\rho(x) = \alpha \in K_x - \{0\}$. Choose, by continuity, a neighborhood N of x such that $\rho(y) \in V_x(\alpha)$ for every $y \in N$. Since field homomorphisms are injective, $N \subseteq C$.) Now define a section τ by letting $\tau(x) = 1/\rho(x)$ if $x \in C$ and $\tau(x) = 0$ if $x \notin C$. Then $\rho\tau\rho = \rho$.

The map $x \mapsto P_x$ is one-to-one: If $x \neq y$ there is a clopen set D containing x but not y. There is then a section taking the value 0 on D and 1 on $X - D$.

Every maximal ideal of R is of the form P_x: For, suppose I is an ideal not contained in any P_x. Choose ρ_x in $I - P_x$, and let C_x be the support of ρ_x. As shown above C_x is clopen, and we can choose a finite disjoint clopen refinement of the covering $\{C_x\colon x \in X\}$, say $\{C_1, \ldots, C_n\}$. We have sections ρ_i with $\mathrm{Supp}(\rho_i) \supseteq C_i$. Let ε_i be the idempotent with support C_i. Then $\varepsilon_1\rho_1 + \cdots + \varepsilon_n\rho_n$ is a unit in I.

Now we have a bijective map $X \to \mathrm{Spec}(R)$. Since both spaces are compact Hausdorff, the map is a homeomorphism if it is continuous. But continuity is easy: Given a basic open set $\mathrm{D}(\rho)$ of $\mathrm{Spec}(R)$, where $\rho \in R$, we have $\mathrm{D}(\rho) = \{P_x\colon \rho \notin P_x\} = \{P_x\colon \rho(x) \neq 0\}$, and the inverse image of this set under the map $X \to \mathrm{Spec}(R)$ is the support of ρ, which we know is clopen.

We still need to define compatible maps on each chain $C := C_x$, where x is now a fixed element of $X - X^{(1)}$. (This is where we need the extra assumption (†).) Suppose first that C is finite, say, $C = \{x_0, \ldots, x_n\}$, with $x = x_0 \# \cdots \# x_n$. We have $x_{i-1} \in V_{x_i}$ for $1 \le i \le n$. Therefore if $0 \le i \le j \le n$ we define $\sigma_{x_i x_j} = f_{x_i x_{i+1}} f_{x_{i+1} x_{i+2}} \cdots f_{x_{j-1} x_j}$.

Finally, suppose that C is infinite. Since C is well-ordered by $\prec$, we will define our F-embeddings on C by transfinite induction. For each $t \in C$, let $E_t = \{y \in C\colon y \prec t\}$. Suppose we have defined F-embeddings σ_{yz} on E_t, for some $t > 0$, in such a way that (a) and (b) are satisfied. We want to extend our definitions to $E_t \cup \{t\}$. If t is a successor in C, let u be its predecessor. Then $u \in V_t$, and if $y \in E_t$ we let $\sigma_{yt} = \sigma_{yu} f_{ut}$ (and we let σ_{tt} be the identity).

If t is not a successor, there is a sequence $y_1 \prec y_2 \prec y_3 \prec \dots$ in E_t with $t = \sup\{y_i\colon i \geq 1\}$. (Countability is used here.) By (4) and assumption (†) there is an m such that $\sigma_{y_m y_n}$ is an isomorphism for every $n \geq m$. If $y_m \preceq p \preceq q \prec t$ we can choose n such that $q \preceq y_n$ and conclude easily that σ_{pq} is an isomorphism. Since $y_m \prec t$ there is an element p such that $y_m \preceq p \# t$. If, now, $q \in E_t$ we define

$$\sigma_{tq} = \begin{cases} \sigma_{pq}^{-1} f_{pt} & \text{if } p \preceq q \\ \sigma_{qp} f_{pt} & \text{if } q \preceq p. \end{cases}$$

We define σ_{tt} to be the identity map. A simple transfinite induction completes the proof of the theorem. □

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