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On a Problem in Neutron Transport Theory*

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The purpose of this paper is to formulate and solve a mixed initial and boundary-valued problem for a slightly generalized transport equation. We look at smooth deformations of a three-dimensional body, $\mathscr{D}$, as described by a Lagrange transformation $x = x(\xi, t)$, where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathscr{D}$ is the Lagrange variable and $x = (x_1, x_2, x_3)$ is the image of (ξ_1, ξ_2, ξ_3) at time t. We ask for the distribution of neutrons in $\mathscr{D}$ as it ($\mathscr{D}$) varies with time. The corresponding neutron density function $u(\xi, v, t)$ should satisfy

$$u_t + [v - \dot{x}] \cdot J^{-1} \operatorname{grad}_\xi u - \sigma(\xi, v, t)u = \int_v K(\xi, v, v', t)u(\xi, v', t)\, dv' \quad (1)$$

$$u(\xi, v, 0) = u_0(\xi, v) \qquad u(\xi, v, t) = 0 \quad \text{for} \quad \xi \in \mathscr{B}(v, t)$$

where

$$\mathscr{B}(v, t) = \{\xi \in \dot{\mathscr{D}} \text{ (boundary of } \mathscr{D}) \mid \xi + s(J^{-1})^1[v - \dot{x}] \in \mathscr{D} \text{ for } 0 < s < \epsilon(\xi, v, t)\}$$

$\dot{x} = \partial x / \partial t$ and $J = J(\xi, t)$ is the Jacobian matrix of the transformation $x = x(\xi, t)$. The points $v = (v_1, v_2, v_3) \in V$ form a bounded subset of E_3 and the Cartesian product $\mathscr{D} \times V$ is the so-called phase space. For the distribution of neutrons in a fixed three-dimensional body, one solves the mixed problem for the linearized Boltzmann equation

$$u_t + v \cdot \operatorname{grad}_x u - \sigma(x, v)u = \int_v K(x, v, v')u(x, v', t)\, dv' \quad (2)$$

$$u(x, v, 0) = u_0(x, v) \qquad u(x, v, t) = 0 \quad \text{for} \quad x \in \mathscr{B}(v)$$

where

$$\mathscr{B}(v) = \{x \in \dot{\mathscr{D}} \mid x + sv \in \mathscr{D} \quad \text{for} \quad 0 < s < \epsilon(x, v)\}$$

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and

$$\int_v K^2 \, dv' < \infty.$$

Problem (1) can be derived from (2) by substituting $x(\xi, t)$ for x and using the chain rule for differentiation. The essential difference between the two problems is that the coefficients and boundary condition are time dependent for problem (1). In particular, the boundary condition is found by solving the system of ordinary differential equations for the characteristics,

$$\frac{\partial w}{\partial s} = (J^{-1})'(w, t)[v - \dot{x}(w, t)]$$
$$w(\xi, v, t, 0) \equiv \xi$$

and then requiring the density function to vanish at those boundary points that are initial values for incoming characteristics. Problem (2) is well posed in view of the Hille-Yosida theorem (see, e.g., Jorgens [1], for existence and the asymptotic behavior of the solution).

An equivalent formulation of the boundary condition for problem (1) is

$$\mathscr{B}(v, t) = \{\xi \in \dot{\mathscr{D}} \mid J^{-1}n(\xi) \cdot [v - \dot{x}] < 0\}$$

and of (2) is

$$\mathscr{B}(v) = \{x \in \dot{\mathscr{D}} \mid n \cdot v < 0\}$$

where n is the outgoing normal vector at the point x. Let

$$B(t)U = u_t + [v - \dot{x}] \cdot J^{-1} \operatorname{grad}_\xi u - \sigma(\xi, v, t)u$$
$$A(t)U = B(t)u + \int_v K(\xi, v, v', t)u(\xi, v', t)\, dv'$$

We need the following theorem which is due, essentially, to T. Kato.

THEOREM 1. *For each* t, $0 \leqslant t \leqslant T$ *let*

(i) $A(t)$ *be a closed linear operator that is densely defined in a Banach space* X,

(ii) $\| [\lambda - A(t)]^{-1} \| \leqslant 1/\lambda$ *for* $\lambda > 0$ *and uniformly in* t,

(iii) $\Delta(t)$ = domain of $A(t)$ *be independent of* t,

(iv) $P(t) = [I - A(t)][I - A(0)]^{-1}$ *be strongly continuously differentiable in* t. *Then the initial valued problem* $du/dt = A(t)u$, $u(0) = u_0$ *has a unique solution, given by* $u(t) = U(t, 0)u_0$ where the "*evolution operator*" $U(t, s)$ *is strongly continuous in the triangle* $0 \leqslant s \leqslant t \leqslant T$ *and* $U(t, s)U(s, n) = U(t, r)$.

If we require the deformation to take place along rays, i.e., $x(\xi, t) = \alpha(t)\xi$ where α is a scalar-valued function, then $\Delta(t)$ is independent of t. Physically, we think of an explosion or expansion of some material because of heating. In order to verify (ii), we get the resolvent of $B(t)$ by the usual technique of integrating along a characteristic from a given point up to the boundary. We take for X, the Banach space of piecewise continuous functions[1] with the maximum norm. The estimate for the norm of the resolvent of $A(t)$ is then easily found. Hypotheses (i) and (iv) are trivially verified.

