Bahadur–Kiefer Theorems for the Product-Limit Process

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Communicated by the Editors

In the random censorship from the right model, strong and weak limit theorems for Bahadur–Kiefer type processes based on the product-limit estimator are established. The main theorem is sharp and may be considered as a final result as far as this type of research is concerned. As a consequence of this theorem a sharp uniform Bahadur representation for product-limit quantiles is obtained. © 1990 Academic Press, Inc.

1. Introduction and Main Results

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. rv's with distribution function $(df) F$ and let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. rv's with $df G$. Both sequences are assumed to be independent. In the random censorship from the right model $X_i$ may be censored on the right by $Y_i$ so that the pair $(Z_i, \delta_i)$, $i = 1, 2, \ldots$ is observed, where $Z_i = \min(X_i, Y_i)$ and $\delta_i = 1_{\{X_i \leq Y_i\}}$. The $df H$ of the $Z_i$ (which are also independent) is then given by $H = 1 - (1 - F)(1 - G)$.

As in most applications all rv's are assumed to be positive. Moreover, we assume throughout that the following condition is satisfied:

**(A)** $F$ is differentiable on $(0, \infty)$ with continuous and positive derivative $f$ and $G$ is continuous on $(0, \infty)$.

Received September 18, 1989.

AMS 1980 subject classifications: primary 60F05, 60F15, 60F17; secondary 62E20, 62G30.
Key words and phrases: Bahadur representation, empirical and quantile processes, limit theorems, product-limit, random censorship.
The product-limit (PL) estimator $F_n$ (at stage $n$) introduced by Kaplan and Meier [9] comes out as the maximum likelihood estimator of $F$:

$$1 - F_n(x) = \prod_{z_{(i)} \leq x} (1 - \delta_{(i)}/(n - i + 1)), \quad x \geq 0,$$

where $0 < z_{(1)} \leq \cdots \leq z_{(n)}$ are the order statistics of the $Z_i$, $1 \leq i \leq n$, and $\delta_{(i)}$ are the corresponding $\delta$'s. The associated PL process will be given by

$$X_n(x) = n^{1/2}(F_n(x) - F(x)), \quad x \geq 0.$$

The quantile function (or inverse) $Q$ of $F$ is naturally estimated by

$$Q_n(t) = \inf\{x: F_n(x) \geq t\}, \quad t \in (0, 1).$$

The PL quantile process is then given by

$$B_n(t) = n^{1/2}f(Q(t))(Q(t) - Q_n(t)), \quad t \in (0, 1).$$

In this paper we will study the so-called Bahadur-Kiefer process associated with the above PL and PL quantile process defined by

$$K(t) = \frac{d}{dt}Q(t) + B_n(t), \quad t \in (0, 1). \quad (1.1)$$

In the uncensored case this process was introduced by Bahadur [2] and further investigated by Kiefer [10, 11]. A discussion of the literature on the subject for the censored as well as the uncensored case is postponed until the end of this section.

Write

$$TG = \inf\{x: G(x) = 1\}$$

and let $\Lambda$ be a Gaussian process defined on $[0, F(T_G))$, with mean zero and covariance function

$$E(\Lambda(s)\Lambda(t)) = (1 - s)(1 - t) h(s \wedge t), \quad 0 \leq s, t < F(T_G),$$

where

$$h(s) = \int_0^s (1 - u)^2 (1 - G(Q(u)))^{-1} du, \quad 0 \leq s < F(T_G).$$

Moreover, let us define the Gaussian process $\bar{\Lambda}_G$ by

$$\bar{\Lambda}_G(s) = \Lambda(s)/(1 - G(Q(s))), \quad 0 \leq s < F(T_G).$$
In the same spirit we write
\[ \beta_{n,G} = \beta_n/(1 - G \circ Q). \]

Our first result gives the weak convergence of the finite dimensional distributions of \( R_n \).

**Theorem 1.** Let condition (A) be satisfied and let \( 0 < \theta < F(T_0) \).
Suppose that for any \( 0 < \varepsilon < Q(\theta) \)
\[ \lim \sup_{t \in [0,1]} \frac{|f(t) - f(s)|}{t - s} = 0. \]
Let \( k \in \mathbb{N} \) and \( 0 < s_1 < \ldots < s_k < \theta \) be fixed. Then as \( n \to \infty \),
\[ n^{1/4}(R_n(s_1), \ldots, R_n(s_k)) \to (Z_1|\bar{A}_G(s_1)|^{1/2}, \ldots, Z_k|\bar{A}_G(s_k)|^{1/2}), \]
where \( Z_1, \ldots, Z_k \) are independent \( N(0, 1) \) rv's independent of \( \bar{A}_G \).

The second result is an almost sure analogue of Theorem 1.

