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On the Hamiltonian analysis of non-linear massive gravity

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ABSTRACT

In this Letter we present a very simple and independent argument for the absence of the Boulware–Deser ghost in the recently proposed potentially ghost-free non-linear massive gravity. The limitation is that, in its simple form, the argument is, in a sense, non-constructive and less explicit than the standard approach. However, the formalism developed here may prove to be useful for discussing the formal aspects of the theory.

1. Introduction

It has been known for a very long time that giving a mass to the graviton in a stable and viable manner is a very difficult task, if not impossible. At the linear level the only healthy mass term is the one found by Fierz and Pauli in their classical paper [1]. It propagates only five degrees of freedom in accordance with the general expectations for a massive spin-2 particle, while generically there would be six independent variables in the theory, with the sixth one representing a ghost. Unfortunately, the linear theory anyway contradicts observations due to the scalar graviton which couples to dust modifying the effective gravitational constant, but not to radiation keeping the bending of light intact. However, it was later argued by Vainshtein [2] that non-linear effects will take over at small scales and save the whole day. But, almost at the same time, Boulware and Deser have shown [3] that even with the Fierz–Pauli mass term, be there any non-linear Vainshtein mechanism or not, the sixth degree of freedom comes back at the non-linear level reintroducing the ghost mode. And therefore a stable theory of massive gravity is probably not possible at all.

However, recently a proposal for a ghost-free massive gravity has appeared [4–6]. This model has been extensively analysed in the perturbation theory, and the absence of the sixth mode was shown explicitly up to the fourth order in perturbations [7,8]. At the same time, a fully non-linear Hamiltonian analysis has been done and proved the existence of the Hamiltonian constraint non-perturbatively [9,10]. After that, some doubts were expressed in the literature [11] as to the existence of the secondary constraint generated by the Hamiltonian one which is needed for a consistent elimination of the sixth degree of freedom. But very recently the secondary constraint in non-linear massive gravity was evaluated (almost) explicitly [12] thus finalising the proof of absence of the Boulware–Deser ghost in this class of models. It justifies the large amount of interest which has been drawn towards understanding the phenomenological consequences of the new model. And now there is a number of papers discussing the black holes [13–16] and cosmological solutions [17–19] in the massive gravity, as well as the first interesting results concerning the cosmological perturbations [20].

The ADM analysis of [9,10,12] is technically quite involved; and given the paramount importance of the topic, we find it necessary to have a thorough understanding of this non-linear phenomenon of ghost exorcision from various vantage points. In this Letter we would like to offer very simple arguments for the absence of the sixth degree of freedom in de Rham–Gabadadze–Tolley (dRGT) gravity at the fully non-perturbative level. In Section 2 we give a brief review of the standard ADM analysis and its application to massive gravity models. In Section 3 we introduce our set-up, and explain a very simple reason for the Hamiltonian constraint to exist, at least in the minimal dRGT-gravity with the flat reference metric. In Section 4 we comment on the general dRGT models. And finally, in Section 5 we conclude.

2. Review of the ADM Hamiltonian analysis

The standard way [21] of doing the Hamiltonian analysis in GR is via the $(3+1)$-decomposition of space–time:

$$ds^2 \equiv g_{\mu\nu}\,dx^\mu\,dx^\nu = -\left(N^2 - N_k N^k\right)dt^2 + 2N_i\,dx^i\,dt + \gamma_{ij}\,dx^i\,dx^j \tag{1}$$

where N and N_i are the lapse and shift functions respectively, and $N_i \equiv \gamma_{ik}N^k$. One can also find the inverse metric in terms of the lapse and shift functions and inverse spatial metric:

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^i}{N^2} \\ \frac{N^j}{N^2} & \gamma^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix}. \tag{2}$$

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And as the M_{00}-minor of the $g_{\mu\nu}$ matrix equals $\gamma \equiv \det \gamma_{ij}$, we can conclude from $g^{00} = -\frac{1}{N^2}$ that $\sqrt{-g} = N\sqrt{\gamma}$ due to the standard rule of inverting the matrices.

