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Asymptotic behavior in time periodic parabolic problems with unbounded coefficients

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ABSTRACT

We study asymptotic behavior in a class of nonautonomous second order parabolic equations with time periodic unbounded coefficients in $\mathbb{R} \times \mathbb{R}^d$. Our results generalize and improve asymptotic behavior results for Markov semigroups having an invariant measure. We also study spectral properties of the realization of the parabolic operator $u \mapsto \mathcal{A}(t)u - u_t$ in suitable L^p spaces.

1. Introduction

We consider linear second-order differential operators,

$$
\begin{aligned}
\big(\mathcal{A}(t)\varphi\big)(x) &= \sum_{i,j=1}^{d} q_{ij}(t,x)D_{ij}\varphi(x) + \sum_{i=1}^{d} b_i(t,x)D_i\varphi(x) \\
&= \mathrm{Tr}\big(Q(t,x)D^2\varphi(x)\big) + \big\langle b(t,x), \nabla\varphi(x)\big\rangle, \qquad (1.1)
\end{aligned}
$$

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with smooth enough coefficients defined in $\mathbb{R}^{1+d}$, satisfying the uniform ellipticity assumption

$$\sum_{i,j=1}^{d} q_{ij}(t,x)\xi_i\xi_j \geqslant \eta_0|\xi|^2, \quad (t,x) \in \mathbb{R}^{1+d},\ \xi \in \mathbb{R}^d. \tag{1.2}$$

Under general assumptions, a Markov evolution operator $P(t,s)$ associated to the family $\{\mathcal{A}(t)\}$ has been constructed and studied in [21]. For every continuous and bounded φ and for any $s \in \mathbb{R}$, the function $(t,x) \mapsto P(t,s)\varphi(x)$ is the unique bounded classical solution u to the Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}(t)u(t,x), & t > s,\ x \in \mathbb{R}^d, \\ u(s,x) = \varphi(x), & x \in \mathbb{R}^d. \end{cases} \tag{1.3}$$

Since the coefficients are allowed to be unbounded, L^p spaces with respect to the Lebesgue measure are not a natural setting for problem (1.3). This is well understood in the autonomous case $\mathcal{A}(t) \equiv A$, where $P(t,s) = e^{(t-s)A}$ and the Lebesgue measure is replaced by an invariant measure, i.e., a Borel probability measure μ such that

$$\int_{\mathbb{R}^d} e^{tA}\varphi\, d\mu = \int_{\mathbb{R}^d} \varphi\, d\mu, \quad t > 0,\ \varphi \in C_b\big(\mathbb{R}^d\big).$$

Under suitable assumptions it is possible to show that there exists a unique invariant measure. In this case, e^{tA} is extended to a contraction semigroup in $L^p(\mathbb{R}^d,\mu)$ for every $p \in [1,\infty)$, and $e^{tA}\varphi$ goes to the mean value $\int_{\mathbb{R}^d}\varphi\, d\mu$ in $L^p(\mathbb{R}^d,\mu)$ for every $\varphi \in L^p(\mathbb{R}^d,\mu)$ as $t \to \infty$, if $p > 1$.

The natural generalization of invariant measures to the time depending case are families of Borel probability measures $\{\mu_s\colon s \in \mathbb{R}\}$, called *evolution systems of measures*, such that

$$\int_{\mathbb{R}^d} P(t,s)\varphi\, d\mu_t = \int_{\mathbb{R}^d} \varphi\, d\mu_s, \quad t > s,\ \varphi \in C_b\big(\mathbb{R}^d\big).$$

A sufficient condition for their existence, similar to a well known sufficient condition for the existence of an invariant measure in the autonomous case, is the following: there exist a C^2 function $V : \mathbb{R}^d \to \mathbb{R}$ such that $\lim_{|x|\to\infty} V(x) = \infty$, and positive numbers a, c such that $\mathcal{A}(s)V(x) \leqslant a - cV(x)$ for each $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$.

If an evolution system of measures exists, then, as in the autonomous case, $P(t,s)$ may be extended to a contraction (still called $P(t,s)$) from $L^p(\mathbb{R}^d,\mu_s)$ to $L^p(\mathbb{R}^d,\mu_t)$, i.e.,

$$\big\|P(t,s)\varphi\big\|_{L^p(\mathbb{R}^d,\mu_t)} \leqslant \|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}, \quad t > s, \tag{1.4}$$

for every $\varphi \in L^p(\mathbb{R}^d,\mu_s)$.

In this paper we treat the case of time periodic coefficients, and we study asymptotic behavior of $P(t,s)$ and spectral properties of the parabolic operator

$$\mathcal{G} := \mathcal{A}(t) - D_t \tag{1.5}$$

in L^p spaces associated to a distinguished evolution system of measures. In fact, the evolution systems of measures are infinitely many, and we consider the unique T-periodic one, i.e. the only one such that $\mu_s = \mu_{s+T}$ for every $s \in \mathbb{R}$, where T is the period of the coefficients. We extend to this setting the convergence results of the autonomous case, showing that for $1 < p < \infty$

$$\lim_{t\to\infty} \big\|P(t,s)\varphi - m_s\varphi\big\|_{L^p(\mathbb{R}^d,\mu_t)} = 0, \quad s \in \mathbb{R},\ \varphi \in L^p\big(\mathbb{R}^d,\mu_s\big), \tag{1.6}$$

and

$$\lim_{s\to-\infty}\big\|P(t,s)\varphi - m_s\varphi\big\|_{L^p(\mathbb{R}^d,\mu_t)} = 0, \quad t\in\mathbb{R},\ \varphi\in C_b\big(\mathbb{R}^d\big), \tag{1.7}$$

where

$$m_s\varphi = \int_{\mathbb{R}^d}\varphi(y)\mu_s(dy).$$

This is done under suitable assumptions, that in the case of C_b^1 diffusion coefficients reduce to

$$\sup_{s\in\mathbb{R},\, x,y\in\mathbb{R}^d,\, x\neq y}\frac{\langle b(s,x)-b(s,y), x-y\rangle}{|x-y|^2} < \infty, \tag{1.8}$$

or, equivalently, $\sup\{\sum_{i,j=1}^d D_i b_j(s,x)\xi_i\xi_j\colon\ (s,x)\in\mathbb{R}^{1+d},\ \xi\in\mathbb{R}^d,\ |\xi|=1\}<\infty$. Inequality (1.8) can be seen as a weak dissipativity condition on the vector fields $b(s,\cdot)$. Under a stronger dissipativity condition, for bounded diffusion coefficients we prove exponential convergence, i.e., for every $p\in(1,\infty)$ there exist $M>0$, $\omega<0$ such that

$$\big\|P(t,s)\varphi - m_s\varphi\big\|_{L^p(\mathbb{R}^d,\mu_t)} \leqslant Me^{\omega(t-s)}\|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}, \quad t>s,\ \varphi\in L^p\big(\mathbb{R}^d,\mu_s\big). \tag{1.9}$$

The stronger dissipativity assumption was used in [21] to prove pointwise gradient estimates for $P(t,s)\varphi$. In fact, we arrive at exponential convergence through gradient estimates. Then, we discuss the rate of convergence; in Theorem 3.6 we show that for $p\geqslant 2$ and $\omega\in\mathbb{R}$ the conditions

(a) $\exists M>0$: $\|P(t,s)\varphi - m_s\varphi\|_{L^p(\mathbb{R}^d,\mu_t)} \leqslant Me^{\omega(t-s)}\|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}$, $t>s$, $\varphi\in L^p(\mathbb{R}^d,\mu_s)$,
(b) $\exists N>0$: $\||\nabla_x P(t,s)\varphi|\|_{L^p(\mathbb{R}^d,\mu_t)} \leqslant Ne^{\omega(t-s)}\|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}$, $t>s+1$, $\varphi\in L^p(\mathbb{R}^d,\mu_s)$,

are equivalent. Therefore, denoting by ω_p (resp. γ_p) the infimum of the $\omega\in\mathbb{R}$ such that (a) (resp. (b)) holds, we have $\omega_p=\gamma_p$.

Such characterization of the convergence rate was proved for time depending Ornstein–Uhlenbeck operators (i.e., when Q is independent of x and B is linear in x) in [18] for $p=2$. Apart from Ornstein–Uhlenbeck operators, it seems to be new even in the autonomous case. For Ornstein–Uhlenbeck operators we have a precise expression of γ_2 in terms of the data, and our Theorem 3.15 shows that $\gamma_p=\gamma_2<0$ for every $p\in(1,\infty)$. In general, γ_p could depend explicitly on p and we only give upper estimates for it.

In the autonomous case, exponential convergence to equilibrium in $L^2(\mathbb{R}^d,\mu)$ is usually obtained through Poincaré inequalities such as

$$\int_{\mathbb{R}^d}\left|\varphi - \int_{\mathbb{R}^d}\varphi\,d\mu\right|^2 d\mu \leqslant C_0\int_{\mathbb{R}^d}\big|Q^{1/2}\nabla\varphi\big|^2 d\mu, \quad \varphi\in D(A), \tag{1.10}$$

where $D(A)$ is the domain of the generator of e^{tA} in $L^2(\mathbb{R}^d,\mu)$. If (1.10) holds we get $\omega_2\leqslant -\eta_0/C_0$ and in the (symmetric) case $A\varphi=\Delta\varphi+\langle\nabla\Phi,\nabla\varphi\rangle$ we have $\omega_2=1/C_0=\eta_0/C_0$, and ω_2 is a minimum. Therefore, the problem is reduced to find the best Poincaré constant C_0, which is a hard task in general. The upper bounds on C_0 that come from gradient estimates yield $\eta_0/C_0\geqslant\gamma_2$, and the equality holds only in very special cases. Therefore, Theorem 3.6 gives a better rate of convergence (see the discussion after Proposition 3.9).

We follow a purely deterministic approach, although the well-known connections between linear second order parabolic equations and nonlinear ordinary stochastic differential equations might be used (such as e.g., in [3,14,20,25]) to get some of our formulae and/or estimates. The key tool of our analysis is the evolution semigroup,

$$\mathcal{T}(t)u(s,x) = P(s,s-t)u(s-t,\cdot)(x), \quad t \geqslant 0,\ s \in \mathbb{R},\ x \in \mathbb{R}^d,$$

that is a Markov semigroup in the space $C_b(\mathbb{T} \times \mathbb{R}^d)$ of the continuous and bounded functions u such that $u(s,\cdot) = u(s+T,\cdot)$ for all $s \in \mathbb{R}$. Its unique invariant measure is

$$\mu(ds,dx) = \frac{1}{T}\mu_s(dx)\,ds.$$

All Markov semigroups having invariant measures have natural extensions to contraction semigroups in L^p spaces with respect to such measures. Dealing with periodic functions, we consider the space $L^p(\mathbb{T} \times \mathbb{R}^d, \mu)$ that consists of all μ-measurable functions u such that $u(s,\cdot) = u(s+T,\cdot)$ for a.e. $s \in \mathbb{R}$, and such that $\int_0^T \int_{\mathbb{R}^d} |u(s,x)|^p \mu_s(dx)\,ds$ is finite. We denote by G_p the infinitesimal generator of $\mathcal{T}(t)$ in $L^p(\mathbb{T} \times \mathbb{R}^d, \mu)$. G_p is a realization of the parabolic operator $\mathcal{G}$, defined in (1.5), in the space $L^p(\mathbb{T} \times \mathbb{R}^d, \mu)$.

We introduce a projection Π on space independent functions,

$$\Pi u(s,x) := \int_{\mathbb{R}^d} u(s,y)\mu_s(dy), \quad s \in \mathbb{R},\ x \in \mathbb{R}^d,$$

and we prove that $\mathcal{T}(t)(I - \Pi)$ is strongly stable in all spaces $L^p(\mathbb{T} \times \mathbb{R}^d, \mu)$, $1 < p < \infty$, that is

$$\lim_{t\to\infty} \big\|\mathcal{T}(t)(u - \Pi u)\big\|_{L^p(\mathbb{T}\times\mathbb{R}^d,\mu)} = 0, \quad u \in L^p\big(\mathbb{T} \times \mathbb{R}^d, \mu\big). \tag{1.11}$$

From this fact we deduce (1.6) and (1.7); if $\mathcal{T}(t)(I - \Pi)$ is exponentially stable we deduce (1.9). We arrive at (1.11) through a similar property of the space gradient of $\mathcal{T}(t)u$, i.e.,

$$\lim_{t\to\infty} \big\|\,|\nabla_x \mathcal{T}(t)u|\,\big\|_{L^p(\mathbb{T}\times\mathbb{R}^d,\mu)} = 0, \quad u \in L^p\big(\mathbb{T} \times \mathbb{R}^d, \mu\big),$$

which is proved using semigroups arguments that seem not to have counterparts for evolution operators. In particular, we use the identity

$$\int_0^T \int_{\mathbb{R}^d} uG_2u\,d\mu = -\int_0^T \int_{\mathbb{R}^d} \langle Q\,\nabla_x u, \nabla_x u\rangle\,d\mu, \quad u \in D(G_2),$$

which is a time dependent version of what is called *identité du carré du champ* by the French mathematicians.

$\mathcal{T}(t)$ is a nice example of a Markov semigroup that is not strong Feller and not irreducible, and that has a unique invariant measure μ. On the other hand, (1.11) shows that in general $\mathcal{T}(t)u$ does not converge to the mean value of u with respect to μ as $t \to \infty$.

Together with asymptotic behavior results, it is natural to get spectral properties of the operators G_p, $1 < p < \infty$. When (1.9) holds, we prove that G_p has a spectral gap, and precisely

$$\sup\big\{\operatorname{Re}\lambda:\ \lambda \in \sigma(G_p) \setminus i\mathbb{R}\big\} = \omega_p < 0.$$

We remark that the equation $\lambda u - \mathcal{G}u = f$, with $f \in L^p(\mathbb{T} \times \mathbb{R}^d, \mu)$, cannot be seen as an evolution equation in a fixed L^p space X,

$$u'(t) - \mathcal{A}(t)u(t) + \lambda u(t) = f(t,\cdot)$$

because our spaces $X(t) = L^p(\mathbb{R}^d, \mu_t)$ vary with time. For the same reason, $\mathcal{T}(t)$ is not a usual evolution semigroup in a fixed Banach space X. However, it exhibits some of the typical features of evolution semigroups in fixed Banach spaces, in particular the Spectral Mapping Theorem holds.

If the diffusion coefficients do not depend on the space variables, and the supremum in (1.8) is equal to some negative number r_0, we get a log-Sobolev type inequality,

$$\int_0^T \int_{\mathbb{R}^d} |u|^2 \log\bigl(u^2\bigr)\, d\mu \leqslant \frac{1}{T} \int_0^T \Pi u^2 \log\bigl(\Pi u^2\bigr)\, ds + \frac{2\Lambda}{|r_0|} \int_0^T \int_{\mathbb{R}^d} |\nabla_x u|^2\, d\mu, \tag{1.12}$$

for every $u \in D(G_2)$. Here, Λ is the supremum of the maximum eigenvalues of the matrices $Q(s)$ when s varies in $[0, T]$. Also this inequality is proved using the evolution semigroup $\mathcal{T}(t)$, through semigroups arguments that have no counterparts for evolution operators. Using (1.12) we show that the domains $D(G_p)$ are compactly embedded in $L^p(\mathbb{T} \times \mathbb{R}^d, \mu)$ for $1 < p < \infty$, and from this fact a lot of nice consequences follow. In particular, the spectrum of each operator G_p consists of eigenvalues and it is independent of p, and the exponential decay rates $\omega_p \leqslant r_0$ are independent of p.

The interest in log-Sobolev estimates goes beyond asymptotic behavior, and much literature has been devoted to them in the autonomous case. See e.g., the surveys [1,19]. Therefore, it is worth to establish them in L^p spaces with time-space variables. In Proposition 3.12 and in Theorem 3.14 we prove L^p versions of (1.12).

The paper ends with illustrations of the asymptotic behavior and spectral results for explicit examples of families of operators $\mathcal{A}(t)$ that satisfy our assumptions.

Except for nonautonomous Ornstein–Uhlenbeck operators, this one seems to be the first systematic study of asymptotic behavior in linear nonautonomous parabolic problems with unbounded coefficients in $\mathbb{R}^d$. A part of our results lends itself to generalizations to some infinite dimensional settings, where $\mathbb{R}^d$ is replaced by a separable Hilbert space H, in the spirit of e.g., [9,10,8].

Notations. We denote by $B_b(\mathbb{R}^d)$ the Banach space of all bounded and Borel measurable functions $f : \mathbb{R}^d \to \mathbb{R}$, and by $C_b(\mathbb{R}^d)$ its subspace of all continuous functions. $B_b(\mathbb{R}^d)$ and $C_b(\mathbb{R}^d)$ are endowed with the sup norm $\|\cdot\|_\infty$. For $k \in \mathbb{N}$, $C_b^k(\mathbb{R}^d)$ is the set of all functions $f \in C_b(\mathbb{R}^d)$ whose derivatives up to the kth-order are bounded and continuous in $\mathbb{R}^d$. We use the subscript "c" instead of "b" for spaces of functions with compact support.

Throughout the paper we consider real valued functions $(s, x) \mapsto f(s, x)$ defined in $\mathbb{R}^{1+d}$, that are T-periodic with respect to time. It is useful to identify such functions with functions defined in $\mathbb{T} \times \mathbb{R}^d$, where $\mathbb{T} = [0, T]$ *mod* T. So, we denote by $C_b(\mathbb{T} \times \mathbb{R}^d)$ the space of the continuous, bounded, and T-time periodic functions $f : \mathbb{R}^{1+d} \to \mathbb{R}$, endowed with the sup norm. Similarly, for any $\alpha \in (0, 1)$, we denote by $C_{\rm loc}^{\alpha/2,\alpha}(\mathbb{T} \times \mathbb{R}^d)$ the set of all functions $f \in C_b(\mathbb{T} \times \mathbb{R}^d)$ which belong to $C^{\alpha/2,\alpha}([0, T] \times B(0, R))$ for any $R > 0$, and by $W^{1,2}_{p,{\rm loc}}(\mathbb{T} \times \mathbb{R}^d, ds \times dx)$ the set of all time periodic functions f such that f, $D_s f$, and the first and second order space derivatives of f belong to $L^p((0, T) \times B(0, R), ds \times dx)$ for any $R > 0$.

2. Preliminaries

2.1. General properties of $P(t, s)$ and of evolution systems of measures

Hypothesis 2.1.

(i) The coefficients q_{ij} and b_i $(i, j = 1, \ldots, d)$ are T-time periodic and belong to $C_{\rm loc}^{\alpha/2,\alpha}(\mathbb{T} \times \mathbb{R}^d)$ for any $i, j = 1, \ldots, d$ and some $\alpha \in (0, 1)$.
(ii) For every $(s, x) \in \mathbb{R}^{1+d}$, the matrix $Q(s, x)$ is symmetric and there exists a function $\eta : \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}$ such that $0 < \eta_0 := \inf_{\mathbb{T}\times\mathbb{R}^d} \eta$ and

$$\bigl\langle Q(s, x)\xi, \xi\bigr\rangle \geqslant \eta(s, x)|\xi|^2, \quad \xi \in \mathbb{R}^d,\ (s, x) \in \mathbb{T} \times \mathbb{R}^d.$$

(iii) There exist a positive function $V \in C^2(\mathbb{R}^d)$ and numbers a, $c > 0$ such that

$$\lim_{|x|\to\infty} V(x) = \infty \quad \text{and} \quad \big(\mathcal{A}(s)V\big)(x) \leqslant a - cV(x), \quad (s,x) \in \mathbb{T} \times \mathbb{R}^d.$$

Here we recall some results from [21] and [22]. The first one is that, under Hypothesis 2.1, for every $f \in C_b(\mathbb{R}^d)$ problem (1.3) has a unique bounded classical solution u (see [21, Thm. 2.2]). The evolution operator $P(t,s)$ is defined by

$$P(t,s)f = u(t,\cdot), \quad t \geqslant s \in \mathbb{R}.$$

Some properties of $P(t,s)$, taken from [21], are summarized in the next theorem and in its corollaries.

Theorem 2.2. *Let Hypothesis* 2.1 *hold. Define* $\Lambda := \{(t,s,x) \in \mathbb{R}^{2+d}\colon\ t > s,\ x \in \mathbb{R}^d\}$. *Then*:

(i) *for every* $\varphi \in C_b(\mathbb{R}^d)$, *the function* $(t,s,x) \mapsto P(t,s)\varphi(x)$ *is continuous in* $\overline{\Lambda}$. *For every* $s \in \mathbb{R}$, *the function* $(t,x) \mapsto P(t,s)\varphi(x)$ *belongs to* $C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}((s,\infty)\times\mathbb{R}^d)$;
(ii) *for every* $\varphi \in C^\infty_c(\mathbb{R}^d)$, *the function* $(t,s,x) \mapsto P(t,s)\varphi(x)$ *is continuously differentiable with respect to* s *in* $\overline{\Lambda}$ *and* $D_s P(t,s)\varphi(x) = -P(t,s)\mathcal{A}(s)\varphi(x)$ *for any* $(t,s,x) \in \overline{\Lambda}$;
(iii) *for each* $(t,s,x) \in \Lambda$ *there exists a Borel probability measure* $p_{t,s,x}$ *in* $\mathbb{R}^d$ *such that*

$$P(t,s)\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)\, p_{t,s,x}(dy), \quad f \in C_b\big(\mathbb{R}^d\big). \tag{2.1}$$

Moreover, $p_{t,s,x}(dy) = g(t,s,x,y)dy$ *for a positive function* g. *In particular,* $P(t,s)$ *is irreducible*;
(iv) $P(t,s)$ *is strong Feller*; *extending it to* $L^\infty(\mathbb{R}^d,dx)$ *through formula* (2.1), *it maps* $L^\infty(\mathbb{R}^d,dx)$ (*and, in particular,* $B_b(\mathbb{R}^d)$) *into* $C_b(\mathbb{R}^d)$ *for* $t > s$, *and*

$$\big\|P(t,s)\varphi\big\|_\infty \leqslant \|\varphi\|_\infty, \quad \varphi \in L^\infty\big(\mathbb{R}^d,dx\big),\ t > s;$$

(v) *there exists a tight*[1] *evolution system of measures* $\{\mu_s\colon\ s \in \mathbb{R}\}$ *for* $P(t,s)$. *Moreover,*

$$P(t,s)V(x) := \int_{\mathbb{R}^d} V(y)\, p_{t,s,x}(dy) \leqslant V(x) + \frac{a}{c}, \quad t > s,\ x \in \mathbb{R}^d, \tag{2.2}$$

and

$$\int_{\mathbb{R}^d} V(y)\mu_t(dy) \leqslant \min V + \frac{a}{c}, \quad t \in \mathbb{R}, \tag{2.3}$$

where the constants a *and* c *are given by Hypothesis* 2.1(iii).

In particular, statement (i) follows from Theorems 3.7 and 2.2 (Step 3), statement (ii) follows from Lemma 3.2, statement (iii) is a consequence of Proposition 2.4 and Corollary 2.5, the strong Feller property of statement (iv) is proved in Corollary 4.3, the existence of the measures μ_s is proved in Theorem 5.4 while estimates (2.2) and (2.3) follow from the proofs of Lemma 5.3 and Theorem 5.4, respectively.

Note that estimate (2.2) implies that the family $\{p_{t,s,x}\colon\ t > s,\ x \in B(0,r)\}$ is tight for every $r > 0$.

[1] i.e., $\forall \varepsilon > 0\ \exists R = R(\varepsilon) > 0$ such that $\mu_s(B(0,R)) \geqslant 1 - \varepsilon$, for all $s \in \mathbb{R}$.