For arbitrary deformations, $x(\xi, t)$ we look at the cylinder Q, over $\mathscr{D}$, that lies between the planes $t = 0$ and $t = h$. We call $\mathscr{D}_\tau$ the intersection of $\mathscr{D}$ with the plane $t = \tau$, $0 \leqslant \tau \leqslant h$. We write problem (1) in the form

$$Lu = u_t + \sum_{l=1}^{3} a_l u_{\xi_l} + \sigma u - \int_v K(\xi, v, v', t) u(\xi, v', t)\, dv' = 0 \qquad (3)$$

$$u(\xi, v, 0) = u_0(\xi, v) \qquad\qquad u(\xi, v, t) = 0$$

at points of $\tilde{Q}$ for which $\sum_{l=1}^{3} a_l n_l < 0$ where $\tilde{Q}$ is the vertical portion of the boundary of Q. $a_l = a_l(\xi, v, t)$ is the lth component of the vector

$$(J^{-1})'(\xi, t)[v - \dot{x}(\xi, t)], \qquad l = 1, 2, 3, \quad \sigma = \sigma(\xi, v, t)$$

and (n_1, n_2, n_3) is the normal vector to $\mathscr{D} = \mathscr{D}_0$. Let $H_r(t)$ be the Hilbert space of functions $u(t) = u(\xi, v, t)$, with scalar product

$$(u(t), w(t))_r = \int_v \int_{\mathscr{D}_t} \sum_{p \leqslant r} D^p u \overline{D^p w}\, d\xi\, dv$$

so that

$$\| u(t) \|_r^2 = \int_v \int_{\mathscr{D}_t} \sum_{|p| \leqslant r} | D^p u |^2\, d\xi\, dv.$$

D^p stands for the pth order derivatives with respect to ξ *only* and $D^0 u = u$. Let

$$(u, w) = \int_{V \times Q} u \bar{w}\, d\xi\, dv\, dt$$

(with no subscript) so that

$$\| u \|^2 = \int_0^h \| u(t) \|_0^2\, dt.$$

Define

$$L^* w = -\Big[w_t + \sum_{l=1}^{3} (a_l w)_{\xi_l} - \sigma w + \int_v K(\xi, v, v', t) w(\xi, v', t)\, dv'\Big].$$

[1] More precisely, X is the space of functions that have discontinuities only at prescribed characteristics, and is thus complete.

The energy inequality

$$\| u(t) \|_r^2 \leqslant c \| u(0) \|_r^2 + \int_0^t \| f(t') \|_r \, dt'$$

follows from the identity

$$(w, Lu) - (L^*w, u) = \int_v \int_{\tilde{Q}} wu \sum_{l=1}^{3} a_l n_l \, ds \, dt + (w(t), u(t))_0 - (w(0), u(0))_0 .$$

Uniqueness is a consequence of the energy inequality. For the existence proof we use a variant of the Schauder method that solves the Cauchy problem for hyperbolic equations. For each integer, n, let $\{V_1, ..., V_n\}$ be a partition of V into measurable sets, such that $\lim_{u\to\infty} \max_{1\leqslant i\leqslant n} m(V_i) = 0$ and let $C_i(v)$ be the characteristic function of V_i, i.e.

$$c_i(v) = \begin{cases} 1 & \text{if} \quad v \in V_i \\ 0 & \text{if} \quad v \notin V_i \end{cases}.$$

For arbitrary, but fixed $v_i \in V_i$, $i = 1, \cdots, n$, define

$$K_{ij} = K(\xi, v_j, v_i', t) c_i(v') c_j(v), \; K_i^n = \sum_{j=1}^{n} K_{ij}$$

$$\sigma_j = \sigma(\xi, v_j, t) c_j(v),$$

$$\sigma^n = \sum_{j=1}^{n} \sigma_j ,$$

$$a_{lj} = a_l(\xi, v_j, t) c_j(v)$$

and

$$a_l^n = \sum_{j=1}^{n} a_{lj} , \quad l = 1, 2, 3$$

We approximate Eq. (3) by an operator L_n defined by

$$L_n u = u_t + \sum_{l=1}^{3} a_l^n u_{\xi_l} + \sigma^n u - \sum_{i=1}^{n} K_i^n u_i$$

where

$$u = \sum_{i=1}^{n} u_i(\xi, t) c_i(v).$$

Let

$$M_j u_j = \frac{\partial u_j}{\partial t} + \sum_{l=1}^{3} a_{lj} \frac{\partial u_j}{\partial \xi_l} + \sigma_j u_j - \sum_{i=1}^{n} K_{ij} u_i .$$

We formulate the following problems

$$L_n U_n = 0, \;\; U_n(\xi, v, 0) = U_{n_0}(\xi, v), \;\; U_n(\xi, v, t) = 0 \tag{4}$$

at points of $\tilde{Q}$, for which

$$\sum_{l=1}^{3} a_l^n n_l < 0$$

and

$$M_j u_j = 0, \quad j = 1, -, n, \quad u_{j_0}(\xi, 0) = u_{j_0}(\xi), \quad u_j(\xi, t) = 0 \tag{5}$$

at points of $\tilde{Q}$, for which

$$\sum_{l=1}^{3} a_{lj} n_l < 0.$$

We solve the mixed problem (5) by power series (see G. F.D. Duff [3], pp. 159).

Now, if we define

$$U_n = \sum_{j=1}^{n} u_j c_j(v),$$

where the u_j's satisfy (5), with

$$U_{n_0} = \sum_{j=1}^{n} u_{j_0} c_j(v)$$

we get

$$0 = \sum_{j=1}^{n} M_j u_j c_j(v) = L_n U_n \, .$$

U_n is a solution of problem (4). It is clear that $\| (L_n - L)u \|_r \to 0$ for any $u \in H_r(t)$. The corresponding solutions and their derivatives converge by using the Sobolev lemma and the energy inequality.

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