The next step is to calculate the Einstein–Hilbert Lagrangian density in terms of the ADM variables which is not so easy a task unless one takes the general geometric relations for embedded geometries directly from the textbooks. We find it most reasonable to recall the definition of the Riemann tensor as a commutator of covariant derivatives, and then to follow the geometric path outlined in the classical volume by Misner, Thorne and Wheeler [22]. But anyway, the result is

$$\sqrt{-g}R = N\sqrt{\gamma}\left(\overset{(3)}{R} + K^i_k K^k_i - \left(K^i_i\right)^2\right)$$
$$+ \text{(total time derivative and covariant divergence terms)}$$

where we have introduced the extrinsic curvatures

$$K_{ik} = \frac{1}{2N}\left(\overset{(3)}{\nabla}_i N_k + \overset{(3)}{\nabla}_k N_i - \dot{\gamma}_{ik}\right), \tag{3}$$

and the three-dimensional scalar curvature and covariant derivatives as well as raising and lowering of the indices are defined by the spatial slice metric γ_{ij}.

Now one can define the canonical momenta $\pi^{ij} \equiv \frac{\partial L}{\partial \dot{\gamma}_{ij}}$ for the physical variables and also find the primary constraints $\pi_N = 0$ and $\pi_{N_i} = 0$ for the momenta of the lapse and shift functions which act as Lagrange multipliers. The Hamiltonian then reads

$$H = -\int d^3x \sqrt{\gamma}\left(N\left(\overset{(3)}{R} + \frac{1}{\gamma}\left(\frac{1}{2}\left(\pi^j_j\right)^2 - \pi_{ik}\pi^{ik}\right)\right) + 2N^i \overset{(3)}{\nabla}{}^k \pi_{ik}\right). \tag{4}$$

The commutation of this Hamiltonian with the unphysical momenta directly gives the four physical constraints, $\mathcal{C} = -\sqrt{\gamma}(R^{(3)} + \frac{1}{\gamma}(\frac{1}{2}(\pi^j_j)^2 - \pi_{ik}\pi^{ik}))$ and $\mathcal{C}_i = -2\sqrt{\gamma}\nabla^{(3)k}\pi_{ik}$, which reduce the number of propagating degrees of freedom from six to two, and make the Hamiltonian equal zero (in the weak sense of Dirac) as it should be in a time-reparametrisation-invariant theory.

2.1. dRGT gravity

In the massive gravity theories one has to introduce an additional reference metric $f_{\mu\nu}$ which can either be taken fixed (for example, Minkowski one) or endowed with its own dynamics thus producing a bigravity model, for otherwise there is no way to construct a non-derivative invariant which would act as a potential term for the graviton. (We will exclusively take the fixed $f_{\mu\nu}$ metric option.) The basic ingredient of the dRGT model is the square-root matrix $(\sqrt{g^{-1}f})^{\mu}_{\nu}$, and the minimal model potential is taken to be [6]

$$V = 2m^2\left(\left(\sqrt{g^{-1}f}\right)^{\mu}_{\mu} - 3\right). \tag{5}$$

In what follows we will ignore the $-6m^2$ term which is introduced in order to avoid a contribution to the cosmological constant. And in the simplest case of Minkowski reference metric we deal with the square root of the following matrix:

$$g^{\mu\alpha}\eta_{\alpha\nu} = \begin{pmatrix} \frac{1}{N^2} & \frac{N^i}{N^2} \\ -\frac{N^j}{N^2} & \gamma^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix}. \tag{6}$$

It is obvious now that the lapse and shift functions enter the Hamiltonian

$$H = -\int d^3x \sqrt{\gamma}\left(N\left(\overset{(3)}{R} + \frac{1}{\gamma}\left(\frac{1}{2}\left(\pi^j_j\right)^2 - \pi_{ik}\pi^{ik}\right) - V\right) + 2N^i \overset{(3)}{\nabla}{}^k \pi_{ik}\right) \tag{7}$$

non-linearly in the $\sqrt{\gamma}NV$ term, and therefore, naively one would expect to get some non-trivial equations for these unphysical variables instead of the physical constraints $\mathcal{C}$ and $\mathcal{C}^i$. If it was the case, we would end up with six degrees of freedom including the Boulware–Deser ghost. However, the very peculiar feature of the potential (5) is that one combination of these constraints does survive as a physical relation leaving us with a healthy number of degrees of freedom [5,7,8]. And we would like to understand the reasons for that.