Since the coefficients q_{ij} and b_i are T-time periodic, uniqueness of the bounded solution to (1.3) implies that $P(t+T, s+T) = P(t,s)$ for $t \geqslant s$. Moreover, looking at the construction of the measures μ_t of [21] one can see that $\mu_s = \mu_{s+T}$ for each $s \in \mathbb{R}$ (see [22, Rem. 6.8(i)]).

The evolution systems of invariant measures are infinitely many, in general. In the case of nonautonomous Ornstein–Uhlenbeck equations, they have been explicitly characterized in [17, Prop. 2.2]. In the next section we shall prove that all the T-periodic families $\{\mu_s\colon s \in \mathbb{R}\}$ constructed in [21] actually coincide, since $P(t,s)$ has a unique T-periodic evolution system of measures.

In the next corollary we prove some consequences of Theorem 2.2. For this purpose, for every $\varphi \in L^1(\mathbb{R}^d, \mu_s)$ we define the mean value

$$m_s\varphi := \int_{\mathbb{R}^d} \varphi(y)\mu_s(dy).$$

Corollary 2.3. *Let Hypothesis* 2.1 *hold. Then*:

(a) *for every $\varphi \in C_b(\mathbb{R}^d)$ the function $s \mapsto m_s\varphi$ is continuous in $\mathbb{R}$. More generally, for every $u \in C_b(\mathbb{R}^{1+d})$ the function $s \mapsto m_s u(s,\cdot)$ is continuous in $\mathbb{R}$;*

(b) *for every $\varphi \in C_b(\mathbb{R}^d)$ and for every bounded sequence $(\varphi_n) \subset C_b(\mathbb{R}^d)$ that converges locally uniformly to φ we have*

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \|\varphi - \varphi_n\|_{L^p(\mathbb{R}^d,\mu_s)} = 0, \quad 1 \leqslant p < \infty, \tag{2.4}$$

and, for every $r > 0$,

$$\lim_{n\to\infty} \sup_{s\leqslant t\in\mathbb{R}} \big\| P(t,s)(\varphi - \varphi_n)\big\|_{L^\infty(B(0,r))} = 0; \tag{2.5}$$

(c) *for $t > s$, $P(t,s)$ may be extended to a bounded operator from $L^p(\mathbb{R}^d, \mu_s)$ to $L^p(\mathbb{R}^d, \mu_t)$ for all $p \in [1,\infty)$, and* (1.4) *holds.*

Proof. The first part of statement (a) is an easy consequence of the continuity of $P(t,s)\varphi$ with respect to s. Indeed, fix $s_0 \in \mathbb{R}$ and $t \geqslant s_0 + 1$. For $s \in (s_0 - 1, s_0 + 1)$ we have

$$m_s\varphi - m_{s_0}\varphi = \int_{\mathbb{R}^d} \big(P(t,s)\varphi(y) - P(t,s_0)\varphi(y)\big)\mu_t(dy).$$

By Theorem 2.2(i), for every $y \in \mathbb{R}^d$ we have $\lim_{s\to s_0} P(t,s)\varphi(y) - P(t,s_0)\varphi(y) = 0$; moreover $|P(t,s)\varphi(y) - P(t,s_0)\varphi(y)| \leqslant 2\|\varphi\|_\infty$. Therefore, $\lim_{s\to s_0} m_s\varphi - m_{s_0}\varphi = 0$.

Let us prove the second part of statement (a). Fix $s_0 \in \mathbb{R}$. Then,

$$\big|m_s u(s,\cdot) - m_{s_0} u(s_0,\cdot)\big| \leqslant \big|m_s\big(u(s,\cdot) - u(s_0,\cdot)\big)\big| + \big|m_s u(s_0,\cdot) - m_{s_0} u(s_0,\cdot)\big|, \quad s \in \mathbb{R}.$$

By the first part of the statement, $\lim_{s\to s_0} |m_s u(s_0,\cdot) - m_{s_0} u(s_0,\cdot)| = 0$. To estimate $|m_s(u(s,\cdot) - u(s_0,\cdot))|$ we use Theorem 2.2(v). Given $\varepsilon > 0$, let $R > 0$ be such that $\mu_s(\mathbb{R}^d \setminus B(0,R)) \leqslant \varepsilon$ for every $s \in \mathbb{R}$. Then,

$$\begin{aligned}\big|m_s\big(u(s,\cdot) - u(s_0,\cdot)\big)\big| &\leqslant \int_{B(0,R)} \big|u(s,y) - u(s_0,y)\big|\mu_s(dy) + \int_{\mathbb{R}^d\setminus B(0,R)} \big|u(s,y) - u(s_0,y)\big|\mu_s(dy) \\ &\leqslant \big\|u(s,\cdot) - u(s_0,\cdot)\big\|_{L^\infty(B(0,R))} + 2\varepsilon\|u\|_\infty.\end{aligned}$$

Since u is continuous, $\|u(s,\cdot) - u(s_0,\cdot)\|_{L^\infty(B(0,R))} \leqslant \varepsilon$ for $|s - s_0|$ small enough. Statement (a) follows.

The proof of statement (b) is similar. Let $M > 0$ be such that $\|\varphi_n\|_\infty \leqslant M$ for each $n \in \mathbb{N}$. For every $\varepsilon > 0$ let $R > 0$ be as above. Then,

$$\int_{\mathbb{R}^d} |\varphi - \varphi_n|^p \, d\mu_s = \int_{B(0,R)} |\varphi - \varphi_n|^p \, d\mu_s + \int_{\mathbb{R}^d \setminus B(0,R)} |\varphi - \varphi_n|^p \, d\mu_s$$
$$\leqslant \|\varphi - \varphi_n\|^p_{L^\infty(B(0,R))} + \big(\|\varphi\|_\infty + M\big)^p \varepsilon,$$

and (2.4) holds. The proof of (2.5) is the same, through the representation formula $P(t,s)\varphi(x) = \int_{\mathbb{R}^d} \varphi(y) p_{t,s,x}(dy)$ and the tightness of $\{p_{t,s,x}\colon\ s < t,\ x \in B(0,r)\}$.

The proof of statement (c) is the same of the autonomous case. Indeed, for every $\varphi \in C_b(\mathbb{R}^d)$ we have, by Theorem 2.2(iii) and the Hölder inequality,

$$\big|P(t,s)\varphi(x)\big|^p = \bigg|\int_{\mathbb{R}^d} \varphi(y) p_{t,s,x}(dy)\bigg|^p \leqslant \int_{\mathbb{R}^d} \big|\varphi(y)\big|^p p_{t,s,x}(dy) = \big(P(t,s)|\varphi|^p\big)(x),$$

so that, integrating with respect to μ_t, we get

$$\int_{\mathbb{R}^d} \big|P(t,s)\varphi\big|^p \, d\mu_t \leqslant \int_{\mathbb{R}^d} P(t,s)|\varphi|^p \, d\mu_t = \int_{\mathbb{R}^d} |\varphi|^p \, d\mu_s, \quad t \geqslant s,$$

i.e., φ satisfies (1.4). Since $C_b(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d, \mu_s)$, (1.4) holds for every $\varphi \in L^p(\mathbb{R}^d, \mu_s)$. □

2.2. Smoothing properties of $P(t,s)$

We recall some global smoothing properties of the evolution operator $P(t,s)$ that have been proved in [21,22] and will be extensively used in this paper.

Hypothesis 2.4.

(i) The first-order space derivatives of the data q_{ij} and b_i $(i, j = 1, \ldots, d)$ exist and belong to $C^{\alpha/2,\alpha}_{\mathrm{loc}}(\mathbb{T} \times \mathbb{R}^d)$;

(ii) there are two upperly bounded functions $\zeta : \mathbb{T} \to \mathbb{R}_+$ and $r : \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\big\langle \nabla_x b(s,x)\xi, \xi \big\rangle \leqslant r(s,x)|\xi|^2, \quad (s,x) \in \mathbb{T} \times \mathbb{R}^d,\ \xi \in \mathbb{R}^d,$$
$$\big|D_k q_{ij}(s,x)\big| \leqslant \zeta(s)\eta(s,x), \quad (s,x) \in \mathbb{T} \times \mathbb{R}^d,\ i,j,k = 1, \ldots, d,$$

where $\eta(s,x)$ is the ellipticity constant at (s,x) in Hypothesis 2.1(ii).

The following theorem has been proved in [21, Thm. 4.1].

Theorem 2.5. *Let Hypotheses* 2.1 *and* 2.4 *hold. Then, there exist positive constants* $C_1, C_2 > 0$, *such that*

(i) *for every* $\varphi \in C^1_b(\mathbb{R}^d)$ *we have*

$$\big\|\nabla_x P(t,s)\varphi\big\|_\infty \leqslant C_1 \|\varphi\|_{C^1_b(\mathbb{R}^d)}, \quad s < t \leqslant s+1;$$

(ii) *for every* $\varphi \in C_b(\mathbb{R}^d)$ *we have*

$$\left\|\nabla_x P(t,s)\varphi\right\|_\infty \leqslant \frac{C_2}{\sqrt{t-s}}\|\varphi\|_\infty, \quad s<t\leqslant s+1. \tag{2.6}$$

As a consequence, we obtain

$$\left\|\nabla_x P(t,s)\varphi\right\|_\infty \leqslant C_2\|\varphi\|_\infty, \quad t\geqslant s+1, \tag{2.7}$$

for every $\varphi \in C_b(\mathbb{R}^d)$. It is sufficient to recall that $P(t,s)\varphi = P(t,t-1)P(t-1,s)\varphi$ and that $\|P(t-1,s)\varphi\|_\infty \leqslant \|\varphi\|_\infty$.

Theorem 2.6. *Let Hypotheses* 2.1 *and* 2.4 *hold and assume in addition that, for some* $p>1$,

$$\ell_p := \sup_{(s,x)\in[0,T]\times\mathbb{R}^d}\left(r(s,x)+\frac{d^3(\zeta(s))^2\eta(s,x)}{4\min\{p-1,1\}}\right)<\infty. \tag{2.8}$$

Then:

(i) *for every* $\varphi \in C_b^1(\mathbb{R}^d)$ *we have*

$$\left|\nabla_x P(t,s)\varphi(x)\right|^p \leqslant e^{p\ell_p(t-s)}\big(P(t,s)|\nabla\varphi|^p\big)(x), \quad t\geqslant s,\ x\in\mathbb{R}^d; \tag{2.9}$$

(ii) *there exists a positive constant* $C_3=C_3(p)$ *such that*

$$\left|\nabla_x P(t,s)\varphi(x)\right|^p \leqslant C_3^p\max\big\{(t-s)^{-p/2},1\big\}e^{p\ell_p(t-s)}\big(P(t,s)|\varphi|^p\big)(x), \tag{2.10}$$

for every $\varphi\in C_b(\mathbb{R}^d)$, $t>s$ *and* $x\in\mathbb{R}^d$;

(iii) *if the diffusion coefficients* q_{ij} $(i,j=1,\ldots,d)$ *are independent of x, then* (2.9) *holds for* $p=1$ *too, with* $\ell_1=r_0:=\sup_{(s,x)\in\mathbb{T}\times\mathbb{R}^d} r(s,x)$. *Moreover, there exists* $C_4>0$, *independent of t and s, such that*

$$\left\|\left|\nabla_x P(t,s)\varphi\right|\right\|_\infty \leqslant C_4 e^{r_0(t-s)}\|\varphi\|_\infty, \quad t\geqslant s+1. \tag{2.11}$$

Proof. Estimates (2.9) and (2.11) have been proved in [21, Thm. 4.5, Cor. 4.6]. To be precise, in [21, Cor. 4.6] estimate (2.11) is stated as $\||\nabla_x P(t,s)\varphi|\|_\infty \leqslant Ce^{\ell_p(t-s)}\|\varphi\|_\infty$, with C independent of p. If the diffusion coefficients are independent of x, we can take $\zeta\equiv 0$. Hence, $\ell_p=r_0$ and (2.11) follows.

In (the proof of) [22, Prop. 3.3], an estimate similar to (2.10) has been proved with a worse exponential term. To get (2.10) it is sufficient to observe that for $t-s\leqslant 1$, [22, Prop. 3.3] gives

$$\left|\nabla_x P(t,s)\varphi(x)\right|^p \leqslant \frac{K^p}{(t-s)^{\frac{p}{2}}}\big(P(t,s)|\varphi|^p\big)(x), \quad x\in\mathbb{R}^d, \tag{2.12}$$

for some positive constant $K=K(p)$, independent of s,t and φ. If $t-s>1$, we write $P(t,s)\varphi = P(t,s+1)P(s+1,s)\varphi$. From (2.9) and (2.12) we obtain

$$\begin{aligned}\left|\nabla_x P(t,s)\varphi(x)\right|^p &\leqslant e^{p\ell_p(t-s-1)}\big(P(t,s+1)\left|\nabla_x P(s+1,s)\varphi\right|^p\big)(x)\\ &\leqslant K^p e^{p\ell_p(t-s-1)}\big(P(t,s+1)P(s+1,s)|\varphi|^p\big)(x)\\ &= K^p e^{p\ell_p(t-s-1)}\big(P(t,s)|\varphi|^p\big)(x),\end{aligned}$$

for any $x\in\mathbb{R}^d$. Estimate (2.10) follows. □

Remark 2.7. Two remarks are in order.

(a) Estimate (2.10) implies that $P(t,s)$ maps $L^p(\mathbb{R}^d,\mu_s)$ into $W^{1,p}(\mathbb{R}^d,\mu_t)$ for $p>1$, and

$$\big\| |\nabla_x P(t,s)\varphi| \big\|_{L^p(\mathbb{R}^d,\mu_t)} \leqslant C_3 \max\big\{(t-s)^{-1/2},1\big\} e^{\ell_p(t-s)} \|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}. \tag{2.13}$$

This is not true in general for $p=1$, even in the autonomous case. See e.g., [23, Cor. 5.1] for a counterexample given by the Ornstein–Uhlenbeck semigroup.

(b) Estimates (2.13) are sharp near $t=s$, but they are not for $t \gg s$ if $\ell_p>0$. In this case for $t>s+1$ we write $P(t,s)=P(t,t-1)P(t-1,s)$, and using (1.4) we obtain

$$\big\| |\nabla_x P(t,s)\varphi| \big\|_{L^p(\mathbb{R}^d,\mu_t)} \leqslant C_3 e^{\ell_p} \|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}, \quad t \geqslant s+1,\ \varphi \in L^p\big(\mathbb{R}^d,\mu_s\big). \tag{2.14}$$

2.3. The evolution semigroup

The evolution semigroup $\mathcal{T}(t)$ is defined on continuous and bounded functions f by

$$\mathcal{T}(t)f(s,x) = P(s,s-t)f(s-t,\cdot)(x), \quad (s,x)\in\mathbb{R}^{1+d},\ t\geqslant 0.$$

In [21, Prop. 6.1] we have shown that $\mathcal{T}(t)$ is a semigroup of positive contractions in $C_b(\mathbb{R}^{1+d})$. Since $P(s+T,s+T-t)=P(s,s-t)$, $\mathcal{T}(t)$ leaves $C_b(\mathbb{T}\times\mathbb{R}^d)$ invariant for every $t>0$.

$\mathcal{T}(t)$ is not strongly continuous in $C_b(\mathbb{T}\times\mathbb{R}^d)$. However, the last part of the proof of [21, Prop. 6.1] implies that, for any $f\in C_b(\mathbb{T}\times\mathbb{R}^d)$ and any $t_0\geqslant 0$, $\mathcal{T}(t)f$ tends to $\mathcal{T}(t_0)f$, locally uniformly in $\mathbb{T}\times\mathbb{R}^d$ as $t\to t_0$.

In the language of [9], $\mathcal{T}(t)$ is a stochastically continuous Markov semigroup. It improves spatial regularity, as the next lemma shows.

Lemma 2.8. *For every $t>0$ and $f\in C_b(\mathbb{T}\times\mathbb{R}^d)$, the derivatives $D_i\mathcal{T}(t)f$, $D_{ij}\mathcal{T}(t)f$ exist and are continuous in $\mathbb{T}\times\mathbb{R}^d$ for $i,j=1,\dots,d$.*

Proof. Since $\mathcal{T}(t)f(s,x)=P(s,s-t)f(s-t,\cdot)(x)$, it is sufficient to show that the first and second order space derivatives of the function $(t,r,x)\mapsto P(t,r)f(r,\cdot)(x)$ are continuous with respect to $(t,r,x)\in\Lambda$. For any $(t_0,r_0,x_0)\in\Lambda$, fix $\delta>0$ such that $t_0-\delta>r_0+\delta$. The classical interior Schauder estimates (e.g., [15, Thm. 3.5]) imply that for any $R>0$ there exists a positive constant C such that

$$\sup_{|t-t_0|\leqslant\delta,\ |r-r_0|\leqslant\delta} \big\|P(t,r)\varphi\big\|_{C^{2+\alpha}(B(x_0,R))} \leqslant C\|\varphi\|_\infty, \tag{2.15}$$

for every $\varphi\in C_b(\mathbb{R}^d)$. Applying the interpolatory estimates

$$\|D_i\psi\|_{C(B(x_0,R))} \leqslant K_1 \|\psi\|_{C(B(x_0,R))}^{\frac{1+\alpha}{2+\alpha}} \|\psi\|_{C^{2+\alpha}(B(x_0,R))}^{\frac{1}{2+\alpha}}, \quad i=1,\dots,d \tag{2.16}$$

(which hold for every $\psi\in C^{2+\alpha}(B(x_0,R))$ and some positive constant $K_1=K_1(\alpha,R)$, see e.g., [26, Sect. 4.5.2, Rem. 2]) to the function $\psi=P(t,r)f(r,\cdot)-P(t_0,r_0)f(r_0,\cdot)$ with $t\in[t_0-\delta,t_0+\delta]$, $r\in[r_0-\delta,r_0+\delta]$, we deduce

$$\begin{aligned}
&\big\|D_iP(t,r)f(r,\cdot)-D_iP(t_0,r_0)f(r_0,\cdot)\big\|_{C(B(x_0,R))}\\
&\quad\leqslant K_1\big\|P(t,r)f(r,\cdot)-P(t_0,r_0)f(r_0,\cdot)\big\|_{C(B(x_0,R))}^{\frac{1+\alpha}{2+\alpha}}\\
&\qquad\times\Big(\big\|P(t,r)f(r,\cdot)\big\|_{C^{2+\alpha}(B(x_0,R))}+\big\|P(t_0,r_0)f(r_0,\cdot)\big\|_{C^{2+\alpha}(B(x_0,R))}\Big)^{\frac{1}{2+\alpha}}\\
&\quad\leqslant K_1\big\|P(t,r)f(r,\cdot)-P(t_0,r_0)f(r_0,\cdot)\big\|_{C(B(x_0,R))}^{\frac{1+\alpha}{2+\alpha}}\big(2C\|f\|_\infty\big)^{\frac{1}{2+\alpha}},
\end{aligned}$$

where the last inequality follows from (2.15). Since $(t,r,x)\mapsto P(t,r)f(r,\cdot)(x)$ is a continuous function in Λ, the right-hand side vanishes as (t,r) tends to (t_0,r_0), and this implies that $D_iP(t,r)f(r,\cdot)(x)$ is continuous in $(t,r,x)\in\Lambda$.

Using the interpolatory estimates (see [26, Sect. 4.5.2, Rem. 2])

$$\|D_{ij}\psi\|_{C(B(x_0,R))}\leqslant K_2\|\psi\|_{C(B(x_0,R))}^{\frac{\alpha}{2+\alpha}}\|\psi\|_{C^{2+\alpha}(B(x_0,R))}^{\frac{2}{2+\alpha}},\quad i,j=1,\dots,d,$$

instead of (2.16), the same procedure yields that $D_{ij}P(t,r)f(r,\cdot)(x)$ is continuous in Λ for any $i,j=1,\dots,d$. $\square$

The generator G_∞ of $\mathcal{T}(t)$ in $C_b(\mathbb{T}\times\mathbb{R}^d)$ may be defined through its resolvent. Namely, for every $\lambda>0$, $D(G_\infty)$ is the range of the operator

$$u\mapsto v(s,x):=\int_0^\infty e^{-\lambda t}\mathcal{T}(t)u(s,x)\,dt,$$

and $G_\infty v=\lambda v-u$. The following result is taken from [22, Prop. 6.3].

Theorem 2.9. *Under Hypothesis* 2.1 *we have*

$$D(G_\infty)=\left\{f\in\bigcap_{q<\infty}W^{1,2}_{q,\mathrm{loc}}\big(\mathbb{T}\times\mathbb{R}^d,ds\times dx\big)\cap C_b\big(\mathbb{T}\times\mathbb{R}^d\big),\ \mathcal{G}f\in C_b\big(\mathbb{T}\times\mathbb{R}^d\big)\right\},$$

where $\mathcal{G}f(s,x)=\mathcal{A}(s)f(s,\cdot)(x)-D_sf(s,x)$. Moreover, $D(G_\infty)$ coincides with the set of the functions $u\in C_b(\mathbb{T}\times\mathbb{R}^d)$ such that $\sup_{0<t\leqslant 1}t^{-1}\|\mathcal{T}(t)u-u\|_\infty<\infty$ and there exists $g\in C_b(\mathbb{T}\times\mathbb{R}^d)$ such that $t^{-1}(\mathcal{T}(t)u-u)\to g$, locally uniformly in $\mathbb{T}\times\mathbb{R}^d$, as $t\to 0$.

For every T-periodic evolution system of measures $\{\nu_s\colon s\in\mathbb{R}\}$ for $P(t,s)$, the measure $\nu(ds,dx):=\frac{1}{T}\nu_s(dx)\,ds$ is invariant for $\mathcal{T}(t)$. Indeed, the first part of the proof of Corollary 2.3 shows that, for each $\varphi\in C_b(\mathbb{R}^d)$, the function $s\mapsto\int_{\mathbb{R}^d}\varphi\,d\nu_s$ is continuous in $\mathbb{R}$. Hence, for every Borel set $\Gamma\subset\mathbb{R}^d$ the function $s\mapsto\nu_s(\Gamma)$ is Lebesgue measurable, and ν is well defined. Moreover, for every $f\in C_b(\mathbb{T}\times\mathbb{R}^d)$ we have

$$\begin{aligned}\int_{(0,T)\times\mathbb{R}^d}\mathcal{T}(t)f\,d\nu&=\frac{1}{T}\int_0^T ds\int_{\mathbb{R}^d}P(s,s-t)f(s-t,\cdot)\,d\nu_s\\&=\frac{1}{T}\int_0^T ds\int_{\mathbb{R}^d}f(s-t,\cdot)\,d\nu_{s-t}\\&=\frac{1}{T}\int_{-t}^{T-t}d\sigma\int_{\mathbb{R}^d}f(\sigma,\cdot)\,d\nu_\sigma\\&=\int_{(0,T)\times\mathbb{R}^d}f\,d\nu,\end{aligned}$$

where the last equality follows from the periodicity of the function $\sigma\mapsto\int_{\mathbb{R}^d}f(\sigma,\cdot)\,d\nu_\sigma$.

Proposition 2.10. *Under Hypothesis* 2.1*, $P(t,s)$ has a unique T-periodic evolution system of measures, and $\mathcal{T}(t)$ has a unique invariant measure.*

Proof. Let $\{\nu_s\colon s\in\mathbb{R}\}$ be any T-periodic evolution system of measures for $P(t,s)$. The arguments in [5, Thm. 4.3] show that $\nu=\frac{1}{T}\nu_s(dx)\,ds$ is ergodic for $\mathcal{T}(t)$, in the sense that, if Γ is a Borel set in $\mathbb{T}\times\mathbb{R}^d$ such that $\mathcal{T}(t)\mathbb{1}_\Gamma=\mathbb{1}_\Gamma$ for every t, then either $\nu(\Gamma)=0$ or $\nu(\Gamma)=1$. This implies that the invariant measures μ and ν corresponding to two different evolution systems of measures $\{\mu_s\colon s\in\mathbb{R}\}$ and $\{\nu_s\colon s\in\mathbb{R}\}$, are either singular or coincide, see e.g., [9, Prop. 3.2.5]. But we know from [21, Prop. 5.2] that μ_t and ν_t are equivalent to the Lebesgue measure in $\mathbb{R}^d$ for every $t\in\mathbb{R}$, hence μ and ν are equivalent to the Lebesgue measure in $\mathbb{T}\times\mathbb{R}^d$ so that they cannot be singular. Therefore, $\nu=\mu$, which implies $\nu_s=\mu_s$ for a.e. $s\in\mathbb{R}$. Since $s\mapsto\int_{\mathbb{R}^d}\varphi(x)\nu_s(dx)$ and $s\mapsto\int_{\mathbb{R}^d}\varphi(x)\mu_s(dx)$ are continuous for each $\varphi\in C_b(\mathbb{R}^d)$ by Corollary 2.3, then $\nu_s=\mu_s$ for every $t\in\mathbb{R}$.

Let us prove that each invariant measure ν for $\mathcal{T}(t)$ comes from an evolution system of measures. Arguing as in the case of time depending Ornstein–Uhlenbeck operators [7, Prop. 4.2] we see that ν is of the type $\nu(ds,dx)=\frac{1}{T}\nu_s(dx)\,dt$, where $\{\nu_s\colon s\in\mathbb{R}\}$ is a family of T-periodic probability measures. Let $f(s,x)=g(s)\varphi(x)$, with $g\in C(\mathbb{T})$ and $\varphi\in C_b(\mathbb{R}^d)$. For $t>0$ the equality $\int_{\mathbb{T}\times\mathbb{R}^d}\mathcal{T}(t)f\,d\nu=\int_{\mathbb{T}\times\mathbb{R}^d}f\,d\nu$ means

$$\begin{aligned}\int_0^T g(s)\int_{\mathbb{R}^d}\varphi(x)\nu_s(dx)\,ds&=\int_0^T g(s-t)\int_{\mathbb{R}^d}P(s,s-t)\varphi(x)\nu_s(dx)\,ds\\&=\int_{-t}^{T-t} g(s)\int_{\mathbb{R}^d}P(s+t,s)\varphi(x)\nu_{s+t}(dx)\,ds\\&=\int_0^T g(s)\int_{\mathbb{R}^d}P(s+t,s)\varphi(x)\nu_{s+t}(dx)\,ds.\end{aligned}$$

Since g is arbitrary, then $\int_{\mathbb{R}^d}P(s+t,s)\varphi(x)\nu_{s+t}(dx)=\int_{\mathbb{R}^d}\varphi(x)\nu_s(dx)$ for every $t>0$, which means that $\{\nu_s\colon s\in\mathbb{R}\}$ is an evolution system of measures. □

From now on we shall consider the invariant measure μ for $\mathcal{T}(t)$ defined by

$$\mu(ds,dx):=\frac{1}{T}\mu_s(dx)\,ds,$$

where $\{\mu_s\colon t\in\mathbb{R}\}$ is the unique T-periodic evolution system of measures for $P(t,s)$.

As a consequence of [2, p. 2067], there exists a continuous positive function $\rho:\mathbb{T}\times\mathbb{R}^d\to\mathbb{R}$ such that $\mu(ds,dx)=\rho(s,x)\,ds\,dx$. The computation at the end of the proof of Proposition 2.10 shows that the family of measures $\nu_s(dx):=\rho(s,x)\,dx$ are a T-periodic evolution system of measures. By uniqueness, $\nu_s=\mu_s/T$ for every s, i.e. the density of μ_s is $T\rho(s,\cdot)$ for every $s\in\mathbb{R}$.

For any $p\in[1,\infty)$, we introduce the space $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$ of all functions f such that $f(s+T,x)=f(s,x)$ for a.e. $(s,x)\in\mathbb{R}^{1+d}$ and

$$\|f\|^p_{L^p(\mathbb{T}\times\mathbb{R}^d,\mu)}:=\int_{(0,T)\times\mathbb{R}^d}|f|^p\,d\mu<\infty.$$

We also use the symbol $\int_{\mathbb{T}\times\mathbb{R}^d}|f|^p\,d\mu$ for $\int_{(0,T)\times\mathbb{R}^d}|f|^p\,d\mu$. If no confusion may arise, we write $\|f\|_p$ for $\|f\|_{L^p(\mathbb{T}\times\mathbb{R}^d,\mu)}$.

As all Markov semigroups having an invariant measure, $\mathcal{T}(t)$ can be extended to a semigroup of positive contractions in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$ for any $p\in[1,\infty)$. We still call $\mathcal{T}(t)$ these extensions, using the notation $\mathcal{T}_p(t)$ only when we deal with different L^p spaces.

It is easy to see that $\mathcal{T}(t)$ is strongly continuous in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$ for $1\leqslant p<\infty$. Indeed, we already know that $\mathcal{T}(t)f$ tends to f locally uniformly as $t\to 0^+$, for any $f\in C_b(\mathbb{T}\times\mathbb{R}^d)$. Moreover, $\|\mathcal{T}(t)f\|_\infty\leqslant\|f\|_\infty$ for any $t>0$. By dominated convergence, $\mathcal{T}(t)f$ tends to f in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$ as $t\to 0^+$. Since $C_b(\mathbb{T}\times\mathbb{R}^d)$ is dense in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$, $\mathcal{T}(t)f$ tends to f in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$ as $t\to 0^+$, for every $f\in L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$.

We denote by G_p the infinitesimal generator of $\mathcal{T}(t)$ in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$. In general, the characterization of the domain $D(G_p)$ of G_p is not obvious, and even determining whether a given smooth function f belongs to $D(G_p)$ is not obvious. In the case of time depending Ornstein–Uhlenbeck operators, $D(G_2)$ has been characterized in [17] as the space of all $f\in L^2(\mathbb{T}\times\mathbb{R}^d,\mu)$ such that there exist D_sf, D_if, $D_{ij}f\in L^2(\mathbb{T}\times\mathbb{R}^d,\mu)$ for $i,j=1,\dots,d$. A similar characterization for $2\neq p\in(1,\infty)$ follows adapting to the periodic case the procedure of [16]. We do not expect the same result in the general case, since for functions f with all derivatives in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$, $\mathcal{A}(s)f(s,\cdot)$ does not necessarily belong to $L^p(\mathbb{R}^d,\mu_s)$, even for bounded diffusion coefficients. Fortunately, the explicit knowledge of $D(G_p)$ is not necessary in several circumstances, provided we know a good core of G_p. In the paper [22] sufficient conditions have been given for $C_c^\infty(\mathbb{T}\times\mathbb{R}^d)$ be a core of G_p, for $1\leqslant p<\infty$. In general, $C_c^\infty(\mathbb{T}\times\mathbb{R}^d)$ is contained in $D(G_p)$ but it is not a core. However, we have the following result (see [22, Thm. 6.7]).

Proposition 2.11. *Under Hypothesis* 2.1, *$\mathcal{T}(t)$ maps $D(G_\infty)$ into itself, and $D(G_\infty)$ is a core of G_p for every $p\in[1,\infty)$ and every $t>0$.*

Adapting to our situation a similar result for evolution semigroups in fixed Banach spaces (e.g., [4, Thm. 3.12]), we determine another core of G_p.

To this purpose, for any $\tau\in\mathbb{R}$, $\chi\in C_c^\infty(\mathbb{R}^d)$ and $\alpha\in C_c^1(\mathbb{R})$ with $\operatorname{supp}\alpha\subset(a,a+T)$ for some $a\geqslant\tau$, we define the function $u_{\tau,\chi,\alpha}:\mathbb{R}^{1+d}\to\mathbb{R}$, as the T-periodic (with respect to s) extension of the function $(s,x)\mapsto\alpha(s)P(s,\tau)\chi(x)$ defined in $[a,a+T)\times\mathbb{R}^d$.

Proposition 2.12. *For each τ, χ and α as above, the function $u_{\tau,\chi,\alpha}$ belongs to $D(G_p)$ and the linear span $\mathcal{C}$ of the functions $u_{\tau,\chi,\alpha}$ is a core for G_p, for each $p\in[1,\infty)$.*

Proof. Any function $u\in\mathcal{C}$ is in $C^{1,2}(\mathbb{R}^{1+d})$ by (the proof of) [21, Thm. 2.2] and, since it is periodic in time and bounded, it belongs to $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$ for every $p\in[1,\infty)$.

Fix τ, α, χ as in the statement and define $u:=u_{\tau,\chi,\alpha}$. For each $s\in[a,a+T)$, $x\in\mathbb{R}^d$ we have

$$\mathcal{G}u(s,x)=-\big\{\alpha'(s)P(s,\tau)\chi(x)+\alpha(s)\mathcal{A}(s)P(s,\tau)\chi(x)\big\}+\alpha(s)\mathcal{A}(s)P(s,\tau)\chi(x),$$

so that

$$\mathcal{G}u(s,x)=-\alpha'(s)P(s,\tau)\chi(x),\quad s\in[a,a+T),\ x\in\mathbb{R}^d,$$

and, for every $k\in\mathbb{Z}$,

$$\mathcal{G}u(s,x)=-\alpha'(s-kT)P(s-kT,\tau)\chi(x),\quad s\in\big[a+kT,a+(k+1)T\big),\ x\in\mathbb{R}^d.$$

Let us prove that, for every $t>0$, $\mathcal{T}(t)u\in\mathcal{C}$. For every $s\in\mathbb{R}$, let $k\in\mathbb{Z}$ be such that $s-t\in[a+kT,a+(k+1)T)$. Then $s-t-kT\geqslant\tau$, and for every $x\in\mathbb{R}^d$ we have

$$\begin{aligned}\big(\mathcal{T}(t)u\big)(s,x) &= \alpha(s-t-kT)P(s,s-t)P(s-t-kT,\tau)\chi(x)\\ &= \alpha(s-t-kT)P(s-kT,s-t-kT)P(s-t-kT,\tau)\chi(x)\\ &= \alpha(s-t-kT)P(s-kT,\tau)\chi(x),\end{aligned}$$

so that $\mathcal{T}(t)u$ is the T-periodic extension of the function $(s,x)\mapsto \alpha(s-t)P(s,\tau)\chi(x)$ defined in $[a+t,a+t+T)\times\mathbb{R}^d$, which belongs to $\mathcal{C}$. Moreover, $t\mapsto \mathcal{T}(t)u$ is differentiable at $t=0$ with values in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$ and we have

$$\left(\frac{d}{dt}\mathcal{T}(t)u\right)_{|t=0}(s,x) = -\alpha'(s)P(s,\tau)\chi(x) = \mathcal{G}u(s,x), \quad s\in[a,a+T),$$

which shows that $u\in D(G_p)$ and $G_pu=\mathcal{G}u$.

Let us prove that $\mathcal{C}$ is dense in the domain of G_p. Since $\mathcal{T}(t)$ maps $\mathcal{C}$ into itself for any $t>0$, it is enough to prove that $\mathcal{C}$ is dense in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$.

We recall that the linear span of the functions $(s,x)\mapsto\beta(s)\chi(x)$, with $\beta\in C^1(\mathbb{T})$, $\chi\in C_c^\infty(\mathbb{R}^d)$, is dense in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$. Therefore, it is enough to approximate any product $g=\beta\chi$ of this type by elements of $\mathcal{C}$.

Fix $\beta\in C^1(\mathbb{T})$, $\chi\in C_c^\infty(\mathbb{R}^d)$ and $\varepsilon>0$. Let $\tau\in\mathbb{R}$. Lemma 3.2 of [21] implies that $P(s,\tau)\chi$ tends to χ, uniformly in $\mathbb{R}^d$, as $s\to\tau^+$. Therefore, there exists $\tau'\in(\tau,\tau+T)$ such that $\|P(s,\tau)\chi-\chi\|_\infty\leqslant\varepsilon$, for each $s\in[\tau,\tau']$, which implies

$$\big\|P(s,\tau)\chi-\chi\big\|_{L^p(\mathbb{R}^d,\mu_s)}\leqslant\varepsilon, \quad \tau\leqslant s\leqslant\tau'.$$

Let us cover $\mathbb{T}$ by a finite number of such intervals (*mod* T) (τ_k,τ_k'), $k=1,\dots,K$, and let (α_k) be an associated partition of unity. Setting

$$u_k(s,x)=\beta(s)\alpha_k(s)P(s,\tau_k)\chi(x), \quad s\in[\tau_k,\tau_k+T)$$

and still denoting by u_k its T-periodic extension, the function u defined by

$$u(s,x):=\sum_{k=1}^K u_k(s,x), \quad s\in\mathbb{R},\ x\in\mathbb{R}^d,$$

belongs to $\mathcal{C}$, and we have

$$\big\|g(s,\cdot)-u(s,\cdot)\big\|_{L^p(\mathbb{R}^d,\mu_s)}\leqslant\|\beta\|_\infty\sum_{k=1}^K\alpha_k(s)\big\|P(s,\tau_k)\chi-\chi\big\|_{L^p(\mathbb{R}^d,\mu_s)}\leqslant\varepsilon\|\beta\|_\infty,$$

for any $s\in[0,T]$. Integrating with respect to s in $(0,T)$ we obtain

$$\|g-u\|_p\leqslant\varepsilon\|\beta\|_\infty,$$

and the statement follows. □

Corollary 2.13. *For $1<p<\infty$, $D(G_p)\subset W^{1,2}_{p,\mathrm{loc}}(\mathbb{T}\times\mathbb{R}^d, ds\times dx)$ and for every $r>0$ the restriction mapping $\mathcal{R}: D(G_p)\to W^{1,2}_p(\mathbb{T}\times B(0,r), ds\times dx)$, defined by $\mathcal{R}u=u_{|\mathbb{T}\times B(0,r)}$, is continuous.*

Proof. Every $u \in \mathcal{C}$ belongs to $C^{1,2}(\mathbb{T}\times\mathbb{R}^d)$, and $G_p u(s,x) = \mathcal{A}(s)u(s,x) - D_s u(s,x)$, so that by classical regularity results for parabolic equations in L^p spaces with respect to the Lebesgue measure there exists $C_1 = C_1(r)$ such that

$$\|u\|_{W^{1,2}_p(\mathbb{T}\times B(0,r),ds\times dx)} \leqslant C_1\big(\|u\|_{L^p(\mathbb{T}\times B(0,2r),ds\times dx)} + \|G_p u\|_{L^p(\mathbb{T}\times B(0,2r),ds\times dx)}\big).$$

On the other hand, since $\mu(ds,dx) = \rho(s,x)\,ds\,dx$ for a positive continuous function ρ, there exists $C_2 = C_2(r)$ such that $\|\cdot\|_{L^p(\mathbb{T}\times B(0,2r),ds\times dx)} \leqslant C_2\|\cdot\|_{L^p(\mathbb{T}\times\mathbb{R}^d,\mu)}$. Then, $\mathcal{R}$ is continuous from $\mathcal{C}$ (endowed with the $D(G_p)$-norm) to $W^{1,2}_p(\mathbb{T}\times B(0,r), ds\times dx)$, and since $\mathcal{C}$ is dense in $D(G_p)$ the statement follows. □

The estimates on the space derivatives of $P(t,s)f$ yield a useful embedding result for $D(G_p)$. We denote by $W^{0,1}_p(\mathbb{T}\times\mathbb{R}^d,\mu)$ the set of the functions $f \in L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$ having space derivatives $D_i f$ in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$, for every $i = 1,\dots,d$. It is a Banach space with the norm

$$\|f\|_{W^{0,1}_p(\mathbb{T}\times\mathbb{R}^d,\mu)} = \|f\|_{L^p(\mathbb{T}\times\mathbb{R}^d,\mu)} + \sum_{i=1}^d \|D_i f\|_{L^p(\mathbb{T}\times\mathbb{R}^d,\mu)}.$$

Similarly, we denote by $C^{0,1}_b(\mathbb{T}\times\mathbb{R}^d)$ the set of the functions $f \in C_b(\mathbb{T}\times\mathbb{R}^d)$ having space derivatives $D_i f$ in $C_b(\mathbb{T}\times\mathbb{R}^d)$, for every $i = 1,\dots,d$. It is a Banach space with the norm

$$\|f\|_{C^{0,1}_b(\mathbb{T}\times\mathbb{R}^d)} = \|f\|_\infty + \sum_{i=1}^d \|D_i f\|_\infty.$$

Proposition 2.14. *Assume that Hypotheses* 2.1 *and* 2.4 *are satisfied. Let C_2 be the constant given by Theorem* 2.5*. Then, for every $t > 0$ and $f \in C_b(\mathbb{T}\times\mathbb{R}^d)$ we have*

$$\big\||\nabla_x \mathcal{T}(t)f|\big\|_\infty \leqslant C_2 \max\{t^{-1/2}, 1\}\|f\|_\infty. \tag{2.17}$$

Moreover, $D(G_\infty)$ is continuously embedded into $C^{0,1}_b(\mathbb{T}\times\mathbb{R}^d)$.

If for some $p > 1$ the constant ℓ_p in (2.8) *is finite, let C_3 be the constant in estimate* (2.10)*. Then, for every $f \in L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$,*

$$\big\||\nabla_x \mathcal{T}(t)f|\big\|_p \leqslant \begin{cases} C_3 e^{\ell_p} t^{-1/2}\|f\|_p, & 0 < t \leqslant 1, \\ C_3 \min\{e^{\ell_p t}, e^{\ell_p}\}\|f\|_p, & t \geqslant 1, \end{cases} \tag{2.18}$$

and $D(G_p)$ is continuously embedded into $W^{0,1}_p(\mathbb{T}\times\mathbb{R}^d,\mu)$. Moreover,

$$\big\||\nabla_x \mathcal{T}(t)f|\big\|_p \leqslant e^{\ell_p t}\big\||\nabla_x f|\big\|_p, \quad t > 0,\ f \in W^{0,1}_p\big(\mathbb{T}\times\mathbb{R}^d,\mu\big). \tag{2.19}$$

Proof. Recalling that $\mathcal{T}(t)f(s,x) = P(s,s-t)f(s-t,\cdot)(x)$, estimate (2.17) follows immediately from (2.7) and (2.6).

Let us prove that $D(G_\infty)$ is continuously embedded into $C^{0,1}_b(\mathbb{R}^{1+d})$. $D(G_\infty)$ coincides with the range of the resolvent $R(\lambda, G_\infty)$, for any $\lambda > 0$. Since $R(\lambda,G_\infty)f(s,x) = \int_0^\infty e^{-\lambda t}\mathcal{T}(t)f(s,x)\,dt$ for any $(s,x) \in \mathbb{R}^{1+d}$, estimate (2.17) implies that the derivatives $D_i R(\lambda, G_\infty)f$ are bounded by $C\|f\|_\infty$ for some $C > 0$. Their continuity follows from the continuity of the space derivatives of $\mathcal{T}(t)f$ (Lemma 2.8) through the dominated convergence theorem.

Assume now that $\ell_p < \infty$ and let $f \in L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$. Using (2.13) we obtain

$$\int_{\mathbb{T}\times\mathbb{R}^d} \left|\nabla_x \mathcal{T}(t) f\right|^p d\mu = \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left|\nabla_x P(s,s-t) f(s-t,\cdot)(x)\right|^p \mu_s(dx)\,ds$$

$$\leqslant \frac{(C_3 e^{\ell_p t})^p}{T}\max\{t^{-p/2},1\}\int_0^T \|f(s-t,\cdot)\|^p_{L^p(\mathbb{R}^d,\mu_{s-t})}\,ds$$

$$= \left(C_3 e^{\ell_p t}\right)^p \max\{t^{-p/2},1\}\|f\|^p_{L^p(\mathbb{T}\times\mathbb{R}^d,\mu)}.$$

If $\ell_p > 0$, for $t \geqslant 1$ we use (2.14) instead of (2.13), and we get

$$\left\|\left|\nabla_x \mathcal{T}(t) f\right|\right\|_p \leqslant C_3 e^{\ell_p}\|f\|_p.$$

In any case, (2.18) holds. The embedding $D(G_p) \subset W^{0,1}_p(\mathbb{T}\times\mathbb{R}^d,\mu)$ follows again from the equality $R(\lambda, G_p) f = \int_0^\infty e^{-\lambda t}\mathcal{T}(t) f\,dt$, for any $\lambda > 0$.

Estimate (2.19) follows from Theorem 2.6(i) and from the density of $C^{0,1}_b(\mathbb{T}\times\mathbb{R}^d,\mu)$ in $W^{0,1}_p(\mathbb{T}\times\mathbb{R}^d,\mu)$. □

If some bounds on Q and on $\langle b, x\rangle$ hold, we can prove important integration formulae in $D(G_\infty)$. They yield the embeddings $D(G_p) \subset W^{0,1}_p(\mathbb{T}\times\mathbb{R}^d,\mu)$, even without the assumption $\ell_p < \infty$.

Proposition 2.15. *Assume that Hypotheses* 2.1 *and* 2.4 *are satisfied. Then*:

(a) *if there exists $C > 0$ such that*

$$\|Q(s,x)\|_{\mathcal{L}(\mathbb{R}^d)} \leqslant C\big(|x|+1\big)V(x),$$
$$\big\langle b(s,x),x\big\rangle \leqslant C\big(|x|^2+1\big)V(x),\quad (s,x)\in\mathbb{R}^{1+d}, \tag{2.20}$$

then for every $p \in (1,\infty)$ and $u \in D(G_\infty)$, $|u|^{p-2}\langle Q\nabla_x u, \nabla_x u\rangle \chi_{\{u\neq 0\}}$ belongs to $L^1(\mathbb{T}\times\mathbb{R}^d,\mu)$, and

$$\int_{\mathbb{T}\times\mathbb{R}^d} |u|^{p-2}\langle Q\nabla_x u,\nabla_x u\rangle\chi_{\{u\neq 0\}}\,d\mu \leqslant -\frac{1}{p-1}\int_{\mathbb{T}\times\mathbb{R}^d} u|u|^{p-2} G_p u\,d\mu; \tag{2.21}$$

(b) *if there exists $C > 0$ such that*

$$\|Q(s,x)\|_{\mathcal{L}(\mathbb{R}^d)} \leqslant C\big(|x|+1\big)V(x),$$
$$\big|\big\langle b(s,x),x\big\rangle\big| \leqslant C\big(|x|^2+1\big)V(x),\quad (s,x)\in\mathbb{R}^{1+d}, \tag{2.22}$$

then for $p \geqslant 2$ and $u \in D(G_\infty)$ we have

$$\int_{\mathbb{T}\times\mathbb{R}^d} |u|^{p-2}\langle Q\nabla_x u,\nabla_x u\rangle\chi_{\{u\neq 0\}}\,d\mu = -\frac{1}{p-1}\int_{\mathbb{T}\times\mathbb{R}^d} u|u|^{p-2} G_p u\,d\mu; \tag{2.23}$$

(c) *if the diffusion coefficients q_{ij} are bounded, then for every $p \in (1,\infty)$ and $u \in D(G_\infty)$,* (2.23) *holds.*

Proof. *Step 1:* $p \geqslant 2$. Let $u \in D(G_\infty)$. Then,

$$\mathcal{G}\big(|u|^p\big) = pu|u|^{p-2}\mathcal{G}u + p(p-1)|u|^{p-2}\langle Q\,\nabla_x u, \nabla_x u\rangle. \tag{2.24}$$

Since u, $D_i u$, $\mathcal{G}u$ are bounded, if the diffusion coefficients are bounded $\mathcal{G}(|u|^p)$ is bounded too. Therefore, $|u|^p \in D(G_\infty) \subset D(G_1)$, so that $\int_{\mathbb{T}\times\mathbb{R}^d} \mathcal{G}(|u|^p)\,d\mu = 0$ and (2.23) holds.