The standard approach to the Hamiltonian analysis [9,10,12] is to explicitly calculate the square root matrix in (5). Note that if it was not for the spatial metric γ, then the square root would have been really easy to find. Indeed, it follows from the very simple relation:

$$\begin{pmatrix} 1 & a^i \\ -a^j & -a^i a^j \end{pmatrix}^2 = \left(1 - a^k a^k\right)\begin{pmatrix} 1 & a^i \\ -a^j & -a^i a^j \end{pmatrix}. \tag{8}$$

However, one cannot just simply take the square root of this part and then combine it with the square root of γ^{-1} as they can never anticommute. But one can hope to redefine the shift functions $N_i = (\delta^j_i + ND^j_i)n_j$ such that the remaining N-dependent part of the $\sqrt{g^{-1}f}$ would still acquire this nice form [9]:

$$\sqrt{g^{-1}f} = \frac{1}{N\sqrt{1 - n^k n^k}}\begin{pmatrix} 1 & n^i \\ -n^j & -n^i n^j \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & X^{ij}(\gamma, n) \end{pmatrix}, \tag{9}$$

and then the anticommutator with the X matrix would account for the difference between N^i and n^i, and between X^2 and γ. If this is so, then after such a redefinition the lapse function will enter the Hamiltonian linearly enforcing a truly physical constraint. Clearly, if there exists a combination of lapse and shifts which enters the Hamiltonian linearly then it must be possible to perform such a decomposition of $\sqrt{g^{-1}f}$ after a linear in N redefinition of shifts.[1] Indeed, in the (singular) limit of $N \to 0$, which is equivalent to $n_i \to N_i$, the $\frac{1}{N}$-part of the square root must tend to satisfying the property (8); and therefore it should always satisfy it because the difference between n_i and N_i is determined by N on which no explicit dependence is allowed. And vice versa, if one finds such a change of variables, calculates the X^{ij} matrix and proves the relation (9), then the theory is free of the Boulware–Deser ghost. And it was actually done in [9] with the only limitation that the D^j_i operator is determined as a non-linear function of n, and therefore, in terms of initial variables, the transformation D is found only as an implicit function of the lapse and shifts.

These results [9] have proven that the dRGT gravity is a potentially healthy deformation of GR at the full non-perturbative level. Moreover, it has been done for an arbitrary reference metric [10] and for non-minimal models too. The latter actually correspond to adding two more potential terms, $V_2 = (\mathrm{Tr}\sqrt{g^{-1}f})^2 - \mathrm{Tr}(\sqrt{g^{-1}f})^2$ and $V_3 = (\mathrm{Tr}\sqrt{g^{-1}f})^3 - 3(\mathrm{Tr}\sqrt{g^{-1}f})(\mathrm{Tr}(\sqrt{g^{-1}f})^2) + 2\,\mathrm{Tr}(\sqrt{g^{-1}f})^3$.

3. The simple argument

In our approach we introduce an extra matrix of auxiliary fields Φ^{μ}_{ν} into the model, so that the potential takes the following form:

[1] If it was not linear then the lapse would definitely go non-linear in front of $\mathcal{C}^i$s.

$$V = \frac{m^2}{N}\left(\Phi^\mu_\mu + \left(\Phi^{-1}\right)^\mu_\nu N^2 g^{\nu\alpha} f_{\alpha\mu}\right) \tag{10}$$

which yields the standard $2m^2 \operatorname{Tr}\sqrt{g^{-1}f}$ term (5) after integrating out the auxiliary fields. With the Minkowski reference metric we can safely demand $\Phi^k_i = \Phi^i_k$ and $\Phi^0_i = -\Phi^i_0$, while in a general case some other combinations of Φs will drop out of the action.[2] Now, if we set out to make every single step explicitly, the primary constraints are $\mathcal{C}_1 = \pi_N$, $\mathcal{C}_{2i} = \pi_{N^i}$ and $\mathcal{C}_3{}^\mu_\nu = \pi_{\Phi^\nu_\mu}$. The constraint $\mathcal{C}_3$ will generate the matrix constraint $\mathcal{C}_4 = \Phi^2 - N^2 g^{-1} f$, and we are particularly interested in the constraints $\mathcal{C}_5$ and $\mathcal{C}_{6i}$ generated by $\mathcal{C}_1$ and $\mathcal{C}_{2i}$ respectively.

With a simple commutation we obtain

$$\mathcal{C}_{6i} = \sqrt{\gamma}\left(-2 \overset{(3)}{\nabla}{}^k \pi_{ik} + m^2 N^2 \left(\Phi^{-1}\right)^\mu_\nu \frac{\partial}{\partial N^i} g^{\nu\alpha} f_{\alpha\mu}\right)$$

where the derivative of the matrix (6) can be easily found to be

$$N^2 \frac{\partial}{\partial N^i} g^{\nu\alpha} f_{\alpha\mu} = \begin{pmatrix} 0 & \delta^j_i \\ -\delta^k_i & -\delta^j_i N^k - \delta^k_i N^j \end{pmatrix}.$$

These constraints allow us to determine the shift functions in terms of γ_{ik}, π^{ik} and Φ; and a naive expectation would be that, in combination with $\mathcal{C}_4$, it is possible to express them solely in terms of γ_{ik} and π^{ik}. And, finally, we find the last remaining constraint at this stage

$$\begin{aligned}\mathcal{C}_5 &= \mathcal{C}^{(GR)} + m^2 \sqrt{\gamma}\left(\Phi^{-1}\right)^\mu_\nu \frac{\partial}{\partial N} N^2 g^{\nu\alpha} f_{\alpha\mu} \\ &= \sqrt{\gamma}\left(-\overset{(3)}{R} - \frac{1}{\gamma}\left(\frac{1}{2}\left(\pi^j_j\right)^2 - \pi_{ik}\pi^{ik}\right) + 2m^2 N \left(\Phi^{-1}\right)^i_j \gamma^{ij}\right)\end{aligned}$$

where one should not worry too much about summing over two upper indices as we just do not write out the unit matrix from the spatial part of the reference metric explicitly.[3]

Up to that stage of analysis, the total Hamiltonian density is

$$\begin{aligned}\mathcal{H} = &-\sqrt{\gamma} N\left(\overset{(3)}{R} + \frac{1}{\gamma}\left(\frac{1}{2}\left(\pi^j_j\right)^2 - \pi_{ik}\pi^{ik}\right)\right) - 2\sqrt{\gamma} N^i \overset{(3)}{\nabla}{}^k \pi_{ik} \\ &+ \sqrt{\gamma} m^2 \left(\Phi^\mu_\mu + \left(\Phi^{-1}\right)^\mu_\nu N^2 g^{\nu\alpha} f_{\alpha\mu}\right) \\ &+ \lambda_1 \pi_N + \lambda_2{}^i \pi_{N^i} + \lambda_3{}^\mu_\nu \pi_{\Phi^\mu_\nu} \\ &+ \sqrt{\gamma} \lambda_4{}^\mu_\nu \left(\Phi^\nu_\alpha \Phi^\alpha_\mu - N^2 g^{\nu\alpha} f_{\alpha\mu}\right) \\ &+ \sqrt{\gamma} \lambda_5 \left(-\overset{(3)}{R} - \frac{1}{\gamma}\left(\frac{1}{2}\left(\pi^j_j\right)^2 - \pi_{ik}\pi^{ik}\right) \right. \\ &\left. + 2m^2 N \left(\Phi^{-1}\right)^i_j \gamma^{ij}\right) \\ &+ \sqrt{\gamma} \lambda_6{}^i \left(-2 \overset{(3)}{\nabla}{}^k \pi_{ik} + 2m^2 \left(\left(\Phi^{-1}\right)^i_0 + \left(\Phi^{-1}\right)^i_j N^j\right)\right),\end{aligned} \tag{11}$$

and it would be its final form, with the set of purely second class constraints, for a generic choice of potential. But in the case at hand one can easily check that a particular combination of unphysical momenta (see the subsection below) does actually commute, in the weak sense, with the total Hamiltonian *irrespective* of the values of Lagrange multipliers. Hence, the constraints $\mathcal{C}_4$, $\mathcal{C}_5$ and $\mathcal{C}_6$ do not allow to unambiguously express the naive unphysical variables, N, N_i and Φ, in terms of the spatial metric and its momenta. And we can nothing but conclude that they do contain a non-trivial constraining equation for the would-be-physical variables γ_{ij} and π^{ij}.[4] This observation is actually the final step in our proof that the dRGT gravity contains strictly less than six degrees of freedom.