Even if the diffusion coefficients are unbounded, we shall show that $\mathcal{G}(|u|^p) \in L^1(\mathbb{T}\times\mathbb{R}^d, \mu)$, and that $\int_{\mathbb{T}\times\mathbb{R}^d} \mathcal{G}(|u|^p)\,d\mu \leqslant 0$ if (2.20) holds, $\int_{\mathbb{T}\times\mathbb{R}^d} \mathcal{G}(|u|^p)\,d\mu = 0$ if (2.22) holds. Note that since $\mathcal{G}(|u|^p)$ is the sum of a bounded function and a positive function, then $\int_{\mathbb{T}\times\mathbb{R}^d} \mathcal{G}(|u|^p)\,d\mu$ is either finite or equal to ∞.

Let $\eta \in C^\infty(\mathbb{R})$ be a nonincreasing function such that $\mathbb{1}_{(-\infty,1]} \leqslant \eta \leqslant \mathbb{1}_{(-\infty,2]}$. For $n \in \mathbb{N}$ define the functions $\theta_n(x) := \eta(|x|/n)$ for any $x \in \mathbb{R}^d$. Let us observe that

$$\int\limits_{\mathbb{T}\times\mathbb{R}^d} \mathcal{G}\big(|u|^p\big)\,d\mu = \lim_{n\to\infty} \int\limits_{\mathbb{T}\times\mathbb{R}^d} \mathcal{G}\big(|u|^p\big)\theta_n\,d\mu. \tag{2.25}$$

Formula (2.25) follows applying the monotone convergence theorem to the positive and increasing sequence $(\theta_n|u|^{p-2}\langle Q\,\nabla_x u, \nabla_x u\rangle)$ and the dominated convergence theorem to the sequence $(p\theta_n u|u|^{p-2}\mathcal{G}u)$.

Let us estimate the integrals $\int_{\mathbb{T}\times\mathbb{R}^d} \mathcal{G}(|u|^p)\theta_n\,d\mu$. For every $n \in \mathbb{N}$, the function $|u|^p\theta_n$ belongs to $\bigcap_{p<\infty} W^{1,2}_{p,\mathrm{loc}}(\mathbb{R}^{1+d}) \cap C_b(\mathbb{T}\times\mathbb{R}^d)$ and

$$\mathcal{G}\big(|u|^p\theta_n\big) = |u|^p\mathcal{G}(\theta_n) + \mathcal{G}\big(|u|^p\big)\theta_n + 2pu|u|^{p-2}\langle Q\,\nabla_x u, \nabla\theta_n\rangle$$

belongs to $C_b(\mathbb{T}\times\mathbb{R}^d)$. Hence, $|u|^p\theta_n \in D(G_\infty)$, so that the mean value of $\mathcal{G}(|u|^p\theta_n)$ vanishes. This means

$$\int\limits_{\mathbb{T}\times\mathbb{R}^d} \mathcal{G}\big(|u|^p\big)\theta_n\,d\mu = \int\limits_{\mathbb{T}\times\mathbb{R}^d} \big(-|u|^p\mathcal{G}(\theta_n) - 2pu|u|^{p-2}\langle Q\,\nabla_x u, \nabla\theta_n\rangle\big)\,d\mu. \tag{2.26}$$

Let us compute $\mathcal{G}(\theta_n)(s,x) = (\mathcal{A}(s)\theta_n)(s,x)$. We have $D_j\theta_n(0) = D_{ij}\theta_n(0) = 0$ and

$$D_j\theta_n(x) = \eta'\left(\frac{|x|}{n}\right)\frac{x_j}{|x|n},$$

$$D_{ij}\theta_n(x) = \eta''\left(\frac{|x|}{n}\right)\frac{x_i x_j}{|x|^2n^2} + \eta'\left(\frac{|x|}{n}\right)\frac{\delta_{ij}}{|x|n} - \eta'\left(\frac{|x|}{n}\right)\frac{x_i x_j}{|x|^3 n},$$

for any $x \in \mathbb{R}^d \setminus \{0\}$ and any $i, j = 1, \ldots, d$. Therefore,

$$\begin{aligned}\big(\mathcal{A}(s)\theta_n\big)(s,x) = {} & \eta''\left(\frac{|x|}{n}\right)\frac{\langle Q\,(s,x)x, x\rangle}{|x|^2n^2} + \eta'\left(\frac{|x|}{n}\right)\frac{\mathrm{Tr}(Q\,(s,x))}{|x|n} \\ & - \eta'\left(\frac{|x|}{n}\right)\frac{\langle Q\,(s,x)x, x\rangle}{|x|^3n} + \eta'\left(\frac{|x|}{n}\right)\frac{\langle b(s,x), x\rangle}{|x|n}.\end{aligned}$$

Since $\eta'(r)$ and $\eta''(r)$ vanish if $r \notin (1,2)$, there exists $C_1 > 0$ such that

$$\left|\eta''\left(\frac{|x|}{n}\right)\frac{\langle Q\,(s,x)x, x\rangle}{|x|^2n^2} + \eta'\left(\frac{|x|}{n}\right)\frac{\mathrm{Tr}(Q\,(s,x))}{|x|n} - \eta'\left(\frac{|x|}{n}\right)\frac{\langle Q\,(s,x)x, x\rangle}{|x|^3n}\right| \leqslant \frac{C_1 V(x)}{n},$$

for any $(s,x) \in \mathbb{R}^{1+d}$. Moreover, $\langle Q\nabla_x u, \nabla\theta_n\rangle$ goes to 0 pointwise as $n \to \infty$, and for each $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$ we have

$$\left|\left\langle Q(s,x)\nabla_x u(s,x), \nabla\theta_n(x)\right\rangle\right| \leqslant C\left(|x|+1\right)V(x)\left\||\nabla_x u|\right\|_\infty \frac{1}{n}\left|\eta'\left(\frac{|x|}{n}\right)\right|$$
$$\leqslant 3C\left\|\eta'\right\|_\infty \||\nabla_x u|\|_\infty V(x).$$

Thus, for $(s,x) \in \mathbb{T} \times (\mathbb{R}^d \setminus \{0\})$ we have

$$-\left(|u|^p\mathcal{G}(\theta_n) + 2pu|u|^{p-2}\langle Q\nabla_x u, \nabla\theta_n\rangle\right)(s,x) = f_n(s,x) - \eta'\left(\frac{|x|}{n}\right)\frac{\langle b(s,x), x\rangle}{|x|n}, \qquad (2.27)$$

where $\lim_{n\to\infty}\|f_n\|_{L^1(\mathbb{T}\times\mathbb{R}^d,\mu)} = 0$. Let us split $\langle b(s,x), x\rangle = \langle b(s,x), x\rangle^+ - \langle b(s,x), x\rangle^-$. Then $\eta'(|x|/n)\frac{\langle b(s,x),x\rangle^-}{|x|n} \leqslant 0$, while $\eta'(|x|/n)\frac{\langle b(s,x),x\rangle^+}{|x|n}$ goes to 0 pointwise as $n \to \infty$, and by (2.20)

$$\left|\eta'\left(\frac{|x|}{n}\right)\frac{\langle b(s,x), x\rangle^+}{|x|n}\right| \leqslant \left|\eta'\left(\frac{|x|}{n}\right)\right|\frac{C(|x|^2+1)V(x)}{|x|n} \leqslant 5C\left\|\eta'\right\|_\infty V(x), \qquad (2.28)$$

for any $(s,x) \in \mathbb{R}^{1+d}$, so that by dominated convergence

$$\lim_{n\to\infty}\int_{\mathbb{T}\times\mathbb{R}^d}\left|\eta'\left(\frac{|x|}{n}\right)\right|\frac{\langle b(s,x), x\rangle^+}{|x|n}\,d\mu = 0. \qquad (2.29)$$

Formulae (2.26) and (2.27) yield, for every $R > 0$,

$$\int_{\mathbb{T}\times\mathbb{R}^d}\mathcal{G}\left(|u|^p\right)\theta_n\,d\mu \leqslant \int_{\mathbb{T}\times\mathbb{R}^d}\left(f_n(s,x) - \eta'\left(\frac{|x|}{n}\right)\frac{\langle b(s,x), x\rangle^+}{|x|n}\right)d\mu, \qquad (2.30)$$

where the right-hand side goes to 0 as $n \to \infty$. Now, taking (2.25) into account, we deduce that $\int_{\mathbb{T}\times\mathbb{R}^d}\mathcal{G}(|u|^p)\,d\mu \leqslant 0$, so that $\mathcal{G}(|u|^p) \in L^1(\mathbb{T}\times\mathbb{R}^d,\mu)$ and (2.21) follows. If in addition (2.22) holds, estimate (2.28) and its consequence (2.29) hold with $\langle b(s,x), x\rangle^+$ replaced by $\langle b(s,x), x\rangle$, so that (2.30) may be replaced by

$$\int_{\mathbb{T}\times\mathbb{R}^d}\mathcal{G}\left(|u|^p\right)\theta_n\,d\mu = \int_{\mathbb{T}\times\mathbb{R}^d}\left(f_n(s,x) - \eta'\left(\frac{|x|}{n}\right)\frac{\langle b(s,x), x\rangle}{|x|n}\right)d\mu$$

and letting $n \to \infty$, (2.25) implies (2.23).

Step 2: $1 < p < 2$. Fix $u \in D(G_\infty)$ and $\delta > 0$. Then, the function $u_\delta := (u^2+\delta)^{\frac{p}{2}} - \delta^{\frac{p}{2}}$ is bounded and continuous in $\mathbb{T}\times\mathbb{R}^d$. A straightforward computation shows that

$$\mathcal{G}u_\delta = pu\left(u^2+\delta\right)^{\frac{p}{2}-1}\mathcal{G}u + p\left(u^2+\delta\right)^{\frac{p}{2}-2}\left[(p-1)u^2+\delta\right]\langle Q\nabla_x u, \nabla_x u\rangle.$$

If (2.20) holds, then

$$\lim_{n\to\infty}\int_{\mathbb{T}\times\mathbb{R}^d}\mathcal{G}(u_\delta)\theta_n\,d\mu = \int_{\mathbb{T}\times\mathbb{R}^d}\mathcal{G}(u_\delta)\,d\mu.$$

Moreover,

$$\mathcal{G}(u_\delta\theta_n) = u_\delta\mathcal{G}(\theta_n) + \mathcal{G}(u_\delta)\theta_n + 2pu\big(u^2+\delta\big)^{\frac{p-2}{2}}\langle Q\,\nabla_x u, \nabla\theta_n\rangle$$

belongs to $C_b(\mathbb{T}\times\mathbb{R}^d)$, and

$$\Big|u\big(u^2+\delta\big)^{\frac{p-2}{2}}\langle Q\,\nabla_x u, \nabla\theta_n\rangle\Big| \leqslant 2C\|\eta'\|_\infty V\big(\|u\|_\infty^2+\delta\big)^{\frac{p-1}{2}}\big\||\nabla_x u|\big\|_\infty.$$

The arguments used in Step 1 show that $\mathcal{G}(|u|^p)\in L^1(\mathbb{T}\times\mathbb{R}^d,\mu)$ and $\int_{\mathbb{T}\times\mathbb{R}^d}\mathcal{G}u_\delta\,d\mu\leqslant 0$, i.e.

$$\int\limits_{\mathbb{T}\times\mathbb{R}^d}\big(u^2+\delta\big)^{\frac{p}{2}-2}\big[(p-1)u^2+\delta\big]\langle Q\,\nabla_x u, \nabla_x u\rangle\,d\mu \leqslant -\int\limits_{\mathbb{T}\times\mathbb{R}^d} u\big(u^2+\delta\big)^{\frac{p}{2}-1}\mathcal{G}u\,d\mu. \tag{2.31}$$

If the diffusion coefficients are bounded, $\mathcal{G}u_\delta$ belongs to $C_b(\mathbb{T}\times\mathbb{R}^d)$, hence $u_\delta\in D(G_\infty)\subset D(G_1)$ and the inequality in (2.31) can be replaced by an equality.

By dominated convergence, the right-hand side of (2.31) converges, as $\delta\to 0$, to the corresponding integral with $\delta=0$. Indeed, for any $\delta>0$, the function $|u|(u^2+\delta)^{\frac{p}{2}-1}|G_p u|$ is bounded from above by $|u|^{p-1}|G_p u|$ since $p<2$, and the μ-a.e. pointwise convergence is obvious. On the other hand, the functions

$$\big(u^2+\delta\big)^{\frac{p}{2}-2}\big[(p-1)u^2+\delta\big]\langle Q\,\nabla_x u, \nabla_x u\rangle$$

converge pointwise a.e. (with respect to the Lebesgue measure) in $\{u\neq 0\}$ to the function $(p-1)|u|^{p-2}\langle Q\,\nabla_x u, \nabla_x u\rangle\chi_{\{u\neq 0\}}$, and $\nabla_x u=0$ a.e. (with respect to the Lebesgue measure) in the set $\{u=0\}$. Since μ is absolutely continuous with respect to the Lebesgue measure, it follows that $(u^2+\delta)^{\frac{p}{2}-2}[(p-1)u^2+\delta]\langle Q\,\nabla_x u, \nabla_x u\rangle$ converges to 0 μ-a.e. in $\{u=0\}$. This shows that $(u^2+\delta)^{\frac{p}{2}-2}[(p-1)u^2+\delta]\langle Q\,\nabla_x u, \nabla_x u\rangle$ converges pointwise to $(p-1)|u|^{p-2}\langle Q\,\nabla_x u, \nabla_x u\rangle\chi_{\{u\neq 0\}}$ μ-a.e. in $\mathbb{T}\times\mathbb{R}^d$. By the Fatou Lemma,

$$\begin{aligned}(p-1)\int\limits_{\{u\neq 0\}}|u|^{p-2}\langle Q\,\nabla_x u, \nabla_x u\rangle\,d\mu &\leqslant \liminf_{\delta\to 0^+}\int\limits_{\mathbb{T}\times\mathbb{R}^d}\big(u^2+\delta\big)^{\frac{p}{2}-2}\big[(p-1)u^2+\delta\big]\langle Q\,\nabla_x u, \nabla_x u\rangle\,d\mu\\ &= -\lim_{\delta\to 0^+}\int\limits_{\mathbb{T}\times\mathbb{R}^d} u\big(u^2+\delta\big)^{\frac{p}{2}-1}G_p u\,d\mu\\ &= -\int\limits_{\mathbb{T}\times\mathbb{R}^d} u|u|^{p-2}G_p u\,d\mu,\end{aligned}$$

and this implies that $|u|^{p-2}\langle Q\,\nabla_x u, \nabla_x u\rangle\chi_{\{u\neq 0\}}$ belongs to $L^1(\mathbb{T}\times\mathbb{R}^d,\mu)$. Now, since

$$\big(u^2+\delta\big)^{\frac{p}{2}-2}\big[(p-1)u^2+\delta\big]\langle Q\,\nabla_x u, \nabla_x u\rangle \leqslant (p-1)|u|^{p-2}\langle Q\,\nabla_x u, \nabla_x u\rangle\chi_{\{u\neq 0\}},$$

we can apply the dominated convergence theorem to both sides of (2.31) (which is an equality if the diffusion coefficients are bounded) and conclude that (2.21) holds if (2.20) is satisfied, and that (2.23) holds if the diffusion coefficients are bounded. □

Corollary 2.16. *Let Hypotheses* 2.1 *and* 2.4 *hold. Then*:

(a) *if the diffusion coefficients are bounded, or if* (2.20) *holds,* $D(G_p) \subset W^{0,1}_p(\mathbb{T}\times\mathbb{R}^d)$ *for each* $p \in (1,\infty)$, *and the mapping* $f \mapsto Q^{1/2}\nabla_x f$ *is continuous from* $D(G_p)$ *into* $(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))^d$ *for* $1 < p \leqslant 2$;
(b) *if the diffusion coefficients are bounded, or if* (2.20) *holds, inequality* (2.21) *holds for every* $p \in (1,\infty)$ *and* $u \in D(G_p)$;
(c) *if the diffusion coefficients are bounded, equality* (2.23) *holds for every* $p \geqslant 2$ *and* $u \in D(G_p)$.

Proof. (a). Let $1 < p \leqslant 2$ and $f \in D(G_\infty)$. Using the Hölder inequality, (2.21) and then the Hölder inequality again, we get

$$\Big(\int_{\mathbb{T}\times\mathbb{R}^d} |Q^{1/2}\nabla_x f|^p\,d\mu\Big)^{\frac{2}{p}} \leqslant \Big(\int_{\mathbb{T}\times\mathbb{R}^d} |f|^p\,d\mu\Big)^{\frac{2}{p}-1}\int_{\mathbb{T}\times\mathbb{R}^d} |f|^{p-2}\big|Q^{1/2}\nabla_x f\big|^2\chi_{\{f\neq 0\}}\,d\mu$$
$$\leqslant \|f\|_p^{2-p}\frac{1}{p-1}\int_{\mathbb{T}\times\mathbb{R}^d}|f|^{p-1}|G_pf|\,d\mu$$
$$\leqslant \frac{1}{p-1}\|f\|_p\|G_pf\|_p.$$

Therefore, $\||Q^{1/2}\nabla_x f|\|_p \leqslant 1/(2\sqrt{p-1})\|f\|_{D(G_p)}$. Since $D(G_\infty)$ is dense in $D(G_p)$, the mapping $f \mapsto Q^{1/2}\nabla_x f$ is bounded from $D(G_p)$ to $(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))^d$ and, since $Q(s,x) \geqslant \eta_0 I$ for each s and x, also the mapping $f \mapsto \nabla_x f$ is bounded from $D(G_p)$ to $(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))^p$, that is $D(G_p) \subset W^{0,1}_p(\mathbb{T}\times\mathbb{R}^d,\mu)$.

If $p > 2$, the embedding follows by interpolation between L^2 and L^∞. Precisely, since for $i = 1,\dots,d$ and $\lambda > 0$ the mappings $f \mapsto D_i\int_0^\infty e^{-\lambda t}\mathcal{T}(t)f\,dt$ are bounded in $L^2(\mathbb{T}\times\mathbb{R}^d,\mu)$ by the above arguments, and in $L^\infty(\mathbb{T}\times\mathbb{R}^d,\mu)$ by Proposition 2.14, they are bounded in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$ for every $p \in (2,\infty)$. On the other hand, $\int_0^\infty e^{-\lambda t}\mathcal{T}(t)f\,dt = R(\lambda,G_p)f$ for every $f \in L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$. Therefore, the range of $R(\lambda,G_p)$, which is the domain of G_p, is continuously embedded in $W^{0,1}_p(\mathbb{T}\times\mathbb{R}^d,\mu)$.

(b). Consider the nonlinear functions on $D(G_p)$ defined by

$$H(u) = |u|^{p-2}\langle Q\nabla_x u, \nabla_x u\rangle\chi_{\{u\neq 0\}}, \qquad K(u) = u|u|^{p-2}G_pu.$$

It is easy to see that K is continuous with values in $L^1(\mathbb{T}\times\mathbb{R}^d,\mu)$. Concerning H, fix $u \in D(G_p)$ and let $(u_n) \subset D(G_\infty)$ converge to u in $D(G_p)$. By statement (a), (u_n) converges to u in $W^{0,1}_p(\mathbb{T}\times\mathbb{R}^d,\mu)$. We may assume (possibly replacing u_n by a suitable subsequence) that u_n, D_iu_n converge, respectively, to u, D_iu pointwise μ-a.e, $i = 1,\dots,d$, so that $H(u_n)$ converges to $H(u)$ pointwise μ-a.e in $\{u \neq 0\}$. For every $n \in \mathbb{N}$, u_n satisfies (2.21) by Proposition 2.15. Letting $n \to \infty$ we get, by the Fatou Lemma,

$$\int_{\mathbb{T}\times\mathbb{R}^d}|u|^{p-2}\langle Q\nabla_x u,\nabla_x u\rangle\chi_{\{u\neq 0\}}\,d\mu \leqslant \liminf_{n\to\infty}\int_{\mathbb{T}\times\mathbb{R}^d}|u_n|^{p-2}\langle Q\nabla_x u_n,\nabla_x u_n\rangle\chi_{\{u_n\neq 0\}}\,d\mu$$
$$\leqslant \lim_{n\to\infty}\int_{\mathbb{T}\times\mathbb{R}^d} u_n|u_n|^{p-2}G_pu_n\,d\mu$$
$$= \int_{\mathbb{T}\times\mathbb{R}^d} u|u|^{p-2}G_pu\,d\mu,$$

that is, (2.21) holds for every $u \in D(G_p)$.

(c). If the diffusion coefficients are bounded, using statement (a) and the Hölder inequality it is easy to see that the function $H : D(G_p) \to L^1(\mathbb{T} \times \mathbb{R}^d, \mu)$ is continuous for $p \geqslant 2$. Since $D(G_\infty)$ is dense in $D(G_p)$ and (2.23) holds for $u \in D(G_\infty)$ by Proposition 2.15, then it holds for $u \in D(G_p)$. □

As in the case of evolution semigroups in fixed Banach spaces, the spectral mapping theorem holds for $\mathcal{T}(t)$. The proof is the same of [18, Prop. 2.1] with obvious changes, and it is omitted.

Theorem 2.17. *Let $1 \leqslant p < \infty$, and denote by $\mathcal{T}_p(t)$ the realization of $\mathcal{T}(t)$ in $L^p(\mathbb{T} \times \mathbb{R}^d, \mu)$. Then we have*

$$\sigma\big(\mathcal{T}_p(t)\big) \setminus \{0\} = e^{t\sigma(G_p)}, \quad t > 0.$$

3. Asymptotic behavior

In this section we prove some asymptotic behavior results for $\mathcal{T}(t)$ that yield asymptotic behavior results for $P(t,s)$.

We introduce a projection Π on functions depending only on time, defined by

$$\Pi f(s,x) := m_s f(s,\cdot) = \int_{\mathbb{R}^d} f(s,y)\mu_s(dy), \quad s \in \mathbb{T},\ x \in \mathbb{R}^d.$$

It is easy to see that $\|\Pi\|_{\mathcal{L}(C_b(\mathbb{T}\times\mathbb{R}^d))} = 1$, and $\|\Pi\|_{\mathcal{L}(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))} = 1$. The ranges of $\Pi(C_b(\mathbb{T} \times \mathbb{R}^d))$ and of $\Pi(L^p(\mathbb{T} \times \mathbb{R}^d, \mu))$ may be identified with $C(\mathbb{T})$ and with $L^p(\mathbb{T}, \frac{ds}{T})$, respectively. $\mathcal{T}(t)$ leaves $C(\mathbb{T})$ and $L^p(\mathbb{T}, \frac{ds}{T})$ invariant, and the part of $\mathcal{T}(t)$ in such spaces is just the translation semigroup $f \mapsto f(\cdot - t)$. Although $\mathcal{T}(t)$ is not strongly continuous in $C_b(\mathbb{T} \times \mathbb{R}^d)$, the part of $\mathcal{T}(t)$ in $C(\mathbb{T})$ is strongly continuous. The infinitesimal generators of the parts of $\mathcal{T}(t)$ in $C(\mathbb{T})$ and in $L^p(\mathbb{T}, \frac{ds}{T})$ have domains (isomorphic to) $C^1(\mathbb{T})$ and $W^{1,p}(\mathbb{T}, \frac{ds}{T})$, respectively, and coincide with $-D_s$.

In the next theorems we relate the asymptotic behavior of $\mathcal{T}(t)$ to the asymptotic behavior of $P(t,s)$.

Theorem 3.1. *Let Hypothesis* 2.1 *hold. For $1 \leqslant p < \infty$, consider the following statements*:

(i) *for each $f \in L^p(\mathbb{T} \times \mathbb{R}^d, \mu)$ we have*

$$\lim_{t\to\infty} \big\|\mathcal{T}(t)(f - \Pi f)\big\|_{L^p(\mathbb{T}\times\mathbb{R}^d,\mu)} = 0; \tag{3.1}$$

(ii) *for each $\varphi \in C_b(\mathbb{R}^d)$ we have*

$$\exists/\forall t \in \mathbb{R}, \quad \lim_{s\to-\infty} \big\|P(t,s)\varphi - m_s\varphi\big\|_{L^p(\mathbb{R}^d,\mu_t)} = 0; \tag{3.2}$$

(iii) *for some/each $s \in \mathbb{R}$ we have*

$$\lim_{t\to\infty} \big\|P(t,s)\varphi - m_s\varphi\big\|_{L^p(\mathbb{R}^d,\mu_t)} = 0, \quad \varphi \in L^p\big(\mathbb{R}^d, \mu_s\big); \tag{3.3}$$

(iv) *for each $\varphi \in C_b(\mathbb{R}^d)$ we have*

$$\exists/\forall t \in \mathbb{R}, \quad \lim_{s\to-\infty} \big\|P(t,s)\varphi - m_s\varphi\big\|_{L^\infty(B(0,R))} = 0, \quad R > 0; \tag{3.4}$$

(v) *for some/each $s \in \mathbb{R}$ we have*

$$\lim_{t\to\infty} \left\| P(t,s)\varphi - m_s\varphi \right\|_{L^\infty(B(0,R))} = 0, \quad \varphi \in C_b\left(\mathbb{R}^d\right),\ R>0. \tag{3.5}$$

For every $p \in [1,\infty)$, statements (i), (ii), (iii) *are equivalent, and they are implied by statements* (iv) *and* (v). *If in addition Hypothesis* 2.4 *holds, for every $p \in [1,\infty)$ statements* (i) *to* (v) *are equivalent.*

Proof. The proof is split in several steps.

Step 1: ∃/∀ parts of statements (ii) to (v). To begin with, let us consider statement (ii). Let $\varphi \in C_b(\mathbb{R}^d)$ and $t_0 \in \mathbb{R}$ be such that $\lim_{s\to-\infty} \|P(t_0,s)\varphi - m_s\varphi\|_{L^p(\mathbb{R}^d,\mu_{t_0})} = 0$. Then, for $t > t_0$, we have $P(t,s)\varphi - m_s\varphi = P(t,t_0)(P(t_0,s)\varphi - m_s\varphi)$ so that $\|P(t,s)\varphi - m_s\varphi\|_{L^p(\mathbb{R}^d,\mu_t)} \leqslant \|P(t_0,s)\varphi - m_s\varphi\|_{L^p(\mathbb{R}^d,\mu_{t_0})}$, which goes to 0 as $s \to -\infty$. For $t < t_0$ fix $k \in \mathbb{N}$ such that $t + kT \geqslant t_0$. Then, $P(t,s)\varphi - m_s\varphi = P(t+kT, s+kT)\varphi - m_{s+kT}\varphi$, and $\mu_t = \mu_{t+kT}$, so that $\|P(t,s)\varphi - m_s\varphi\|_{L^p(\mathbb{R}^d,\mu_t)} = \|P(t+kT, s+kT)\varphi - m_{s+kT}\varphi\|_{L^p(\mathbb{R}^d,\mu_{t+kT})}$ vanishes as $s \to -\infty$ by the first part of the proof.

The same arguments yield the ∃/∀ part of statement (iv). Indeed, let $\varphi \in C_b(\mathbb{R}^d)$ and $t_0 \in \mathbb{R}$ be such that $\lim_{s\to-\infty} \|P(t_0,s)\varphi - m_s\varphi\|_{L^\infty(B(0,R))} = 0$ for each $R > 0$. For $t > t_0$ we have $P(t,s)\varphi - m_s\varphi = P(t,t_0)\varphi_s$, where $\varphi_s := P(t_0,s)\varphi - m_s\varphi$ goes to 0 locally uniformly as $s \to -\infty$. Corollary 2.3(b) yields $\lim_{s\to-\infty} \sup_{t>t_0} \|P(t,t_0)\varphi_s\|_{L^\infty(B(0,R))} = 0$ for each $R > 0$, that is (3.4) holds for $t > t_0$ (even uniformly with respect to t). If $t < t_0$ it is sufficient to fix $k \in \mathbb{N}$ such that $t + kT \geqslant t_0$ and to argue as above.

Concerning statement (iii), if (3.3) holds for $s = s_0$, then it holds for each $s \in \mathbb{R}$. Indeed, for $s < s_0$ and $\varphi \in L^p(\mathbb{R}^d, \mu_s)$ we have $P(t,s)\varphi = P(t,s_0)P(s_0,s)\varphi$ and

$$m_{s_0} P(s_0,s)\varphi = \int_{\mathbb{R}^d} P(s_0,s)\varphi \, d\mu_{s_0} = \int_{\mathbb{R}^d} \varphi \, d\mu_s = m_s\varphi,$$

so that

$$\left\| P(t,s)\varphi - m_s\varphi \right\|_{L^p(\mathbb{R}^d,\mu_t)} = \left\| P(t,s_0)\psi - m_{s_0}\psi \right\|_{L^p(\mathbb{R}^d,\mu_t)},$$

with $\psi = P(s_0,s)\varphi$. Since $\psi \in L^p(\mathbb{R}^d, \mu_{s_0})$, the right-hand side vanishes as $t \to \infty$.

For $s > s_0$ fix $k \in \mathbb{N}$ such that $s - kT < s_0$. Then,

$$\left\| P(t,s)\varphi - m_s\varphi \right\|_{L^p(\mathbb{R}^d,\mu_t)} = \left\| P(t-kT, s-kT)\varphi - m_{s-kT}\varphi \right\|_{L^p(\mathbb{R}^d,\mu_{t-kT})}.$$

Since $s - kT < s_0$, by the first part of the proof the right-hand side vanishes as $t \to \infty$.

The same arguments show that if (3.5) holds for some s_0, then it holds for each $s \in \mathbb{R}$.

Step 2: (i) implies (ii). Let us fix $\varphi \in C_c^\infty(\mathbb{R}^d)$. By Step 1, it is enough to show that $\lim_{s\to\infty} \|P(0,-s)\varphi - m_{-s}\varphi\|_{L^p(\mathbb{R}^d,\mu_0)} = 0$. To this aim we shall prove that, for every sequence $(t_n) \to \infty$, there exists a subsequence (s_n) such that $\lim_{n\to\infty} \|P(0,-s_n)\varphi - m_{-s_n}\varphi\|_{L^p(\mathbb{R}^d,\mu_0)} = 0$.

Set $f(s,x) := \varphi(x)$ for any $(s,x) \in \mathbb{R}^{1+d}$. Then, $f \in C_b(\mathbb{T} \times \mathbb{R}^d)$, and for every $t > 0$, $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$ we have

$$\mathcal{T}(t)(I - \Pi)f(s,x) = P(s,s-t)\varphi(x) - m_{s-t}\varphi.$$

Formula (3.1) implies

$$\lim_{t\to\infty}\int_{-T}^{0}\big\|P(s,s-t)\varphi-m_{s-t}\varphi\big\|^p_{L^p(\mathbb{R}^d,\mu_s)}\,ds=0.$$

Since $P(0,s-t)\varphi(x)-m_{s-t}\varphi=P(0,s)[P(s,s-t)\varphi-m_{s-t}\varphi]$ for $s\in[-T,0]$ and $t\geqslant 0$, and (1.4) holds, then

$$\lim_{t\to\infty}\int_{-T}^{0}\big\|P(0,s-t)\varphi-m_{s-t}\varphi\big\|^p_{L^p(\mathbb{R}^d,\mu_0)}\,ds=0. \tag{3.6}$$

It follows that for every sequence $(t_n)\to\infty$ there exist a subsequence (s_n) and a set $\Gamma\subset[-T,0]$, with negligible complement, such that

$$\lim_{n\to\infty}\big\|P(0,s-s_n)\varphi-m_{s-s_n}\varphi\big\|_{L^p(\mathbb{R}^d,\mu_0)}=0,\quad s\in\Gamma. \tag{3.7}$$

Our aim is to show that $0\in\Gamma$. This follows from the uniform continuity in $(-\infty,0]$ of the $C_b(\mathbb{R}^d)$-valued function $s\mapsto P(0,s)\varphi$ (see Theorem 2.2(ii)) and of the real-valued function $s\mapsto m_s\varphi$ (see Corollary 2.3). Indeed, for each $s\in\Gamma$ we have

$$\begin{aligned}\big\|P(0,-s_n)\varphi-m_{-s_n}\varphi\big\|_{L^p(\mathbb{R}^d,\mu_0)}&\leqslant\big\|P(0,-s_n)\varphi-P(0,s-s_n)\varphi\big\|_{L^p(\mathbb{R}^d,\mu_0)}\\&\quad+\big\|P(0,s-s_n)\varphi-m_{s-s_n}\varphi\big\|_{L^p(\mathbb{R}^d,\mu_0)}+|m_{s-s_n}\varphi-m_{-s_n}\varphi|,\end{aligned}$$

and $\|P(0,-s_n)\varphi-P(0,s-s_n)\varphi\|_{L^p(\mathbb{R}^d,\mu_0)}\leqslant\|P(0,-s_n)\varphi-P(0,s-s_n)\varphi\|_\infty$. Then, for every $\varepsilon>0$ there exists $\delta>0$ such that $\|P(0,-s_n)\varphi-P(0,s-s_n)\varphi\|_{L^p(\mathbb{R}^d,\mu_0)}\leqslant\varepsilon$ and $|m_{s-s_n}\varphi-m_{-s_n}\varphi|\leqslant\varepsilon$, for each $s\in(-\delta,0)$ and $n\in\mathbb{N}$. Fix $s\in\Gamma\cap(-\delta,0)$. By (3.7), $\|P(0,s-s_n)\varphi-m_{s-s_n}\varphi\|_{L^p(\mathbb{R}^d,\mu_0)}\leqslant\varepsilon$ for n large enough, say $n\geqslant n(s,\varepsilon)$. Summing up, $\|P(0,-s_n)\varphi-m_{-s_n}\varphi\|_{L^p(\mathbb{R}^d,\mu_0)}\leqslant 3\varepsilon$ for $n\geqslant n(s,\varepsilon)$. Therefore, $0\in\Gamma$ and (3.2) holds for $\varphi\in C_c^\infty(\mathbb{R}^d)$.

Let us now fix $\varphi\in C_b(\mathbb{R}^d)$, and let (φ_n) be a bounded sequence of test functions that converges to φ locally uniformly. By Corollary 2.3(b), $\lim_{n\to\infty}\sup_{s\in\mathbb{R}}\|\varphi_n-\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}=0$. Since

$$\begin{aligned}\big\|P(t,s)\varphi-m_s\varphi\big\|_{L^p(\mathbb{R}^d,\mu_t)}&\leqslant\big\|P(t,s)(\varphi-\varphi_n)\big\|_{L^p(\mathbb{R}^d,\mu_t)}+\big\|P(t,s)\varphi_n-m_s\varphi_n\big\|_{L^p(\mathbb{R}^d,\mu_t)}\\&\quad+|m_s\varphi_n-m_s\varphi|\\&\leqslant 2\sup_{s\in\mathbb{R}}\|\varphi-\varphi_n\|_{L^p(\mathbb{R}^d,\mu_s)}+\big\|P(t,s)\varphi_n-m_s\varphi_n\big\|_{L^p(\mathbb{R}^d,\mu_t)},\end{aligned} \tag{3.8}$$

for every $n\in\mathbb{N}$, then $\|P(t,s)\varphi-m_s\varphi\|_{L^p(\mathbb{R}^d,\mu_t)}$ goes to 0 as $s\to-\infty$.

Step 3: (i) implies (iii). Let $\varphi\in C_b(\mathbb{R}^d)$. Changing variable in (3.6) we get

$$\lim_{t\to\infty}\int_0^T\int_{\mathbb{R}^d}\big|P(s+t,s)\varphi(x)-m_s\varphi\big|^p\mu_{s+t}(dx)\,ds=0,$$

so that there exists a sequence $(t_n)\to\infty$ such that, for almost every $s\in(0,T)$ and by periodicity for almost every $s\in\mathbb{R}$, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\big|P(s+t_n,s)\varphi(x)-m_s\varphi\big|^p\mu_{s+t_n}(dx)=0. \tag{3.9}$$

Let Γ' be the set of all $s \in \mathbb{R}$ such that (3.9) holds. For $s \in \Gamma'$ and for $t \in [t_n, t_{n+1})$ we have

$$P(s+t,s)\varphi - m_s\varphi = P(s+t,s+t_n)\big[P(s+t_n,s)\varphi - m_s\varphi\big],$$

so that, from (1.4),

$$\big\|P(s+t,s)\varphi - m_s\varphi\big\|_{L^p(\mathbb{R}^d,\mu_{s+t})} \leqslant \big\|P(s+t_n,s)\varphi - m_s\varphi\big\|_{L^p(\mathbb{R}^d,\mu_{s+t_n})}.$$

Hence, $\lim_{t\to\infty}\|P(t,s)\varphi - m_s\varphi\|_{L^p(\mathbb{R}^d,\mu_t)} = 0$ for $s \in \Gamma'$. To prove that the limit is zero for every $s \in \mathbb{R}$, we argue as at the end of Step 2, replacing φ_n by $P(s,r_n)\varphi$, with $r_n \in \Gamma'$, $r_n \uparrow s$ as $n \to \infty$. Indeed, by Theorem 2.2(ii), $P(s,r_n)\varphi$ converges to φ locally uniformly. Estimates (3.8) imply the statement.

If $\varphi \in L^p(\mathbb{R}^d,\mu_s)$, (3.3) follows approaching φ by a sequence of functions in $C_b(\mathbb{R}^d)$ and recalling that $P(t,s)$ and m_s are contractions from $L^p(\mathbb{R}^d,\mu_s)$ to $L^p(\mathbb{R}^d,\mu_t)$. So, statement (iii) holds.

Step 4: if Hypothesis 2.4 holds, (ii) and (iii) imply (iv) and (v), respectively. Let $\varphi \in C_b(\mathbb{R}^d)$. Then, for every $t \in \mathbb{R}$, the functions $P(t,s)\varphi - m_s\varphi$ ($s \leqslant t-1$) are equibounded and equicontinuous by estimate (2.7). By the Arzelà–Ascoli Theorem, for every $R > 0$ there exist a sequence $(s_n) \to -\infty$ and a function $g \in C_b(B(0,R))$ such that $\lim_{n\to\infty}\|P(t,s_n)\varphi - m_{s_n}\varphi - g\|_{L^\infty(B(0,R))} = 0$.

Let ρ be the continuous positive version of the density of μ with respect to the Lebesgue measure. Then, $\mu_s = \rho(s,x)dx$ for any $s \in \mathbb{R}$, by the remark after the proof of Proposition 2.10. For each $n \in \mathbb{N}$ we have

$$\begin{aligned}\inf_{\mathbb{R}\times B(0,R)}\rho \int_{B(0,R)} \big|P(t,s_n)\varphi(x) - m_{s_n}\varphi\big|\,dx &\leqslant \int_{B(0,R)} \big|P(t,s_n)\varphi(x) - m_{s_n}\varphi\big|\rho(t,x)\,dx\\ &= \int_{B(0,R)} \big|P(t,s_n)\varphi(x) - m_{s_n}\varphi\big|\,d\mu_t\\ &\leqslant \bigg(\int_{\mathbb{R}^d}\big|P(t,s_n)\varphi(x) - m_{s_n}\varphi\big|^p\,d\mu_t\bigg)^{\frac{1}{p}}.\end{aligned}$$

If (ii) holds, the last term vanishes as $n \to \infty$. Therefore, $\int_{B(0,R)}|g(x)|\,dx = 0$, so that $g \equiv 0$ and (v) holds.

The proof that (iii) implies (v) is the same.

Step 5: (ii), (iii), (iv), (v) imply (i). Let $f(s,x) = \alpha(s)\varphi(x)$, with $\alpha \in C(\mathbb{T})$ and $\varphi \in C_b(\mathbb{R}^d)$. Then $\mathcal{T}(t)(f - \Pi f)(s,x) = \alpha(s-t)(P(s,s-t)\varphi(x) - m_{s-t}\varphi)$, so that

$$\big\|\mathcal{T}(t)(f-\Pi f)\big\|_p^p = \frac{1}{T}\int_0^T\big|\alpha(s-t)\big|^p\int_{\mathbb{R}^d}\big|P(s,s-t)\varphi(x) - m_{s-t}\varphi\big|^p\mu_s(dx)\,ds. \tag{3.10}$$

If (ii) holds, then $\lim_{t\to\infty}\int_{\mathbb{R}^d}|P(s,s-t)\varphi(x) - m_{s-t}\varphi|^p\mu_s(dx) = 0$ for each $s \in (0,T)$. Moreover, $\int_{\mathbb{R}^d}|P(s,s-t)\varphi(x) - m_{s-t}\varphi|^p\mu_s(dx) \leqslant (2\|\varphi\|_\infty)^p$, for each $s \in (0,T)$. If (iv) holds, $|\alpha(s-t)(P(s,s-t)\varphi(x) - m_{s-t}\varphi)|^p$ goes to zero pointwise, and it does not exceed $(2\|\alpha\|_\infty\|\varphi\|_\infty)^p$, for each $s \in (0,T)$. In both cases, by dominated convergence $\lim_{t\to\infty}\|\mathcal{T}(t)(f-\Pi f)\|_{L^p(\mathbb{T}\times\mathbb{R}^d,\mu)} = 0$.

If (iii) or (v) holds, let us rewrite (3.10) as

$$\big\|\mathcal{T}(t)(f-\Pi f)\big\|_{L^p(\mathbb{T}\times\mathbb{R}^d,\mu)}^p = \frac{1}{T}\int_0^T\big|\alpha(s)\big|^p\int_{\mathbb{R}^d}\big|P(s+t,s)\varphi(x) - m_s\varphi\big|^p\mu_{s+t}(dx)\,ds.$$

Then, $\lim_{t\to\infty}\int_{\mathbb{R}^d}|P(s+t,s)\varphi(x)-m_s\varphi|^p\mu_{s+t}(dx)=0$ for each $s\in(0,T)$ if (iii) or (v) hold. If (iii) holds, this is immediate. If (v) holds, it is sufficient to use the uniform convergence of $|P(s+t,s)\varphi-m_s\varphi|^p$ to zero as $t\to\infty$, on each ball $B(0,R)$ and Corollary 2.3. In both cases, we have again $\int_{\mathbb{R}^d}|P(s+t,s)\varphi(x)-m_s\varphi|^p\mu_{s+t}(dx)\leqslant(2\|\varphi\|_\infty)^p$, for each $s\in(0,T)$. By dominated convergence, $\lim_{t\to\infty}\|\mathcal{T}(t)(f-\Pi f)\|_{L^p(\mathbb{T}\times\mathbb{R}^d,\mu)}=0$.

Since the linear span of the functions $f(s,x)=\alpha(s)\varphi(x)$, with $\alpha\in C(\mathbb{T})$ and $\varphi\in C_b(\mathbb{R}^d)$, is dense in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$, (i) follows. □

Theorem 3.2. *Let Hypothesis* 2.1 *hold. Fix* $1\leqslant p\leqslant\infty$, $M>0$, $\omega\in\mathbb{R}$. *The following conditions are equivalent*:

(a) *for every* $t>0$ *and* $u\in L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$,

$$\left\|\mathcal{T}(t)(I-\Pi)u\right\|_p\leqslant Me^{\omega t}\|u\|_p,\quad t>0,\ u\in L^p\big(\mathbb{T}\times\mathbb{R}^d,\mu\big);$$

(b) *for every* $t>s$ *and* $\varphi\in L^p(\mathbb{R}^d,\mu_s)$,

$$\left\|P(t,s)\varphi-m_s\varphi\right\|_{L^p(\mathbb{R}^d,\mu_t)}\leqslant Me^{\omega(t-s)}\|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)},\quad t>s,\ \varphi\in L^p\big(\mathbb{R}^d,\mu_s\big).$$

Proof. For $p=\infty$ the equivalence is immediate.

The proof that (a) $\Rightarrow$ (b) for $p<\infty$ is quite similar to the proof of Step 2 of [18, Thm. 2.17], that concerns $p=2$ and backward Ornstein–Uhlenbeck evolution operators. In our periodic case we do not need the localization function ξ of [18], it is sufficient to define $u(s,\cdot)=\varphi$ for every s. We omit the details of the proof, leaving them to the reader.

Still for $p<\infty$, (b) $\Rightarrow$ (a) is easy. For, if (b) holds, then for $s\in\mathbb{R}$, $t>0$, and $u\in L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$, we have

$$\int_{\mathbb{R}^d}\left|P(s,s-t)u(s-t,\cdot)-m_{s-t}u(s-t,\cdot)\right|^p d\mu_s\leqslant M^pe^{\omega pt}\int_{\mathbb{R}^d}\left|u(s-t,\cdot)\right|^p d\mu_{s-t},$$

and integrating over $[0,T]$ we obtain

$$\begin{aligned}\int_{\mathbb{T}\times\mathbb{R}^d}\left|\mathcal{T}(t)(I-\Pi)u\right|^p d\mu&\leqslant M^pe^{\omega pt}\frac{1}{T}\int_0^T\int_{\mathbb{R}^d}\left|u(s-t,\cdot)\right|^p d\mu_{s-t}\,ds\\&=M^pe^{\omega pt}\frac{1}{T}\int_0^T\int_{\mathbb{R}^d}\left|u(\tau,\cdot)\right|^p d\mu_\tau\,d\tau\\&=M^pe^{\omega pt}\|u\|_p^p.\end{aligned}$$ □

Remark 3.3. It is also possible to relate the asymptotic behavior of $\nabla_x\mathcal{T}(t)$ to the asymptotic behavior of $\nabla_xP(t,s)$. Namely, some of the results of Theorems 3.1 and 3.2 hold, with $|\nabla_x\mathcal{T}(t)u|$ and $|\nabla_xP(t,s)\varphi|$ replacing $\mathcal{T}(t)(I-\Pi)u$ and $P(t,s)\varphi-m_s\varphi$, respectively. The details are left to the reader.

In view of Theorems 3.1 and 3.2, we study the decay to zero of $\mathcal{T}(t)(I-\Pi)$. The starting point is the decay of $|\nabla_x\mathcal{T}(t)f|$ as $t\to\infty$, for every $f\in L^2(\mathbb{T}\times\mathbb{R}^d,\mu)$. Since everything relies on formula (2.21), we need that the assumptions of Proposition 2.15 hold. The proof of the following proposition is an extension to the evolution semigroup of a similar proof for Markov semigroups generated by elliptic operators (see e.g., [6]).

Proposition 3.4. *Let Hypotheses* 2.1 *and* 2.4 *hold. If the diffusion coefficients are bounded, or if* (2.20) *is satisfied, then for every* $f \in D(G_2)$ *we have*

$$\lim_{t\to\infty} \big\| \big|\nabla_x \mathcal{T}(t) f\big| \big\|_2 = 0. \tag{3.11}$$

If moreover the constant ℓ_2 *in* (2.8) *is finite* (*which is always the case if the diffusion coefficients are bounded*), *then* (3.11) *holds for every* $f \in L^2(\mathbb{T} \times \mathbb{R}^d, \mu)$.