3.1. Some technical details

We now proceed to explicitly calculate the commutators of the unphysical momenta with the total Hamiltonian. Obviously, within the constraint surface, we only need to commute them with the other constraints because the commutations with the first two lines in the Hamiltonian (11) have already been done, and the very meaning of the other constraints is that those commutators do weakly vanish. Therefore, we can find

$$\begin{aligned}\frac{1}{\sqrt{\gamma}}\{\mathcal{C}_1, H\} &= -2\lambda_4{}^i_k N \gamma^{ik} + 2\lambda_5 m^2 \left(\Phi^{-1}\right)^i_j \gamma^{ij}, \\ \frac{1}{\sqrt{\gamma}}\{\mathcal{C}_{2i}, H\} &= -2\lambda_4{}^i_0 + 2\lambda_4{}^i_k N^k + 2m^2 \lambda_6{}^k \left(\Phi^{-1}\right)^i_k, \\ \frac{1}{\sqrt{\gamma}}\left\{\mathcal{C}_3{}^\mu_\nu, H\right\} &= 2\lambda_4{}^\mu_\alpha \Phi^\alpha_\nu + 2m^2 \lambda_5 \gamma^{ij} N \frac{\partial}{\partial \Phi^\nu_\mu}\left(\Phi^{-1}\right)^i_j \\ &\quad + 2m^2 \lambda_6{}^i \frac{\partial}{\partial \Phi^\nu_\mu}\left(\left(\Phi^{-1}\right)^i_0 + \left(\Phi^{-1}\right)^i_j N^j\right)\end{aligned}$$

where we use the symmetry of the auxiliary fields and their Lagrange multipliers, $\lambda_4{}^i_k = \lambda_4{}^i_k$ and $\lambda_4{}^0_i = -\lambda_4{}^i_0$. Normally, these commutators would give us fourteen independent linear equations for fourteen Lagrange multipliers, so that all of them would be set to zero. However, using the $\mathcal{C}_4$ constraint and a simple formula $\left(\Phi^{-1}\right)^\alpha_\mu \frac{\partial}{\partial \Phi^\alpha_\nu}\left(\Phi^{-1}\right)^\beta_\kappa = -\left(\Phi^{-2}\right)^\beta_\mu \left(\Phi^{-1}\right)^\nu_\kappa$, one can deduce from the last commutator the following relations:

$$\begin{aligned}\lambda_4{}^i_j &= \frac{1}{2\sqrt{\gamma}}\left(\Phi^{-1}\right)^i_\alpha \left\{\mathcal{C}_3{}^\alpha_j, H\right\} + m^2 \left(\frac{\lambda_5}{N}\left(\Phi^{-1}\right)^i_j \right. \\ &\quad \left. + \frac{\lambda_6{}^k \gamma_{kj}}{N^2}\left(\left(\Phi^{-1}\right)^i_0 + \left(\Phi^{-1}\right)^i_l N^l\right)\right), \\ \lambda_4{}^k_0 &= \frac{1}{2\sqrt{\gamma}}\left(\Phi^{-1}\right)^k_\alpha \left\{\mathcal{C}_3{}^\alpha_0, H\right\} + m^2 \left(\frac{\lambda_5 N^j}{N}\left(\Phi^{-1}\right)^k_j \right. \\ &\quad \left. + \frac{\lambda_6{}^i N_i}{N^2}\left(\left(\Phi^{-1}\right)^k_0 + \left(\Phi^{-1}\right)^k_l N^l\right)\right).\end{aligned}$$

And we see that the combination of $\lambda_4{}^k_0 - \lambda_4{}^k_j N^j$ depends only on the commutators with $\mathcal{C}_3$, while both λ_5 and λ_6 completely drop out of this expression. And therefore, using the commutator with $\mathcal{C}_2$, we obtain

$$2m^2 \lambda_6{}^k \left(\Phi^{-1}\right)^i_k = \frac{1}{\sqrt{\gamma}}\left\{\mathcal{C}_{2i} + \left(\Phi^{-1}\right)^i_\alpha \mathcal{C}_3{}^\alpha_0 - \left(\Phi^{-1}\right)^i_\alpha N^j \mathcal{C}_3{}^\alpha_j\ ,\ H\right\}$$

[2] If it does not seem so obvious as for how to arrive at the form (10), one may start with $NV = 2m^2 \Phi^\mu_\mu + \kappa^\mu_\nu \left(\Phi^\nu_\alpha \Phi^\alpha_\mu - N^2 g^{\nu\alpha} f_{\alpha\mu}\right)$ and integrate out the κs.