Proof. Let $f \in D(G_2)$. From the equality

$$\frac{d}{dt}\big\|\mathcal{T}(t) f\big\|_2^2 = 2\big\langle \mathcal{T}(t) f, G_2 \mathcal{T}(t) f\big\rangle_{L^2(\mathbb{T}\times\mathbb{R}^d,\mu)}, \quad t > 0,$$

we obtain

$$\big\|\mathcal{T}(t) f\big\|_2^2 - \|f\|_2^2 = 2\int_0^t \int_{\mathbb{T}\times\mathbb{R}^d} \mathcal{T}(s) f\, G_2 \mathcal{T}(s) f\, d\mu\, ds, \quad t > 0,$$

and using (2.21), that holds for the functions in $D(G_2)$ by Corollary 2.16(b), we get

$$\big\|\mathcal{T}(t) f\big\|_2^2 + 2\int_0^t \int_{\mathbb{T}\times\mathbb{R}^d} \big\langle Q\, \nabla_x \mathcal{T}(s) f, \nabla_x \mathcal{T}(s) f\big\rangle d\mu\, ds \leqslant \|f\|_2^2, \quad t > 0. \tag{3.12}$$

Therefore, the function

$$\chi_f(s) := \int_{\mathbb{T}\times\mathbb{R}^d} \big|\nabla_x \mathcal{T}(s) f\big|^2 d\mu, \quad s \geqslant 0,$$

is in $L^1(0,\infty)$, and its L^1-norm does not exceed $\|f\|_2^2/\eta_0$. Its derivative is

$$\chi_f'(s) = \int_{\mathbb{T}\times\mathbb{R}^d} 2\big\langle \nabla_x \mathcal{T}(s) f, \nabla_x \mathcal{T}(s) G_2 f\big\rangle d\mu$$

so that, if $f \in D((G_2)^2)$,

$$\big|\chi_f'(s)\big| \leqslant 2\bigg(\int_{\mathbb{T}\times\mathbb{R}^d} \big|\nabla_x \mathcal{T}(s) f\big|^2 d\mu\bigg)^{\frac{1}{2}} \bigg(\int_{\mathbb{T}\times\mathbb{R}^d} \big|\nabla_x \mathcal{T}(s) G_2 f\big|^2 d\mu\bigg)^{\frac{1}{2}} \leqslant \chi_f(s) + \chi_{G_2 f}(s),$$

for any $s \geqslant 0$. Therefore, also χ_f' is in $L^1(0,\infty)$. This implies that $\lim_{s\to\infty} \chi_f(s) = 0$, and (3.11) holds for every $f \in D((G_2)^2)$. For $f \in D(G_2)$, (3.11) follows approaching f by a sequence of functions in $D((G_2)^2)$, which is dense in $(D(G_2), \|\cdot\|_{D(G_2)})$, and using Corollary 2.16(a), which implies that $\nabla_x \mathcal{T}(\cdot)$ is bounded in $(0,\infty)$ with values in $\mathcal{L}(D(G_2), (L^2(\mathbb{T} \times \mathbb{R}^d, \mu))^d)$.

The last assertion follows again by density, approaching any $f \in L^2(\mathbb{T} \times \mathbb{R}^d, \mu)$ by a sequence of functions in $D((G_2)^2)$ and using estimate (2.18) with $p = 2$. □

Theorem 3.5. *Let Hypotheses* 2.1 *and* 2.4 *hold. Further, assume that the diffusion coefficients are bounded, or that* (2.20) *is satisfied. Then, for every* $p \in [1, \infty)$

$$\lim_{t\to\infty} \left\| \mathcal{T}(t)(I - \Pi) f \right\|_p = 0, \quad f \in L^p\left(\mathbb{T} \times \mathbb{R}^d, \mu\right). \tag{3.13}$$

Therefore, statements (ii) *to* (v) *of Theorem* 3.1 *hold.*

Proof. Let $f \in \mathcal{C} \subset D(G_2)$. Then, f is a linear combination of functions $u_{\tau,\chi,\alpha}$, defined before Proposition 2.12.

Let us prove that, for each $u = u_{\tau,\chi,\alpha}$, the set of functions $\{\mathcal{T}(t)(I - \Pi)u\colon t > 0\}$ is equicontinuous and equibounded in $\mathbb{R} \times B(0, R)$, for each $R > 0$.

Since $\Pi u(s, x) = \alpha(s) m_\tau \chi$, then $\mathcal{T}(t)\Pi u(s, x) = \alpha(s - t) m_\tau \chi$ is equicontinuous and equibounded. Concerning $\mathcal{T}(t)u$, we recall that it is the time periodic extension of the function $(s, x) \mapsto \alpha(s - t)P(s, \tau)\chi(x)$ defined for $s \in [a + t, a + t + T)$, $x \in \mathbb{R}^d$, if the support of α is contained in $(a, a + T)$ with $a \geqslant \tau$. We have to prove only equicontinuity, since $\|\mathcal{T}(t)u\|_\infty \leqslant \|\alpha\|_\infty \|\chi\|_\infty$. By Theorem 2.5, $\| |\nabla_x \alpha(s - t)P(s, \tau)\chi| \|_{L^\infty(\mathbb{R}^d)} \leqslant C_1 \|\alpha\|_\infty \|\chi\|_{C^1_b(\mathbb{R}^d)}$, so that $\mathcal{T}(t)u$ is equi-Lipschitz continuous in x. To prove that it is equi-Lipschitz continuous in s we show preliminarily that, for every $R > 0$,

$$\sup_{s\geqslant\tau,\, |x|\leqslant R} \left|\mathcal{A}(s)P(s, \tau)\chi(x)\right| < \infty. \tag{3.14}$$

From the proof of [21, Thm. 2.2] we know that the function $(s, x) \mapsto P(s, \tau)\chi(x)$ belongs to $C^{1+\alpha/2,2+\alpha}_{\text{loc}}([\tau, \infty) \times \mathbb{R}^d)$ and, therefore,

$$\sup_{\tau\leqslant s\leqslant\tau+2T,\, |x|\leqslant R} \left|\mathcal{A}(s)P(s, \tau)\chi(x)\right| < \infty.$$

If $s \in (\tau + kT, \tau + (k + 1)T]$ with $k \geqslant 2$ we write

$$P(s, \tau)\chi = P\big(s, \tau + (k - 1)T\big)P\big(\tau + (k - 1)T, \tau\big)\chi := P(\sigma, \tau)\varphi,$$

with $\sigma = s - (k - 1)T \in (\tau + T, \tau + 2T]$, $\varphi = P(\tau + (k - 1)T, \tau)\chi \in C_b(\mathbb{R}^d)$, $\|\varphi\|_\infty \leqslant \|\chi\|_\infty$. By Theorem 2.2(i),

$$\sup\big\{\left|\mathcal{A}(\sigma)P(\sigma, \tau)\varphi\right|\colon \tau + T \leqslant \sigma < \tau + 2T,\ |x| \leqslant R\big\} \leqslant C(R)\|\varphi\|_\infty,$$

and (3.14) follows.

From the equality

$$D_s\mathcal{T}(t)u(s, \cdot) = \alpha'(s - t)P(s, \tau)\chi + \alpha(s - t)\mathcal{A}(s)P(s, \tau)\chi, \quad s \in [a + t, a + t + T),$$

using (3.14) we obtain that $D_s\mathcal{T}(t)u$ is bounded in $[a + t, a + t + T) \times B(0, R)$. Since it is periodic in s, it is bounded in $\mathbb{R} \times B(0, R)$.

Therefore, for each $f \in \mathcal{C}$ the set of functions $\{\mathcal{T}(t) f\colon t > 0\}$ is equicontinuous and equibounded in $\mathbb{R} \times B(0, R)$, for each $R > 0$. By the Arzelà–Ascoli Theorem and the usual diagonal procedure, there exist a sequence $t_n \to \infty$ and a function $g \in C_b(\mathbb{T} \times \mathbb{R}^d)$ such that $\mathcal{T}(t_n)(I - \Pi) f$ converges to g uniformly on $\mathbb{T} \times B(0, R)$, for each $R > 0$. Since $\|\mathcal{T}(t_n)(I - \Pi) f\|_\infty \leqslant \|f\|_\infty$, by dominated convergence $\mathcal{T}(t_n)(I - \Pi) f$ converges to g in $L^p(\mathbb{T} \times \mathbb{R}^d, \mu)$, for every $p \in [1, \infty)$.

Let us prove that $g \equiv 0$. We have $\lim_{n\to\infty} \|\mathcal{T}(t_n)(I - \Pi) f - g\|_2 = 0$, moreover, by Proposition 3.4, $\lim_{n\to\infty} \| |\nabla_x \mathcal{T}(t_n)(I - \Pi) f| \|_2 = 0$. Since the density ρ of μ with respect to the Lebesgue measure is positive, the space derivatives are closed operators in $L^2(\mathbb{T} \times \mathbb{R}^d, \mu)$. This implies that $g \in$

$W_2^{0,1}(\mathbb{T}\times\mathbb{R}^d,\mu)$ has null space derivatives, so that it depends only on s. On the other hand, $g\in(I-\Pi)(L^2(\mathbb{T}\times\mathbb{R}^d,\mu))$ because it is the limit of the sequence $(\mathcal{T}(t_n)(I-\Pi)f)$ that has values in $(I-\Pi)(L^2(\mathbb{T}\times\mathbb{R}^d,\mu))$. If a function in $(I-\Pi)(L^2(\mathbb{T}\times\mathbb{R}^d,\mu))$ is independent of the space variables, it vanishes. Therefore, $g\equiv 0$. Since the only possible limit g is zero, then $\lim_{t\to\infty}\|\mathcal{T}(t)(I-\Pi)f\|_p=0$, for every $p\in[1,\infty)$.

Since $\mathcal{C}$ is dense in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$, by Proposition 2.12, and $\|\mathcal{T}(t)(I-\Pi)\|_{\mathcal{L}(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))}\leqslant 1$, (3.13) follows. Theorem 3.1 yields the other statements. □

For $\varphi\in C_b(\mathbb{R}^d)$, the convergence of $P(t,s)\varphi-m_s\varphi$ to 0 is not uniform in $\mathbb{R}^d$ in general, even in the autonomous case. Take for instance any Ornstein–Uhlenbeck operator $\mathcal{A}$,

$$\mathcal{A}\varphi=\frac{1}{2}\operatorname{Tr}\big(QD^2\varphi\big)+\langle Bx,\nabla\varphi\rangle,$$

where Q is symmetric and positive definite and all the eigenvalues of B have negative real part. Then, the Ornstein–Uhlenbeck semigroup $T(t)$ has a unique invariant measure μ, which is the Gaussian measure with zero mean and covariance operator $Q_\infty:=\int_0^\infty e^{sB}Qe^{sB^*}\,ds$. We have $P(t,s)=T(t-s)$ and $\mu_t=\mu$ for every $t\in\mathbb{R}$.

Take an exponential function $g=e^{i\langle\cdot,h\rangle}$ ($h\in\mathbb{R}^d\setminus\{0\}$). Then

$$T(t)g=\exp\biggl(-\frac{1}{2}\langle Q_th,h\rangle+i\bigl\langle\cdot,e^{tB^*}h\bigr\rangle\biggr),\quad t\geqslant 0,$$

where $Q_t:=\int_0^t e^{sB}Qe^{sB^*}\,ds$. A simple computation shows that $\int_{\mathbb{R}^d}g\,d\mu=e^{-\langle Q_\infty h,h\rangle/2}$. Therefore,

$$\begin{aligned}T(t)g-\int\limits_{\mathbb{R}^d}g\,d\mu&=\biggl\{\exp\biggl(-\frac{1}{2}\langle Q_th,h\rangle\biggr)-\exp\biggl(-\frac{1}{2}\langle Q_\infty h,h\rangle\biggr)\biggr\}e^{i\langle\cdot,e^{tB^*}h\rangle}\\&\quad+\exp\biggl(-\frac{1}{2}\langle Q_\infty h,h\rangle\biggr)\bigl(\exp\bigl(i\bigl\langle\cdot,e^{tB^*}h\bigr\rangle\bigr)-1\bigr),\end{aligned}$$

for any $t>0$. The sup norm of the first addendum in the right-hand side vanishes as $t\to\infty$ but the second one does not, since, for any $t>0$, $\sup_{x\in\mathbb{R}^d}|\exp(i\langle x,e^{tB^*}h\rangle)-1|=\sup_{\theta\in\mathbb{R}}|\exp(i\theta)-1|=2$.

Concerning exponential rates of convergence, for every $p\in[1,\infty)$ let us define the right halflines

$$\begin{aligned}A_p&:=\bigl\{\omega\in\mathbb{R}\colon\ \exists M_\omega>0\text{ s.t. }\bigl\|\mathcal{T}(t)(f-\Pi f)\bigr\|_p\leqslant M_\omega e^{\omega t}\|f-\Pi f\|_p\text{ for any }t\geqslant 0,\\&\qquad f\in L^p\bigl(\mathbb{T}\times\mathbb{R}^d,\mu\bigr)\bigr\},\\B_p&:=\bigl\{\omega\in\mathbb{R}\colon\ \exists N_\omega>0\text{ s.t. }\bigl\|\bigl|\nabla_x\mathcal{T}(t)f\bigr|\bigr\|_p\leqslant N_\omega e^{\omega t}\|f\|_p\text{ for any }t\geqslant 1,\ f\in L^p\bigl(\mathbb{T}\times\mathbb{R}^d,\mu\bigr)\bigr\},\end{aligned}$$

and their infima

$$\omega_p:=\inf A_p,\qquad\gamma_p:=\inf B_p.\tag{3.15}$$

Then, $\omega_p\leqslant 0$ is the growth bound of the part of $\mathcal{T}(t)$ in $(I-\Pi)(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))$. We recall that if $\ell_p<\infty$, then $\ell_p\in B_p$ by Proposition 2.14, hence $\gamma_p\leqslant\min\{\ell_p,0\}$.

Theorem 3.6. *Let Hypotheses* 2.1 *and* 2.4 *hold. Then $A_p\subset B_p$ for every $p\in(1,\infty)$ such that $\ell_p<\infty$. If the diffusion coefficients are bounded, $B_p\subset A_p$ for every $p\geqslant 2$.*

Proof. Let $\ell_p < \infty$. By Proposition 2.14, $\mathcal{T}(t)$ maps $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$ into $W^{0,1}_p(\mathbb{T}\times\mathbb{R}^d,\mu)$ for every $t>0$.

Fix $f\in L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$ and $\omega\in A_p$. Since Πf is independent of x, $\nabla_x\mathcal{T}(t)f=\nabla_x\mathcal{T}(t)(f-\Pi f)$. Taking (2.18) into account, for $t>1$ we estimate

$$\begin{aligned}\big\|\nabla_x\mathcal{T}(t)(f-\Pi f)\big\|_p &= \big\|\nabla_x\mathcal{T}(1)\big(\mathcal{T}(t-1)(f-\Pi f)\big)\big\|_p\\ &\leqslant C_3e^{\ell_p}\big\|\mathcal{T}(t-1)(f-\Pi f)\big\|_p\\ &\leqslant C_3e^{\ell_p}M_\omega e^{\omega(t-1)}\|f-\Pi f\|_p,\end{aligned}$$

so that $\omega\in B_p$, and the first part of the statement is proved.

If the diffusion coefficients are bounded, set

$$\Lambda := \sup\big\{\big\langle Q(s,x)\xi,\xi\big\rangle:\ s\in\mathbb{T},\ x\in\mathbb{R}^d,\ \xi\in\mathbb{R}^d,\ |\xi|=1\big\}. \tag{3.16}$$

Since $A_p\supset[0,\infty)$, if $B_p\subset[0,\infty)$ the inclusion $B_p\subset A_p$ is obvious. So, we may assume that $B_p\cap(-\infty,0)\neq\emptyset$.

Fix $f\in(I-\Pi)(D(G_p))$ and $\omega\in B_p$, $\omega<0$. Then,

$$\frac{d}{dt}\int\limits_{\mathbb{T}\times\mathbb{R}^d}\big|\mathcal{T}(t)f\big|^p\,d\mu = p\int\limits_{\mathbb{T}\times\mathbb{R}^d}\big|\mathcal{T}(t)f\big|^{p-2}\mathcal{T}(t)f\,G_p\mathcal{T}(t)f\,d\mu,$$

so that, by (2.23) and the Hölder inequality,

$$\begin{aligned}\frac{d}{dt}\int\limits_{\mathbb{T}\times\mathbb{R}^d}\big|\mathcal{T}(t)f\big|^p\,d\mu &= -p(p-1)\int\limits_{\mathbb{T}\times\mathbb{R}^d}\big|\mathcal{T}(t)f\big|^{p-2}\big\langle Q\,\nabla_x\mathcal{T}(t)f,\nabla_x\mathcal{T}(t)f\big\rangle d\mu\\ &\geqslant -p(p-1)\Lambda\big\|\mathcal{T}(t)f\big\|_p^{p-2}\big\|\big|\nabla_x\mathcal{T}(t)f\big|\big\|_p^2\\ &\geqslant -p(p-1)\Lambda\big\|\mathcal{T}(t)f\big\|_p^{p-2}N_\omega^2e^{2\omega t}\|f\|_p^2.\end{aligned}$$

Therefore, the function

$$\beta(t) := \big\|\mathcal{T}(t)f\big\|_p^2 = \bigg(\int\limits_{\mathbb{T}\times\mathbb{R}^d}\big|\mathcal{T}(t)f\big|^p\,d\mu\bigg)^{\frac{2}{p}},\quad t\geqslant 1,$$

either vanishes in a halfline, or is strictly positive in $[1,\infty)$ and satisfies

$$\beta'(t) = \frac{2}{p}\big\|\mathcal{T}(t)f\big\|_p^{2-p}\frac{d}{dt}\int\limits_{\mathbb{T}\times\mathbb{R}^d}\big|\mathcal{T}(t)f\big|^p\,d\mu \geqslant -2(p-1)\Lambda N_\omega^2e^{2\omega t}\|f\|_p^2.$$

Since $\lim_{t\to\infty}\beta(t)=0$ by Theorem 3.5, then

$$\beta(t) = -\int_t^\infty\beta'(s)\,ds \leqslant 2(p-1)\Lambda N_\omega^2\|f\|_p^2\int_t^\infty e^{2\omega s}\,ds = \frac{(p-1)\Lambda N_\omega^2}{|\omega|}e^{2\omega t}\|f\|_p^2,$$

for any $t \geqslant 1$, that is,

$$\left\|\mathcal{T}(t)f\right\|_p^2 \leqslant \widetilde{M}_p e^{2\omega t}\|f\|_p^2, \quad t \geqslant 1.$$

Since $(I-\Pi)(D(G_p))$ is dense in $(I-\Pi)(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))$, the above estimate holds for any $f \in (I-\Pi)(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))$, and this implies that $\omega \in A_p$. It follows that $B_p \subset A_p$. □

The second part of the proof of Theorem 3.6 may be easily adapted to the case of unbounded diffusion coefficients, and it yields, for $p \geqslant 2$,

$$\left\|\mathcal{T}(t)f\right\|_p^2 \leqslant Ce^{2\omega t}\|f\|_p^2, \quad f \in (I-\Pi)\big(L^p\big(\mathbb{T}\times\mathbb{R}^d,\mu\big)\big),\ t \geqslant 1,$$

for every $\omega < 0$ such that

$$\exists M:\ \left\|\left\langle Q\,\nabla_x\mathcal{T}(t)f, \nabla_x\mathcal{T}(t)f\right\rangle\right\|_p \leqslant Me^{\omega t}\|f\|_p, \quad f \in L^p\big(\mathbb{T}\times\mathbb{R}^d,\mu\big),\ t \geqslant 1. \tag{3.17}$$

But at the moment we are not able to give any sufficient conditions for (3.17) to hold, while a sufficient condition for $\gamma_p < 0$ is $\ell_2 < 0$, by estimate (2.18).

Theorem 3.6 has two important consequences. The first one is about the spectral gap of G_p and the solvability of the equation $\lambda u - G_p u = f$; the second one is about the asymptotic behavior of the evolution operator $P(t,s)$.

Corollary 3.7. *Let Hypotheses* 2.1 *and* 2.4 *hold. Assume that the diffusion coefficients are bounded and that* $\ell_2 < 0$. *Then*:

(a) $\sigma(G_p)\cap i\mathbb{R} = \{2\pi ik/T\colon k \in \mathbb{Z}\}$ *consists of simple isolated eigenvalues for every* $p \in (1,\infty)$. *In particular, for every* $f \in L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$ *the parabolic problem* $G_p u = f$ *is solvable if and only if* $\int_{\mathbb{T}\times\mathbb{R}^d} f\,d\mu = 0$. *In this case, it has infinite solutions, and the difference of two solutions is constant.*

(b) *For every* $p \in (1,\infty)$ G_p *has a spectral gap. Specifically,*

$$\sup\big\{\operatorname{Re}\lambda\colon \lambda \in \sigma(G_p)\setminus i\mathbb{R}\big\} \leqslant \begin{cases} \ell_2, & \text{if } p \geqslant 2,\\ 2\ell_2(1-1/p), & \text{if } 1 < p < 2.\end{cases}$$

Proof. By Proposition 2.14, $\gamma_2 \leqslant \ell_2$. Since $\|\mathcal{T}(t)(I-\Pi)f\|_1 \leqslant 2\|f\|_1$ for every $t > 0$ and $f \in L^1(\mathbb{T}\times\mathbb{R}^d,\mu)$, using estimate (2.18) with $p = 2$ and interpolating between L^1 and L^2 we get

$$\left\|\mathcal{T}(t)(I-\Pi)f\right\|_p \leqslant M_p e^{2\ell_2(1-1/p)t}\|f\|_p,$$

for every $f \in L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$ and every $t > 0$. Therefore, the spectrum of the part of G_p in $(I-\Pi)(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))$ is contained in the halfplane $\operatorname{Re}\lambda \leqslant \ell_2$, if $p \geqslant 2$, and in the halfplane $\operatorname{Re}\lambda \leqslant 2\ell_2(1-1/p)$, if $1 < p < 2$. For the other values of λ, it is convenient to write the equation $\lambda u - G_p u = f$ as the system

$$\begin{cases} \lambda\Pi u - G_p\Pi u = \Pi f,\\ \lambda(I-\Pi)u - G_p(I-\Pi)u = (I-\Pi)f,\end{cases}$$

where the second equation is uniquely solvable. Setting $\Pi u = \beta$, the first equation may be rewritten as

$$\beta \in W^{1,p}\left(\mathbb{T}, \frac{ds}{T}\right), \quad \lambda\beta(s) + \beta'(s) = m_s f(s,\cdot), \quad s \in \mathbb{T},$$

and it is uniquely solvable if and only if $\lambda \neq 2\pi ik/T$ for every $k \in \mathbb{Z}$. Since the eigenvalues $2\pi ik/T$ of the realization of the first order derivative in $L^p(\mathbb{T}, \frac{ds}{T})$ are simple, the eigenvalues $2\pi ik/T$ of G_p are simple too. In particular, for $\lambda = 0$ the above equation is solvable if and only if $\int_0^T m_s f(s,\cdot)\,ds = 0$, which means $\int_{\mathbb{T}\times\mathbb{R}^d} f\,d\mu = 0$, and in this case the solutions differ by constants. The statements follow. □

Corollary 3.8. *Let Hypotheses* 2.1 *and* 2.4 *hold. Assume that the diffusion coefficients are bounded and that* $\ell_2 < 0$. *Then, for every* $p > 1$ *there exists* $M_p > 0$ *such that*

$$\left\| P(t,s)\varphi - m_s\varphi \right\|_{L^p(\mathbb{R}^d,\mu_t)} \leqslant M_p e^{\ell_2(t-s)} \|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}, \quad t > s,\ \varphi \in L^p\big(\mathbb{R}^d, \mu_s\big), \tag{3.18}$$

if $p \geqslant 2$, *and*

$$\left\| P(t,s)\varphi - m_s\varphi \right\|_{L^p(\mathbb{R}^d,\mu_t)} \leqslant M_p e^{\theta_p(t-s)} \|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}, \quad t > s,\ \varphi \in L^p\big(\mathbb{R}^d, \mu_s\big), \tag{3.19}$$

if $1 < p < 2$, *with* $\theta_p = 2\ell_2(1 - 1/p)$.

Proof. By estimate (2.18), $\ell_2 \in B_p$ for $p \geqslant 2$, and (3.18) follows applying Theorem 3.2 and Theorem 3.6. For $1 < p < 2$, the estimate $\|P(t,s)\varphi - m_s\varphi\|_{L^p(\mathbb{R}^d,\mu_t)} \leqslant M_p e^{2\ell_2(1-1/p)(t-s)} \|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}$ follows interpolating between L^1 and L^2, since $\|P(t,s)\varphi - m_s\varphi\|_{L^1(\mathbb{R}^d,\mu_t)} \leqslant 2\|\varphi\|_{L^1(\mathbb{R}^d,\mu_s)}$ for $t > s$ and $\varphi \in L^1(\mathbb{R}^d, \mu_s)$. □

To get a better decay estimate in L^p spaces with $p < 2$ we need more refined arguments. An important tool is a logarithmic Sobolev estimate, that will be proved in the next subsection.

We end this subsection with a remark. Spectral gaps of elliptic differential operators with unbounded coefficients and asymptotic behavior of the associated semigroups are usually proved through Poincaré inequalities. We may prove a Poincaré type inequality in our nonautonomous setting, and precisely

Proposition 3.9. *Let Hypotheses* 2.1 *and* 2.4 *hold. Assume that the diffusion coefficients are bounded and that* $\ell_2 < 0$. *Then*

$$\int_{\mathbb{T}\times\mathbb{R}^d} |f - \Pi f|^2\,d\mu \leqslant \frac{\Lambda}{|\ell_2|} \int_{\mathbb{T}\times\mathbb{R}^d} |\nabla_x f|^2\,d\mu, \quad f \in W_2^{0,1}\big(\mathbb{T}\times\mathbb{R}^d, \mu\big), \tag{3.20}$$

where Λ *is defined in* (3.16).

Proof. Let $f \in (I - \Pi)(D(G_2))$. By Corollary 2.16(c), inequality (3.12) is in fact an equality. Letting $t \to \infty$ in (3.12) and recalling that $\lim_{t\to\infty} \mathcal{T}(t)f = 0$ by Theorem 3.5, we obtain

$$\|f\|_2^2 = 2\int_0^\infty \int_{\mathbb{T}\times\mathbb{R}^d} \big\langle Q\,\nabla_x\mathcal{T}(s)f, \nabla_x\mathcal{T}(s)f\big\rangle\,d\mu\,ds$$

and therefore, using (2.19),

$$\|f\|_2^2 \leqslant 2\Lambda \int_0^\infty \int_{\mathbb{T}\times\mathbb{R}^d} \big|\nabla_x\mathcal{T}(s)f\big|^2\,d\mu\,ds \leqslant 2\Lambda \int_0^\infty \int_{\mathbb{T}\times\mathbb{R}^d} e^{2\ell_2 s}|\nabla_x f|^2\,d\mu\,ds = \frac{\Lambda}{|\ell_2|} \int_{\mathbb{T}\times\mathbb{R}^d} |\nabla_x f|^2\,d\mu,$$

so that (3.20) holds for every $f \in D(G_2)$. Since $D(G_2)$ is dense in $W_2^{0,1}(\mathbb{T} \times \mathbb{R}^d, \mu)$, (3.20) holds for every $f \in W_2^{0,1}(\mathbb{T} \times \mathbb{R}^d, \mu)$. $\square$

Once the Poincaré inequality (3.20) is established, arguing as in [7, Prop. 6.4] we obtain

$$\left\| \mathcal{T}(t)(f - \Pi f) \right\|_2 \leqslant e^{\eta_0 \ell_2 t/\Lambda} \| f - \Pi f \|_2, \quad t > 0,$$

so that $\omega_2 \leqslant \eta_0 \ell_2/\Lambda$. Since $\eta_0 \leqslant \Lambda$, the estimate $\omega_2 \leqslant \ell_2$ obtained through Theorem 3.6 is sharper. Such estimates coincide only if $\eta_0 = \Lambda$, that is if the diffusion matrix Q is a scalar multiple of the identity.

3.1. A log-Sobolev type inequality

Throughout the whole subsection we assume that Hypotheses 2.1 and 2.4 hold, that $r_0 < 0$, and that the diffusion coefficients are independent of x. We recall that $r_0 = \sup_{(t,x) \in \mathbb{T} \times \mathbb{R}^d} r(t,x)$ where r is the function in Hypothesis 2.4(ii). This is an important restriction, due to the fact that in the proof (which is an adaptation to the nonautonomous case of the method of [12, Thm. 6.2.42]) we use the estimate

$$\left| \nabla_x \mathcal{T}(t) f(s,x) \right| \leqslant e^{r_0(t-s)} \left(\mathcal{T}(t) |\nabla_x f| \right)(s,x), \quad t > 0, \ (s,x) \in \mathbb{R}^{1+d}, \tag{3.21}$$

obtained from Theorem 2.6(iii), which is not obvious (and, in general, not true) if the diffusion coefficients are not independent of x. We refer the reader to [27] for a discussion about the validity of an estimate similar to (3.21) in the autonomous case.

Lemma 3.10. *For any $f \in D(G_\infty)$ such that $f \geqslant \delta$ for some $\delta > 0$, we have*

$$\lim_{t \to \infty} \int_{\mathbb{T} \times \mathbb{R}^d} \mathcal{T}(t) f \log\left(\mathcal{T}(t) f \right) d\mu = \frac{1}{T} \int_0^T \Pi f \log(\Pi f)\, ds.$$

Proof. By the definition of $\mathcal{T}(t)$ we have

$$\frac{1}{T} \int_0^T \Pi f \log(\Pi f)\, ds = \int_{\mathbb{T} \times \mathbb{R}^d} \Pi f(\cdot - t) \log\left(\Pi f(\cdot - t) \right) d\mu = \int_{\mathbb{T} \times \mathbb{R}^d} \mathcal{T}(t) \Pi f \log\left(\mathcal{T}(t) \Pi f \right) d\mu.$$

On the other hand, using Hölder inequality and recalling that the function $y \mapsto y \log(y)$ is Hölder continuous on bounded sets, we can determine $C > 0$ and $\alpha \in (0,1)$ such that

$$\begin{aligned} &\left| \int_{\mathbb{T} \times \mathbb{R}^d} \left(\mathcal{T}(t) f \log\left(\mathcal{T}(t) f \right) - \mathcal{T}(t) \Pi f \log\left(\mathcal{T}(t) \Pi f \right) \right) d\mu \right| \\ &\quad \leqslant C \int_{\mathbb{T} \times \mathbb{R}^d} \left| \mathcal{T}(t)(f - \Pi f) \right|^\alpha d\mu \leqslant C \left\| \mathcal{T}(t)(f - \Pi f) \right\|_2^\alpha, \end{aligned}$$

for any $t > 0$, and Theorem 3.5 yields the assertion. $\square$

We recall that Λ is the supremum of the eigenvalues of the matrices $Q(s)$.

Theorem 3.11. *For any $p \in [1, \infty)$ and any $f \in D(G_\infty)$ with positive infimum we have*

$$\int_{\mathbb{T}\times\mathbb{R}^d} f^p \log\bigl(f^p\bigr)\,d\mu \leqslant \frac{1}{T}\int_0^T \Pi f^p \log\bigl(\Pi f^p\bigr)\,ds + \frac{p^2 \Lambda}{2|r_0|}\int_{\mathbb{T}\times\mathbb{R}^d} f^{p-2}|\nabla_x f|^2\,d\mu. \tag{3.22}$$

Proof. Let $f \in D(G_\infty)$ satisfy $f \geqslant \delta$ for some $\delta > 0$. We first prove (3.22) with $p = 1$. By Proposition 2.11, $\mathcal{T}(t)f \in D(G_\infty)$ for any $t > 0$. Moreover, $\mathcal{T}(t)f \geqslant \mathcal{T}(t)\delta \equiv \delta$ for any $t \geqslant 0$.

Let us consider the function $F : [0, \infty) \to \mathbb{R}$ defined by

$$F(t) = \int_{\mathbb{T}\times\mathbb{R}^d} \mathcal{T}(t)f \log\bigl(\mathcal{T}(t)f\bigr)\,d\mu, \quad t \geqslant 0.$$

By Lemma 3.10 we have

$$\lim_{t\to\infty} F(t) = \frac{1}{T}\int_0^T \Pi f \log(\Pi f)\,ds.$$

We want to show that F is differentiable, and to compute $F'(t)$. First of all we remark that, since $\mathcal{T}(t)f \in D(G_\infty)$ and $\mathcal{T}(t)f \geqslant \delta$, the function $\log(\mathcal{T}(t)f)$ is in $D(G_\infty)$ for any $t \geqslant 0$. Indeed, it belongs to $C_b(\mathbb{T}\times\mathbb{R}^d) \cap W^{1,2}_q(\mathbb{T}\times B(0,R))$ for every q and R, and

$$\mathcal{G}\bigl(\log\bigl(\mathcal{T}(t)f\bigr)\bigr) = \frac{1}{\mathcal{T}(t)f}\mathcal{G}\mathcal{T}(t)f - \frac{1}{(\mathcal{T}(t)f)^2}\bigl\langle Q\,\nabla_x\mathcal{T}(t)f, \nabla_x\mathcal{T}(t)f\bigr\rangle$$

is continuous and bounded. Taking Proposition 2.14 into account, it follows that the function $\mathcal{T}(t)f \log(\mathcal{T}(t)f)$ belongs to $D(G_\infty)$ and

$$\begin{aligned}
\mathcal{G}\bigl[\mathcal{T}(t)f \log\bigl(\mathcal{T}(t)f\bigr)\bigr] &= \mathcal{T}(t)f\left(\frac{1}{\mathcal{T}(t)f}\mathcal{G}\mathcal{T}(t)f - \frac{1}{(\mathcal{T}(t)f)^2}\bigl\langle Q\,\nabla_x\mathcal{T}(t)f, \nabla_x\mathcal{T}(t)f\bigr\rangle\right)\\
&\quad + \bigl(\mathcal{G}\mathcal{T}(t)f\bigr)\log\bigl(\mathcal{T}(t)f\bigr) + 2\left\langle Q\,\nabla_x\mathcal{T}(t)f, \frac{\nabla_x\mathcal{T}(t)f}{\mathcal{T}(t)f}\right\rangle\\
&= \mathcal{G}\mathcal{T}(t)f + \frac{1}{\mathcal{T}(t)f}\bigl\langle Q\,\nabla_x\mathcal{T}(t)f, \nabla_x\mathcal{T}(t)f\bigr\rangle\\
&\quad + \bigl(\mathcal{G}\mathcal{T}(t)f\bigr)\log\bigl(\mathcal{T}(t)f\bigr).
\end{aligned} \tag{3.23}$$

A straightforward computation shows that

$$\frac{d}{dt}\bigl(\mathcal{T}(t)f \log\bigl(\mathcal{T}(t)f\bigr)\bigr) = \mathcal{G}\mathcal{T}(t)f \log\bigl(\mathcal{T}(t)f\bigr) + \mathcal{G}\mathcal{T}(t)f, \quad t \geqslant 0.$$

Using (3.23) we get

$$\frac{d}{dt}\bigl(\mathcal{T}(t)f \log\bigl(\mathcal{T}(t)f\bigr)\bigr) = \mathcal{G}\bigl[\mathcal{T}(t)f \log\bigl(\mathcal{T}(t)f\bigr)\bigr] - \frac{1}{\mathcal{T}(t)f}\bigl\langle Q\,\nabla_x\mathcal{T}(t)f, \nabla_x\mathcal{T}(t)f\bigr\rangle,$$

which is continuous and bounded. Therefore, F is differentiable and, since the integral of $\mathcal{G}[\mathcal{T}(t)f\log(\mathcal{T}(t)f)]$ vanishes, we have

$$F'(t) = -\int_{\mathbb{T}\times\mathbb{R}^d} \frac{1}{\mathcal{T}(t)f}\big\langle Q\,\nabla_x\mathcal{T}(t)f, \nabla_x\mathcal{T}(t)f\big\rangle\,d\mu, \quad t \geqslant 0.$$

Let us estimate $F'(t)$. The pointwise estimate (3.21) implies that

$$\big|\nabla_x\mathcal{T}(t)f(s,x)\big|^2 \leqslant e^{2r_0t}\big(\mathcal{T}(t)|\nabla_x f|(s,x)\big)^2, \quad t>0,\ (s,x)\in\mathbb{R}^{1+d},$$

so that

$$\int_{\mathbb{T}\times\mathbb{R}^d} \frac{1}{\mathcal{T}(t)f}\big\langle Q\,\nabla_x\mathcal{T}(t)f, \nabla_x\mathcal{T}(t)f\big\rangle\,d\mu \leqslant e^{2r_0t}\int_{\mathbb{T}\times\mathbb{R}^d} \frac{\Lambda}{\mathcal{T}(t)f}\big(\mathcal{T}(t)|\nabla_x f|\big)^2\,d\mu, \quad t\geqslant 0.$$

Moreover, using the Hölder inequality in the representation formula

$$\mathcal{T}(t)f(s,x) = \int_{\mathbb{R}^d} f(s-t,y)\,p_{s,s-t,x}(dy), \quad t>0,\ (s,x)\in\mathbb{T}\times\mathbb{R}^d$$

(see Theorem 2.2), we get

$$\big(\mathcal{T}(t)|\nabla_x f|\big)^2 = \left(\mathcal{T}(t)\left(\sqrt{f}\,\frac{|\nabla_x f|}{\sqrt{f}}\right)\right)^2 \leqslant \big(\mathcal{T}(t)f\big)\left(\mathcal{T}(t)\left(\frac{|\nabla_x f|^2}{f}\right)\right).$$

Therefore, for each $t\geqslant 0$ we have

$$\int_{\mathbb{T}\times\mathbb{R}^d} \frac{1}{\mathcal{T}(t)f}\big\langle Q\,\nabla_x\mathcal{T}(t)f, \nabla_x\mathcal{T}(t)f\big\rangle\,d\mu \leqslant \Lambda e^{2r_0t}\int_{\mathbb{T}\times\mathbb{R}^d}\mathcal{T}(t)\left(\frac{|\nabla_x f|^2}{f}\right)d\mu = \Lambda e^{2r_0t}\int_{\mathbb{T}\times\mathbb{R}^d}\frac{|\nabla_x f|^2}{f}\,d\mu,$$

that is

$$F'(t) \geqslant -\Lambda e^{2r_0t}\int_{\mathbb{T}\times\mathbb{R}^d}\frac{|\nabla_x f|^2}{f}\,d\mu, \quad t\geqslant 0.$$

Integrating with respect to t in $(0,\infty)$ we get

$$\frac{1}{T}\int_0^T \Pi f\log(\Pi f)\,ds - F(0) = \int_0^\infty F'(t)\,dt \geqslant -\frac{\Lambda}{2|r_0|}\int_{\mathbb{T}\times\mathbb{R}^d}\frac{|\nabla_x f|^2}{f}\,d\mu,$$

that is formula (3.22) with $p=1$.

Let now fix $p\in(1,\infty)$. We have

$$\mathcal{G}\big(f^p\big) = pf^{p-1}\mathcal{G}f + p(p-1)f^{p-2}\langle Q\,\nabla_x f, \nabla_x f\rangle,$$

where $f^p \geqslant \delta^p > 0$, and then, again by Proposition 2.14, $f^p \in D(G_\infty)$. The first part of the proof applied to the function f^p yields the conclusion. □

Proposition 3.12. *For every $p \in (1,\infty)$ and for every $u \in D(G_\infty)$ we have*

$$\int_{\mathbb{T}\times\mathbb{R}^d} |u|^p \log\bigl(|u|^p\bigr)\,d\mu \leqslant \frac{1}{T}\int_0^T \Pi|u|^p \log\bigl(\Pi|u|^p\bigr)\,ds + \frac{p^2\Lambda}{2|r_0|}\int_{\mathbb{T}\times\mathbb{R}^d} |u|^{p-2}|\nabla_x u|^2\,d\mu. \tag{3.24}$$

In addition, if $u \in D(G_\infty)$ satisfies

$$\int_{\mathbb{T}\times\mathbb{R}^d} \frac{|\nabla_x u|^2}{|u|}\,d\mu < \infty,$$

then (3.24) *holds also for $p = 1$.*

Proof. Fix $u \in D(G_\infty)$ and define the sequence

$$u_n := \sqrt{u^2 + \frac{1}{n}}, \quad n \in \mathbb{N}.$$

A straightforward computation shows that

$$\mathcal{G}u_n = \frac{u}{u_n}\mathcal{G}u + \frac{1}{n}\frac{\langle Q\,\nabla_x u, \nabla_x u\rangle}{u_n^3},$$

so that $u_n \in D(G_\infty)$ and, moreover, $u_n \geqslant \frac{1}{\sqrt{n}}$ for any $n \in \mathbb{N}$. Therefore, by Theorem 3.11, we have

$$\int_{\mathbb{T}\times\mathbb{R}^d} u_n^p \log\bigl(u_n^p\bigr)\,d\mu \leqslant \frac{1}{T}\int_0^T \Pi u_n^p \log\bigl(\Pi u_n^p\bigr)\,ds + \frac{p^2\Lambda}{2|r_0|}\int_{\mathbb{T}\times\mathbb{R}^d} u_n^{p-2}|\nabla_x u_n|^2\,d\mu, \tag{3.25}$$

for any $p \in [1,\infty)$ and for any $n \in \mathbb{N}$.

Since $0 < u_n^p \leqslant \|(u^2+1)^{p/2}\|_\infty$ for any $n \in \mathbb{N}$ and the function $x \mapsto x\log x$ is continuous in $[0,\infty)$, the left-hand side of (3.25) converges to $\int_{\mathbb{T}\times\mathbb{R}^d} |u|^p \log(|u|^p)\,d\mu$. Similarly, since $\Pi u_n^p \leqslant \|(u^2+1)^{p/2}\|_\infty$, by the dominated convergence theorem $\Pi|u|^p\log(\Pi|u|^p) \in L^1((0,T),ds)$, and

$$\lim_{n\to\infty}\frac{1}{T}\int_0^T \Pi u_n^p \log\bigl(\Pi u_n^p\bigr)\,ds = \frac{1}{T}\int_0^T \Pi|u|^p \log\bigl(\Pi|u|^p\bigr)\,ds.$$

Concerning the second integral in the right-hand side of (3.25), if $p \in [1,2)$ we have

$$0 < u_n^{p-2}|\nabla_x u_n|^2 \leqslant |u|^{p-2}|\nabla_x u|^2\chi_{\{u\neq 0\}}, \quad \text{a.e. in } \mathbb{T}\times\mathbb{R}^d,$$

and the right-hand side is in $L^1(\mathbb{T}\times\mathbb{R}^d,\mu)$ by Proposition 2.15(c) for $p > 1$ and by assumption for $p = 1$; if $p \geqslant 2$ we have

$$0 < u_n^{p-2}|\nabla_x u_n|^2 \leqslant u_1^{p-2}|\nabla_x u|^2,$$

which is bounded by Proposition 2.14. In any case, the dominated convergence theorem yields

$$\lim_{n\to\infty} \int_{\mathbb{T}\times\mathbb{R}^d} u_n^{p-2}|\nabla_x u_n|^2\, d\mu = \int_{\mathbb{T}\times\mathbb{R}^d} |u|^{p-2}|\nabla_x u|^2\, d\mu$$

and the statement follows. □

Proposition 3.12 with $p = 2$ would be enough to prove next compactness Theorem 3.16. However, it is interesting to extend logarithmic Sobolev inequalities as far as possible. To extend estimate (3.24) to all functions $u \in D(G_p)$ for $p \geqslant 2$, we use the following lemma.

Lemma 3.13. *For $p \geqslant 2$ and $u \in D(G_p)$, $|u|^p \in D(G_1)$, and the mapping $u \mapsto |u|^p$ is continuous from $D(G_p)$ into $D(G_1)$. Moreover, there exists $C_p > 0$ such that $\||u|^p\|_{D(G_1)} \leqslant C_p\|u\|^p_{D(G_p)}$, for every $u \in D(G_p)$.*

Proof. In the proof of Proposition 2.15 we have shown that $|u|^p \in D(G_\infty)$ for any $u \in D(G_\infty)$ and

$$G_\infty\big(|u|^p\big) = pu|u|^{p-2}\mathcal{G}u + p(p-1)|u|^{p-2}\langle Q\,\nabla_x u, \nabla_x u\rangle. \tag{3.26}$$

Recalling that $D(G_p)$ is continuously embedded in $W_p^{0,1}(\mathbb{T}\times\mathbb{R}^d, \mu)$ by Proposition 2.14, formula (3.26) implies that the nonlinear operator $u \mapsto |u|^p$ is continuous from $D(G_\infty)$ (endowed with the $D(G_p)$-norm) to $D(G_1)$. Estimate $\||u|^p\|_{D(G_1)} \leqslant C_p\|u\|^p_{D(G_p)}$ follows using the Hölder inequality in the right-hand side of (3.26). Since $D(G_\infty)$ is dense in $D(G_p)$, the statement follows. □

Theorem 3.14. *For every $p \geqslant 2$ and for every $u \in D(G_p)$, estimate* (3.24) *holds true.*

Proof. Fix $p \geqslant 2$ and $u \in D(G_p)$. Then, there exists a sequence $(u_n) \subset D(G_\infty)$ such that $u_n \to u$ in the graph norm of G_p. Possibly replacing (u_n) by a subsequence, we may assume that $u_n \to u$ pointwise a.e. By Proposition 3.12, for any $n \in \mathbb{N}$ we have

$$\int_{\mathbb{T}\times\mathbb{R}^d} |u_n|^p \log\big(|u_n|^p\big)\, d\mu \leqslant \frac{1}{T}\int_0^T \Pi|u_n|^p \log\big(\Pi|u_n|^p\big)\, ds + \frac{p^2\Lambda}{2|r_0|}\int_{\mathbb{T}\times\mathbb{R}^d} |u_n|^{p-2}|\nabla_x u_n|^2\, d\mu.$$

As a first step, we prove that

$$\lim_{n\to\infty}\int_0^T \Pi|u_n|^p \log\big(\Pi|u_n|^p\big)\, ds = \int_0^T \Pi|u|^p \log\big(\Pi|u|^p\big)\, ds. \tag{3.27}$$

By Lemma 3.13, $|u_n|^p \to |u|^p$ in $D(G_1)$, as $n \to \infty$. Therefore, $\lim_{n\to\infty} \Pi(|u_n|^p) = \Pi(|u|^p)$ in $D(G_1)$. As we already mentioned at the beginning of the section, the part of G_1 in $\Pi(L^1(\mathbb{T}\times\mathbb{R}^d, \mu))$ is the time derivative $-D_s$ with domain isomorphic to $W^{1,1}(\mathbb{T}, \frac{ds}{T})$. It follows that $\lim_{n\to\infty} \Pi(|u_n|^p) = \Pi(|u|^p)$ in $W^{1,1}(\mathbb{T}, \frac{ds}{T})$ and, since $W^{1,1}((0,T), \frac{ds}{T})$ is continuously embedded in $L^\infty(0,T)$, and the function $y \mapsto y\log y$ is α-Hölder-continuous for any $\alpha \in (0,1)$ on bounded sets of $[0,\infty)$, we get

$$\begin{aligned}\frac{1}{T}\int_0^T \big|\varPi|u_n|^p\log\big(\varPi|u_n|^p\big)-\varPi|u|^p\log\big(\varPi|u|^p\big)\big|\,ds &\leqslant \frac{C_1}{T}\int_0^T\big|\varPi|u_n|^p-\varPi|u|^p\big|^{\alpha}\,ds\\ &\leqslant C_1\big\|\varPi|u_n|^p-\varPi|u|^p\big\|_{\infty}^{\alpha}\\ &\leqslant C_2\big\|\varPi|u_n|^p-\varPi|u|^p\big\|_{W^{1,1}((0,T),ds/T)}^{\alpha},\end{aligned}$$

for some positive constants C_i $(i=1,2)$. Then, (3.27) follows. Since the function $u\mapsto H(u)=\int_{\mathbb{T}\times\mathbb{R}^d}|u|^{p-2}|\nabla_x u|^2\,d\mu$ is continuous in $D(G_p)$ by the proof of Corollary 2.16(c), $H(u_n)$ tends to $H(u)$ as $n\to\infty$. Now, denote by $\log_-(y)$ and $\log_+(y)$ the negative and the positive parts of $\log(y)$, i.e.,

$$\log_-(y):=\max\big\{0,-\log(y)\big\},\qquad \log_+(y):=\max\big\{0,\log(y)\big\},\quad y>0.$$

Taking into account that the function $y\mapsto y^p\log_-(y^p)$ is Lipschitz continuous, we get

$$\int_{\mathbb{T}\times\mathbb{R}^d}\big||u|^p\log_-\big(|u|^p\big)-|u_n|^p\log_-\big(|u_n|^p\big)\big|\,d\mu\leqslant C_3\int_{\mathbb{T}\times\mathbb{R}^d}\big||u|^p-|u_n|^p\big|\,d\mu,$$

for some constant $C_3>0$, and the right-hand side tends to 0 as $n\to\infty$.