[3] Note also that we could define $\Phi^2 = g^{-1} f$ instead of $\Phi^2 = N^2 g^{-1} f$, and as one can easily show, it would have resulted in the contribution to the constraint $\mathcal{C}_5$ which could be brought to a very simple form, $2m^2 \sqrt{\gamma}\left(\Phi^{-1}\right)^i_j \gamma^{ij}$, by use of the $\mathcal{C}_4 = 0$ equation. This form contains neither the lapse nor shifts, although the lapse would have appeared in $\mathcal{C}_{6i}$. And it would have been the first instance to face a crash of our naive expectations, that is, modulo some mixing, there are roughly three systems of second class constraints: $\mathcal{C}_1$ and $\mathcal{C}_5$, $\mathcal{C}_2$ and $\mathcal{C}_6$, and $\mathcal{C}_3$ and $\mathcal{C}_4$. However, despite its apparently striking form, this fact is not that easy in being promoted to an actual proof, and therefore we will proceed with the definition (10) which allows for more pleasant calculations.

[4] This is parallelled by the fact that in the standard approach the total Hamiltonian commutes, at this stage of analysis, with π_N with no restrictions on the values of Lagrange multipliers, and indeed we have a physical constraint instead of an equation for N.

in the weak sense. (Note that one should not worry about commuting the Hamiltonian with the coefficients in front of the constraints because the momenta do vanish on the constraint surface.) On the other hand, one can compute ${\lambda_4}^i_j \gamma^{ij}$ and compare it with the $\{\mathcal{C}_1, H\}$ commutator. It (weakly) determines a linear combination of λ_6s:

$$2m^2 {\lambda_6}^k \left(\left(\Phi^{-1} \right)^k_0 + \left(\Phi^{-1} \right)^k_l N^l \right)$$
$$= -\frac{1}{\sqrt{\gamma}} \left\{ N^2 \left(\Phi^{-1} \right)^i_\alpha \gamma^{ij} {\mathcal{C}_3}^\alpha_j + N\mathcal{C}_1 \ , \ H \right\}.$$

These two results must agree, and it singles out a combination of momenta

$$\pi_N + N\left(\Phi^{-1}\right)^i_\alpha \gamma^{ik} \pi_{\Phi^k_\alpha} + \frac{((\Phi^{-1})^i_0 + (\Phi^{-1})^i_l N^l)}{N}$$
$$\times \left(\left(\overset{(3)}{\Phi}{}^{-1} \right)^{-1} \right)^k_i \left(\pi_{N^k} + \left(\Phi^{-1}\right)^k_\alpha \pi_{\Phi^0_\alpha} - \left(\Phi^{-1}\right)^k_\alpha N^j \pi_{\Phi^j_\alpha} \right)$$

which weakly commutes with the Hamiltonian for any values of the Lagrange multipliers.

In our approach, this combination determines the direction in the space of unphysical variables along which there has been no restriction so far, under any of the constraints. This corresponds to the independence of N in the standard treatment. In either approach, a constraint which commutes with all the other constraints (and leaves one combination of the Lagrange multipliers undetermined) may appear in two distinct situations: either it is a genuine first class constraint and corresponds to a gauge freedom in the model, or some extra constraints are needed for the self-consistency so that the whole set of constraints is non-degenerate second class. As the former seems not to be the case, the generation of a one more constraint is unavoidable (and, at the end of the day, the values of all non-dynamical fields should be somehow determined unless there is a gauge freedom indeed), and therefore the scenario foreseen in the reference [11] is a priori highly implausible.

Technically, what follows at the next step is that the Lagrange multipliers λ_1, λ_2, λ_3 should serve to ensure the preservation of $\mathcal{C}_4$, $\mathcal{C}_5$, $\mathcal{C}_6$ constraints. But a factor in front of the all-commuting combination of momenta drops out of this game; and there would not be enough freedom to make all the necessary commutators weakly vanish, if there is no gauge symmetry. (And in our case the commutator with the $\mathcal{C}_5$ constraint has two independent parts, proportional to the delta-function and to its derivative, and the one undetermined combination of the other Lagrange multipliers would be used for fixing the latter part.) This is how an extra constraint is generated in the model. However, it is not *a priori* clear that it would be a physical one making the number of degrees of freedom not more than the healthy amount of five (as opposed to five and a half) because the next constraint could just add the fourth equation for the lapse and shift functions without any information on the spatial sector. Due to this reason it is harder to proof that the theory contains five degrees of freedom than just to show that its number is less than six. But a plausible argument would be that, after integrating out all but one undetermined unphysical variable, the remaining combination should go linearly in the action with the coefficient equal to the remaining constraint which would therefore commute with this part of the Hamiltonian (this is subject to the criticism in [11]). And as is known by now [12], the strange $5\frac{1}{2}$-situation is not the case, and *two* extra constraints do appear in the model, one is the "secondary" one for the spatial variables, and the other finally fixes the lapse and shifts.

We would not proceed with explicit derivations in this Letter because the most important result is already known [9,12], and our only purpose was to present an alternative and fairly simple method of analysis. However, all the necessary calculations are very straightforward even if time consuming.

4. On arbitrary reference metrics and non-minimal dRGT models

For the sake of simplicity, up to now we have considered only the simplest choice of the reference metric, i.e. the Minkowski one. However, the non-linear massive gravity has been proven to be free of the Boulware–Deser ghost for any choice of the reference metric [10], and even in its bigravity version too [12]. Incorporation of an arbitrary lapse and an arbitrary spatial metric is actually trivial (and the latter even makes the location of the spatial indices nicer), while shifts do produce some problems and affect the simple form of the decomposition (9), see [10]. It can be readily seen by taking the general reference metric

$$f_{\mu\nu} = \begin{pmatrix} -(M^2 - M_k M^k) & M_i \\ M_j & s_{ij} \end{pmatrix} \tag{12}$$

and calculating the basic building block of the model:

$$g^{\mu\alpha} f_{\alpha\nu}$$
$$= \begin{pmatrix} \frac{M^2 - M_k(M^k - N^k)}{N^2} & \frac{s_{ij}N^j - M_j}{N^2} \\ -\frac{N^j(M^2 - M_k(M^k - N^k))}{N^2} + \gamma^{ij} M_j & s_{ik}\gamma^{kj} - \frac{s_{ik}N^k N^j - N_i M^j}{N^2} \end{pmatrix}. \tag{13}$$

However, the complication is a relatively mild one: the $\mathcal{C}_5$ constraint receives a more involved contribution of $2m^2 N\sqrt{\gamma} \times ((\Phi^{-1})^0_i M_j + (\Phi^{-1})^k_i s_{kj})\gamma^{ij}$ instead of the simple $2m^2 N\sqrt{\gamma} \times (\Phi^{-1})^i_j \gamma^{ij}$, and the form of $\mathcal{C}_6$ is also changed in an obvious way. Nevertheless, it simply corresponds to a rotation of the variables, and the subsequent calculations become a bit more complicated only due to the form of the coefficients with no crucial change to the results. It actually should have been the case because, at least in the class of coordinate-independent reference metrics, a general metric can be transformed to zero shifts by a (linear) change of coordinates, e.g. by the one which diagonalises the matrix f.[5]

The models with general potentially ghost-free potentials can, in principle, be treated in the same way.[6] Indeed, in order to check that the constraints $\mathcal{C}_1, \ldots, \mathcal{C}_6$ are preserved during the evolution, one has to perform a straightforward computation of the commutators and to decide upon the solvability of a system of linear equations for $\lambda_1, \ldots, \lambda_6$. And if some more constraints are required then they definitely convey a non-trivial piece of information about the spatial sector of the model.

However, once we have understood the reasons for the sixth mode to be absent in the minimal dRGT model, the ghost-free nature of the higher potentials can be better explained with the standard argument of Refs. [9,10]. We know that, after a linear in N change of variables, the minimal model action contains the lapse function only linearly. And if one has convinced himself that it implies the decomposition (9), then it is also obvious [9,10] that the same is true of the V_2 and V_3 potentials because for a matrix A with the property (8) we have $\mathrm{Tr}(A^n) = (\mathrm{Tr}\, A)^n$, and all unwanted

[5] Note that one *can*, of course, make a coordinate transformation, even in massive gravity, as long as both the physical and the reference metric are being changed accordingly. And recall that the very property of $\sqrt{g^{-1}f}$ to be decomposable into the sum of N^{-1} and N^0 parts can be proven at each point of the space–time manifold separately, with no reference to its coordinate dependence or independence.