By the Fatou Lemma we have

$$\begin{aligned}\int_{\mathbb{T}\times\mathbb{R}^d}|u|^p\log_+\big(|u|^p\big)\,d\mu &\leqslant \liminf_{n\to\infty}\int_{\mathbb{T}\times\mathbb{R}^d}|u_n|^p\log_+\big(|u_n|^p\big)\,d\mu\\ &\leqslant \lim_{n\to\infty}\Bigg(\int_{\mathbb{T}\times\mathbb{R}^d}|u_n|^p\log_-\big(|u_n|^p\big)\,d\mu+\frac{1}{T}\int_0^T\varPi|u_n|^p\log\big(\varPi|u_n|^p\big)\,ds\\ &\qquad+\frac{p^2\varLambda}{2|r_0|}\int_{\mathbb{T}\times\mathbb{R}^d}|u_n|^{p-2}|\nabla_x u_n|^2\,d\mu\Bigg)\\ &=\int_{\mathbb{T}\times\mathbb{R}^d}|u|^p\log_-\big(|u|^p\big)\,d\mu+\frac{1}{T}\int_0^T\varPi|u|^p\log\big(\varPi|u|^p\big)\,ds\\ &\qquad+\frac{p^2\varLambda}{2|r_0|}\int_{\mathbb{T}\times\mathbb{R}^d}|u|^{p-2}|\nabla_x u|^2\,d\mu,\end{aligned}$$

which implies (3.24) and concludes the proof. □

3.2. Compactness in L^p spaces

If the domain $D(G_{p_0})$ is compactly embedded in $L^{p_0}(\mathbb{T}\times\mathbb{R}^d,\mu)$ for some p_0, a lot of nice consequences follow.

Theorem 3.15. *Under Hypothesis* 2.1, *assume that the domain of G_{p_0} is compactly embedded in $L^{p_0}(\mathbb{T}\times\mathbb{R}^d,\mu)$ for some $p_0\in[1,\infty]$. Then, for every $p\in(1,\infty)$ the domain of G_p is compactly embedded in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$, and*

(i) *the spectrum of* G_p *consists of isolated eigenvalues independent of* p, *for* $p \in (1, \infty)$. *The associated spectral projections are independent of* p, *too*;

(ii) *the growth bounds* ω_p *defined in* (3.15) *are independent of* $p \in (1, \infty)$. *Denoting by* ω_0 *their common value, for every* $p \in (1, \infty)$ *we have*

$$\omega_0 = \sup\{\operatorname{Re}\lambda\colon \lambda \in \sigma(G_p) \setminus i\mathbb{R}\}.$$

If in addition Hypothesis 2.4 *too is satisfied, then*

(iii) *statement* (a) *of Corollary* 3.7 *holds*;

(iv) $\omega_0 < 0$. *Moreover, for every* $\omega > \omega_0$, $p \in (1, \infty)$ *there exists* $M > 0$ *such that*

$$\left\| P(t,s)\varphi - m_s\varphi \right\|_{L^p(\mathbb{R}^d, \mu_t)} \leqslant M e^{\omega(t-s)} \|\varphi\|_{L^p(\mathbb{R}^d, \mu_s)}, \quad t > s,\ \varphi \in L^p(\mathbb{R}^d, \mu_s). \tag{3.28}$$

Proof. Suppose that $D(G_{p_0})$ is compactly embedded in $L^{p_0}(\mathbb{T} \times \mathbb{R}^d, \mu)$. Then, for any $\lambda > 0$ the resolvent operator $u \mapsto \int_0^\infty e^{-\lambda t}\mathcal{T}(t)u\,dt$ is compact in $L^{p_0}(\mathbb{T} \times \mathbb{R}^d, \mu)$, and since it is bounded in all spaces $L^p(\mathbb{T} \times \mathbb{R}^d, \mu)$, $1 \leqslant p \leqslant \infty$, it is compact in all spaces $L^p(\mathbb{T} \times \mathbb{R}^d, \mu)$, $1 < p < \infty$, by interpolation. See e.g., [11, (proof of) Thm. 1.6.1], for $p_0 < \infty$, and [24, Prop. 4.6] for $p_0 = \infty$. Since the domain of G_p coincides with the range of $R(\lambda, G_p)$, it is compactly embedded in $L^p(\mathbb{T} \times \mathbb{R}^d, \mu)$.

Let us now prove statements (i) to (iv).

(i). By the general spectral theory, the spectrum of G_p consists of isolated eigenvalues. Applying [11, Cor. 1.6.2] to the resolvent $R(\lambda, G_p)$ for a fixed $\lambda > 0$, it follows that the spectrum of $R(\lambda, G_p)$ is independent of p and, hence, the spectrum of G_p is independent of p. It also follows that the spectral projections are independent of p.

(ii). Fix any $p \in (1, \infty)$ and denote by G_Π, $\mathcal{T}_\Pi(t)$, respectively, the parts of G_p, $\mathcal{T}(t)$ in $\Pi(L^p(\mathbb{T} \times \mathbb{R}^d, \mu))$, and by $G_{I-\Pi}$, $\mathcal{T}_{I-\Pi}(t)$, respectively, the parts of G_p, $\mathcal{T}(t)$ in $(I - \Pi)(L^p(\mathbb{T} \times \mathbb{R}^d, \mu))$.

Since Π commutes with $\mathcal{T}(t)$, then $\sigma(G_p) = \sigma(G_\Pi) \cup \sigma(G_{I-\Pi})$. The spectrum of G_Π is the set $\{2k\pi i/T\colon k \in \mathbb{Z}\}$, since $\Pi(L^p(\mathbb{T} \times \mathbb{R}^d, \mu))$ is isometric to $L^p(\mathbb{T}, \frac{ds}{T})$ and $G_\Pi = -D_s$ on $D(G_\Pi) = \Pi(D(G_p))$. Therefore,

$$\sup\{\operatorname{Re}\lambda\colon \lambda \in \sigma(G_p), \operatorname{Re}\lambda < 0\} = \sup\{\operatorname{Re}\lambda\colon \lambda \in \sigma(G_{I-\Pi})\}.$$

Let us prove that such suprema coincide with ω_p. This will imply that ω_p is independent of p, because the left-hand supremum is independent of p.

To this aim, we remark that, although the operators $\mathcal{T}(t)$ are not compact, $\sigma(\mathcal{T}(t)) \setminus \{0\}$ consists of eigenvalues. Indeed, by the Spectral Mapping Theorem 2.17, $\sigma(\mathcal{T}(t)) \setminus \{0\} = e^{t\sigma(G_p)}$, and by the general theory of semigroups (e.g., [13, Thm. IV.3.7]) $P\sigma(\mathcal{T}(t)) \setminus \{0\} = e^{tP\sigma(G_p)}$, where $P\sigma$ denotes the point spectrum. Since $\sigma(G_p) = P\sigma(G_p)$, then $\sigma(\mathcal{T}(t)) \setminus \{0\} = P\sigma(\mathcal{T}(t)) \setminus \{0\}$. As a consequence, also $\sigma(\mathcal{T}_{I-\Pi}(t)) \setminus \{0\}$ consists of eigenvalues, because the elements of $\sigma(\mathcal{T}_{I-\Pi}(t))$, which are not eigenvalues, are contained in $\sigma(\mathcal{T}(t)) \setminus P\sigma(\mathcal{T}(t))$, which does not contain nonzero elements. Again by the spectral mapping theorem for the point spectrum, $\sigma(\mathcal{T}_{I-\Pi}(t)) \setminus \{0\} = e^{t\sigma(G_{I-\Pi})}$ i.e., the semigroup $\mathcal{T}_{I-\Pi}(t)$ satisfies the spectral mapping theorem. This implies that $\omega_p = \sup\{\operatorname{Re}\lambda\colon \lambda \in \sigma(G_{I-\Pi})\}$, because ω_p coincides with the logarithm of the spectral radius of $\mathcal{T}_{I-\Pi}(1)$ (see e.g., [13, Prop. IV.2.2]).

(iii). Since $\mathcal{T}_{I-\Pi}(t)$ is strongly stable by Theorem 3.5, $G_{I-\Pi}$ cannot have eigenvalues on the imaginary axis. Therefore, $i\mathbb{R}$ is contained the resolvent set of $G_{I-\Pi}$. The arguments used in the proof of the statement (a) of Corollary 3.7 yield the statement.

(iv). We already remarked that $\sigma(G_{I-\Pi}) \cap i\mathbb{R} = \emptyset$. Consequently, the spectrum of $\mathcal{T}_{I-\Pi}(1)$ does not intersect the unit circle. It follows that

$$\sup\{|\zeta|\colon \zeta \in \sigma(\mathcal{T}_{I-\Pi}(1))\} < 1. \tag{3.29}$$

Indeed, if there were a sequence of eigenvalues (ζ_n) of $\mathcal{T}_{I-\Pi}(1)$ such that $\lim_{n\to\infty}|\zeta_n| = 1$, a subsequence would converge to an element ζ with modulus 1, and since the spectrum is closed, $\zeta \in \sigma(\mathcal{T}_{I-\Pi}(1))$. But this is impossible. Hence, (3.29) holds.

It follows that there exists $a < 1$ such that

$$\big\|\mathcal{T}_{I-\Pi}(n)\big\|_{\mathcal{L}(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))} = \big\|\big(\mathcal{T}_{I-\Pi}(1)\big)^n\big\|_{\mathcal{L}(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))} \leqslant a^n,$$

for n large, and since

$$\begin{aligned}\big\|\mathcal{T}_{I-\Pi}(t)\big\|_{\mathcal{L}(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))} &= \big\|\mathcal{T}_{I-\Pi}(t-n)\mathcal{T}_{I-\Pi}(n)\big\|_{\mathcal{L}(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))}\\ &\leqslant \big\|\mathcal{T}_{I-\Pi}(n)\big\|_{\mathcal{L}(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))},\end{aligned}$$

for $n \leqslant t < n+1$, $\|\mathcal{T}_{I-\Pi}(t)\|_{\mathcal{L}(L^p(\mathbb{T}\times\mathbb{R}^d,\mu))}$ decays exponentially as $t \to \infty$, i.e., $\omega_p < 0$.

Estimate (3.28) follows from Theorem 3.2. □

As in the autonomous case, log-Sobolev inequalities imply that $D(G_p)$ is compactly embedded in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$, for every $p \in (1,\infty)$.

Theorem 3.16. *Let Hypotheses* 2.1 *and* 2.4 *hold. Assume that* (3.24) *holds for $p = 2$ and for every $f \in D(G_2)$. Then, for any $p \in (1,\infty)$, $D(G_p)$ is compactly embedded in $L^p(\mathbb{T}\times\mathbb{R}^d,\mu)$.*

Proof. By Theorem 3.15, it is enough to prove that $D(G_2)$ is compactly embedded in $L^2(\mathbb{T}\times\mathbb{R}^d,\mu)$.

We shall show that, for every $\varepsilon > 0$, the unit ball B of $D(G_2)$ may be covered by a finite number of balls of $L^2(\mathbb{T}\times\mathbb{R}^d,\mu)$ with radius not greater than ε.

Fix $u \in B$, $k > 1$ and set $E := \{|u| < k\}$. For every $R > 0$ we have, by (3.24),

$$\begin{aligned}\int_0^T\int_{B(0,R)^c} u^2\,d\mu &\leqslant \int_0^T\int_{B(0,R)^c} \mathbb{1}_E k^2\,d\mu + \frac{1}{\log(k^2)}\int_0^T\int_{B(0,R)^c} \mathbb{1}_{E^c} u^2\log\big(u^2\big)\,d\mu\\ &\leqslant \frac{k^2}{T}\int_0^T ds\int_{B(0,R)^c} d\mu_s + \frac{1}{\log(k^2)}\bigg(\frac{1}{T}\int_0^T \Pi\big(u^2\big)\log\big(\Pi u^2\big)\,ds + C_1\int_{\mathbb{T}\times\mathbb{R}^d}|\nabla_x u|^2\,d\mu\bigg),\end{aligned}$$

for some positive constant C_1, independent of u and k.

Fix $\varepsilon > 0$. By Lemma 3.13, $u^2 \in D(G_1)$, and $\|u^2\|_{D(G_1)} \leqslant C_2\|u\|^2_{D(G_2)} \leqslant C_2$, with C_2 independent of u. Therefore, Πu^2 belongs to the domain of the part of G_1 in $\Pi(L^1(\mathbb{T}\times\mathbb{R}^d,\mu))$, which is isomorphic to $W^{1,1}(\mathbb{T},\frac{ds}{T})$. By the Sobolev embedding for the Lebesgue measure, $\|\Pi u^2\|_{L^\infty(0,T)}$ is bounded by a constant independent of u, so that $\int_0^T \Pi u^2\log(\Pi u^2)\,ds$ is bounded by a constant independent of u. Also the integral $\int_{\mathbb{R}^{1+d}}|\nabla_x u|^2\,d\mu$ is bounded by a constant independent of u, by Proposition 2.14. So, there exists $M > 0$ such that $\frac{1}{T}\int_0^T \Pi u^2\log(\Pi u^2)\,ds + C_1\int_{\mathbb{T}\times\mathbb{R}^d}|\nabla_x u|^2\,d\mu \leqslant M$, for every $u \in B$. Taking k large enough, we get

$$\frac{1}{\log(k^2)}\bigg(\frac{1}{T}\int_0^T \Pi u^2\log\big(\Pi u^2\big)\,ds + C_1\int_{\mathbb{T}\times\mathbb{R}^d}|\nabla_x u|^2\,d\mu\bigg) \leqslant \frac{\varepsilon}{2}.$$

By Theorem 2.2(v) the measures μ_s are tight, so that there exists $R > 0$ such that

$$\frac{k^2}{T}\int_0^T ds \int_{B(0,R)^c} d\mu_s \leqslant \frac{\varepsilon}{2}.$$

Summing up,

$$\int_{(0,T)\times B(0,R)^c} u^2\, d\mu \leqslant \varepsilon.$$

By Corollary 2.13, $D(G_2)$ is contained in $W^{1,2}_{2,\mathrm{loc}}(\mathbb{T}\times\mathbb{R}^d, ds\times dx)$ and the restriction operator $\mathcal{R}: D(G_2)\to W^{1,2}_2(\mathbb{T}\times B(0,R), ds\times dx)$, $\mathcal{R}u = u_{|\mathbb{T}\times B(0,R)}$, is continuous. Since the embedding of $W^{1,2}_2(\mathbb{T}\times B(0,R), ds\times dx)$ in $L^2(\mathbb{T}\times B(0,R), ds\times dx)$ is compact, there exist $f_1,\dots,f_k\in L^2(\mathbb{T}\times B(0,R), ds\times dx)$ such that the balls $B(f_i,\varepsilon)$ cover the restrictions of the functions of B to $\mathbb{T}\times B(0,R)$. Let $\tilde f_i$ denote the null extension of f_i to $\mathbb{T}\times\mathbb{R}^d$. Then $B\subset\bigcup_{i=1}^k B(\tilde f_i, 2\varepsilon)$, and the statement follows. □

Remark 3.17. Under Hypotheses 2.1 and 2.4, if the diffusion coefficients are independent of x and $r_0 < 0$, then the assumptions of Theorem 3.16 are satisfied, hence all the statements of Theorem 3.15 hold, as well as the statements of Corollaries 3.7 and 3.8. Since $r_0 = \ell_2 = \omega_0$, statement (ii) of Theorem 3.15 is sharper than the statements of Corollaries 3.7 and 3.8 for $1 < p < 2$, while estimate (3.18) is sharper than statement (iv) of Theorem 3.15 for $p \geqslant 2$.

4. Examples

4.1. Time dependent Ornstein–Uhlenbeck operators

Let us consider the operators

$$\big(\mathcal{A}(t)\varphi\big)(x) = \frac{1}{2}\operatorname{Tr}\big(B(t)B^*(t)D_x^2\varphi(x)\big) + \big\langle A(t)x + f(t), \nabla\varphi(x)\big\rangle, \quad x\in\mathbb{R}^d,$$

with continuous and T-periodic data $A, B:\mathbb{R}\to\mathcal{L}(\mathbb{R}^d)$ and $f:\mathbb{R}\to\mathbb{R}^d$. The ellipticity condition (1.2) is satisfied provided $\det B(t)\neq 0$ for every $t\in\mathbb{R}$.

In [18] asymptotic behavior results for the backward evolution operator $P(s,t)$, $s\leqslant t$, associated to the family $\{\mathcal{A}(t)\}$ in L^2 spaces have been proved, as well as spectral properties of the parabolic operator $u\mapsto\mathcal{A}(s)u + D_s u$. Here, we consider forward evolution operators $P(t,s)$, $t\geqslant s$, and the parabolic operator $u\mapsto\mathcal{A}(s)u - D_s u$. Reverting time, there is no difficulty to pass from backward to forward.

Let $U(t,s)$ be the evolution operator in $\mathbb{R}^d$, solution of $\frac{\partial}{\partial t}U(t,s) = -A(t)U(t,s)$, $U(s,s) = I$. A (unique) T-periodic evolution system of measures $\{\mu_s\colon s\in\mathbb{R}\}$ exists provided the growth bound $\omega_0(U)$ of $U(t,s)$ is negative; in this case the measures μ_t are explicit Gaussian measures.

The results of this paper allow to extend most of the L^2 asymptotic behavior results of [18] to the L^p setting, with $p\in(1,\infty)$. In fact, the log-Sobolev inequality (3.24) holds for $p = 2$, for every $u\in D(G_2)$. It was proved in [7] for every $u\in C^{1,2}_b(\mathbb{T}\times\mathbb{R}^d)$ which is dense in $D(G_2)$, and the procedure of Theorem 3.14 allows to extend it to all the functions $u\in D(G_2)$. Moreover, Proposition 2.4 of [18] shows that $\omega_2 = \omega_0(U)$. Therefore, all the statements of Theorem 3.15 hold, with $\omega_0 = \omega_0(U)$.

Note that our assumption of Hölder regularity of the coefficients is not needed here, because the proof of Theorem 3.14 is independent of time regularity of the coefficients.

4.2. Diffusion coefficients independent of x

Let now consider the operators $\mathcal{A}(t)$ defined in (1.1) with T-periodic diffusion coefficients depending only on time, under the regularity and ellipticity assumptions of Hypothesis 2.1 (i) and (ii). For every $n \in \mathbb{N}$ the function $V(x) := 1 + |x|^{2n}$ satisfies Hypothesis 2.1(iii) provided that there exists $R > 0$ such that

$$\sup_{s\in\mathbb{R},\,|x|\geqslant R} \frac{\langle b(s,x), x\rangle}{|x|^2} < 0.$$

In this case the statements of Theorem 2.2 and of Proposition 2.10 hold. So, there exists a Markov evolution operator $P(t,s)$ with a unique T-periodic evolution system of measures $\{\mu_s\colon s \in \mathbb{R}\}$. The measures μ_s have uniformly bounded moments of every order, i.e.,

$$\sup_{s\in\mathbb{R}} \int_{\mathbb{R}^d} |x|^k \mu_s(dx) < \infty, \quad k \in \mathbb{N}.$$

If moreover the derivatives $D_i b_j$ belong to $C^{\alpha/2,\alpha}_{\mathrm{loc}}(\mathbb{T}\times\mathbb{R}^d)$ and there exists $r_0 \in \mathbb{R}$ such that

$$\langle \nabla_x b(s,x)\xi, \xi\rangle \leqslant r_0|\xi|^2, \quad (s,x) \in \mathbb{T}\times\mathbb{R}^d,\ \xi \in \mathbb{R}^d,$$

then Hypothesis 2.4 holds too.

Applying Theorem 3.5, statements (ii) to (iv) of Theorem 3.1 hold.

In the case that $r_0 < 0$, we have $\ell_2 = r_0 < 0$, and the log-Sobolev inequalities of Subsection 3.1 hold. By Theorem 3.16 the domain $D(G_p)$ is compactly embedded in $L^p(\mathbb{T}\times\mathbb{R}^d, \mu)$ for $p \in (1,\infty)$ and all the statements of Theorem 3.15 hold. Moreover the statements of Corollaries 3.7 and 3.8 hold. See Remark 3.17.

4.3. General diffusion coefficients

In the general case, setting again $V(x) := 1 + |x|^{2n}$, we have

$$\mathcal{A}(s)V(x) = 2n|x|^{2n}\left[(2n-2)\frac{\langle Q(s,x)x, x\rangle}{|x|^4} + \frac{\operatorname{Tr} Q(s,x)}{|x|^2} + \frac{\langle b(s,x), x\rangle}{|x|^2}\right],$$

for any $(s,x) \in \mathbb{R}^{1+d}$, so that Hypothesis 2.1(iii) is satisfied by V provided there exists $R > 0$ such that

$$\sup_{s\in\mathbb{R},\,|x|\geqslant R}\left((2n-2+d)\frac{\Lambda(s,x)}{|x|^2} + \frac{\langle b(s,x), x\rangle}{|x|^2}\right) < 0, \tag{4.1}$$

where $\Lambda(s,x)$ is the greatest eigenvalue of $Q(s,x)$. If also the regularity and ellipticity assumptions of Hypothesis 2.1 (i) and (ii) are satisfied, by Theorem 2.2 and Proposition 2.10 there exists a Markov evolution operator $P(t,s)$ with a unique T-periodic evolution system of measures $\{\mu_s\colon s \in \mathbb{R}\}$; the measures μ_s satisfy

$$\sup_{s\in\mathbb{R}} \int_{\mathbb{R}^d} |x|^{2n} \mu_s(dx) < \infty.$$

If moreover Hypothesis 2.4 is satisfied and there exists $C > 0$ such that

$$\left\| Q(s,x) \right\|_{\mathcal{L}(\mathbb{R}^d)} \leqslant C\left(1+|x|\right)^{2n+1}, \quad s \in \mathbb{R},\ x \in \mathbb{R}^d,$$

then the assumptions of Theorem 3.5 hold. Indeed, since $\Lambda(s,x)$ is positive, (4.1) implies that $\langle b(s,x), x\rangle < 0$ for $|x| \geqslant R$ and $s \in \mathbb{R}$, so that the second condition of (2.20) is satisfied. Then, Theorem 3.5 yields that statements (ii) to (iv) of Theorem 3.1 hold.

If in addition the diffusion coefficients are bounded and the number ℓ_2 in (2.8) is negative, then all the assumptions of Corollaries 3.7 and 3.8 are satisfied, and we have the exponential decay rates given by Corollary 3.8 and the spectral properties of the operators G_p given by Corollary 3.7.

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