[6] There is a somewhat subtle case of the purely quadratic potential V_2 for which the variation of the $\Phi^\alpha_\alpha (\Phi^{-1})^\mu_\nu (g^{-1}f)^\nu_\mu - (g^{-1}f)^\alpha_\alpha$ term with respect to Φ would leave an overall scalar factor undetermined, much like the conformal invariance appears in the Polyakov action of the bosonic string.

powers of the lapse in the potential do cancel. This is how the symmetric polynomials of the eigenvalues come into play; and in four dimensions there is only one more of them, $\det(g^{-1}f)$, which being multiplied by $\sqrt{-g}$ produces nothing but a constant shift of the action. Our method does not respect the decomposition (9), and therefore it is not so elegant in generalising to non-minimal models.

5. Conclusions

We have presented a new method of non-perturbative analysis of the non-linear massive gravity. And in particular, we give a very simple argument for the absence of the Boulware–Deser ghost in this theory. The limitation of our approach is that the argument is non-constructive, and does not proceed directly in terms of the metric components. However, in principle, one can make all the derivations explicitly, and calculate the number of constraints and independent degrees of freedom. In general, the power of this approach is in the fact that many things can be done at the level of a bit bulky but absolutely straightforward calculations, very automatically, with no need of making more qualified and creative jobs such as taking the square root of a matrix.

Admittedly, the introduction of ten more configuration space dimensions (and ten more pairs of constraints to eliminate them) is not very helpful for the actual physical calculations. But, given the relative ease with which this argument shows the presence of extra constraints in the model, it is reasonable to hope that it can be useful for discussing the formal aspects of the theory. And anyway, for the real calculations a somewhat different language is better suited [5,7,8]. Obviously, it would be very interesting to find out whether any kind of such formal tricks with auxiliary fields like Φ could produce a similarly simple argument for the full non-linear stability in order-by-order perturbation theory.

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References

[1] M. Fierz, W. Pauli, Proc. Royal Soc. London A 173 (1939) 211.
[2] A.I. Vainshtein, Phys. Lett. B 39 (1972) 393.
[3] D.G. Boulware, S. Deser, Phys. Rev. D 6 (1972) 3368.
[4] C. de Rham, G. Gabadadze, Phys. Rev. D 82 (2010) 044020, arXiv:1007.0443.
[5] C. de Rham, G. Gabadadze, A. Tolley, Phys. Rev. Lett. 106 (2011) 231101, arXiv:1011.1232.
[6] S.F. Hassan, R.A. Rosen, JHEP 1107 (2011) 009, arXiv:1103.6055.
[7] C. de Rham, G. Gabadadze, A. Tolley, arXiv:1107.3820.
[8] C. de Rham, G. Gabadadze, A. Tolley, JHEP 1111 (2011) 093, arXiv:1108.4521.
[9] S.F. Hassan, R.A. Rosen, arXiv:1106.3344.
[10] S.F. Hassan, R.A. Rosen, A. Schmidt-May, arXiv:1109.3230.
[11] J. Kluson, arXiv:1109.3052.
[12] S.F. Hassan, R.A. Rosen, arXiv:1111.2070.
[13] K. Koyama, G. Niz, G. Tasinato, Phys. Rev. Lett. 107 (2011) 131101, arXiv:1103.4708.
[14] Th.M. Nieuwenhuizen, Phys. Rev. D 84 (2011) 024038, arXiv:1103.5912.
[15] K. Koyama, G. Niz, G. Tasinato, Phys. Rev. D 84 (2011) 064033, arXiv:1104.2143.
[16] L. Berezhiani, G. Chkareuli, C. de Rham, G. Gabadadze, A.J. Tolley, arXiv:1111.3613.
[17] G. D'Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava, A.J. Tolley, arXiv:1108.5231.
[18] A.E. Gumrukcuoglu, Ch. Lin, Sh. Mukohyama, JCAP 1111 (2011) 030, arXiv:1109.3845.
[19] M. von Strauss, A. Schmidt-May, J. Enander, E. Mortsell, S.F. Hassan, arXiv:1111.1655.
[20] A.E. Gumrukcuoglu, Ch. Lin, Sh. Mukohyama, arXiv:1111.4107.
[21] R. Arnowitt, S. Deser, C.W. Misner, The dynamics of generel relativity, in: L. Witten (Ed.), Gravitation: An Introduction to Current Research, Wiley, 1962, p. 227, arXiv:gr-qc/0405109.
[22] C.W. Misner, K.S. Thorne, J.A. Wheeler, Gravitation, W.H. Freeman Publishers, 1973.