**Theorem 2.** Under the conditions of Theorem 1, we have for \( s \in (0, \theta) \) fixed, almost surely,
\[ \lim \sup_{n \to \infty} n^{1/4}(\log \log n)^{-3/4} |R_n(s)| \leq 2^{3/4}(1 - s)^{1/2} h^{1/4}(s)(1 - G(Q(s)))^{-1/2}. \]

We now present our main result, which is so powerful that it has a lot of interesting results as a corollary. For its presentation we use the notation \( \|\varphi\|_{\infty} = \sup_{t \in [a, b]} |\varphi(t)| \), when \( \varphi \) is a real valued function on \([a, b]\).

**Theorem 3.** Let condition (A) be satisfied and let \( 0 < \theta < F(T_0) \).
Suppose there exists a \( C \in (0, \infty) \) such that
\[ \lim \sup_{\Delta \downarrow 0} \sup_{s, t : |t - s| \leq \Delta} \frac{|f(t) - f(s)|}{t - s} < C \]
and let \( f \) be right-continuous at 0. In case \( \lim_{x \downarrow 0} f(x) = 0 \), suppose that, in addition, for some \( a \in (0, \infty) \),
\[ \lim_{x \downarrow 0} f(x) |f'(x)| f(x)^{-2} = a. \]

Then we have
\[ \lim_{n \to \infty} n^{1/4}(\log n)^{-1/2} \|R_n\|_{\infty}^{\theta}/(\|\beta_{n,G}\|_{\theta})^{1/2} = 1 \quad a.s. \]
Combination of Theorem 3 with the results in Aly, Csörgő, and Horváth [1] yields:

**Corollary 1.** Under the conditions of Theorem 3 we have

\[ n^{1/4}(\log n)^{-1/2} \| R_n \|_0^{\theta} \xrightarrow{d} (\| \tilde{A}_G \|_0^{\theta})^{1/2} \quad \text{as} \quad n \to \infty; \]  

\[ \limsup_{n \to \infty} n^{1/4}(\log n)^{-1/2} (\log \log n)^{-1/4} \| R_n \|_0^{\theta} = 2^{1/4} \left( \frac{\| (1-I) \ h^{1/2} \|_0^{\theta}}{1-G \circ Q} \right)^{1/2} \quad \text{a.s.,} \]  

where I denotes the identity function;

\[ 2^{-3/4} \pi^{1/2}(1-\theta)^{1/2} h^{1/4}(\theta) \leq \liminf_{n \to \infty} n^{1/4}(\log n)^{-1/2} (\log \log n)^{1/4} \| R_n \|_0^{\theta} \quad \leq 2^{-3/4} \pi^{1/2} h^{1/4}(\theta)(1-G(Q(\theta)))^{-1/2} \quad \text{a.s.} \]  

If \( \lim_{x \to 0} f(x) > 0 \), then (1.5) entails that uniformly over all \( s \in (0, \theta) \) we have

\[ Q_n(s) = Q(s) + \frac{s - F_n(Q(s))}{f(Q(s))} + O(n^{-3/4}(\log n)^{1/2} (\log \log n)^{1/4}) \quad \text{a.s.} \]

**Discussion and Bibliography.** In the uncensored case Kiefer [10] proved both Theorem 1 for the case \( k = 1 \) and Theorem 2 (with the right constant). Note that in the uncensored case \( (G \equiv 0) \) the constant on the right in Theorem 2 reduces to \( 2^{3/4}(s(1-s))^{1/4} \), whereas in that case the actual value of the limsup is equal to \( 25/4 \cdot 3^{-3/4}(s(1-s))^{1/4} \). Theorem 1 for arbitrary \( k \in \mathbb{N} \) and \( G \equiv 0 \) is presented in Beirlant et al. [3]. An in probability version of Theorem 3 (with \( \theta = 1 \)) in the uncensored case is established in Kiefer [11], where the author claims that the statement holds true almost surely; he did not publish a proof, however. Recently, his claim has been proved in Shorack [15, upper bound]) and Deheuvels and Mason [8, lower bound].

In the literature on the random censorship model only the type of problem discussed in Theorem 2 and (1.5) has been considered. A version of the statement in (1.5) can be found in Cheng [6], but with a worse rate. Aly et al. [1] derived the exact rate in (1.5), but did not find the right constant. Comparing Theorem 3 with its uncensored analogue (Theorem 1A in Deheuvels and Mason [8]), it is striking that \( \hat{\beta}_{n,G} \) instead of \( \beta_n \) shows up in the denominator. Finally, note that the assumptions on \( F \) are somewhat milder than the usual Csörgő–Révész conditions, cf. Theorem 4.3
in Aly et al. [11]. Hence, for positive random variables, the main result in that paper (Theorem 4.4; a Kiefer process type strong approximation of $\beta_n$) is improved as far as the assumptions on $F$ are considered.

2. PROOFS

Consider the new set of rv's

$U_i = F(X_i), \quad V_i = F(Y_i), \quad W_i = F(Z_i) = U_i \wedge V_i.$

Then the $U_i$ are i.i.d. uniform $(0, 1)$ rv's, independent of the $V_i$; the $V_i$ are also i.i.d. with df $G \circ Q$. The PL estimator based on these reduced rv's is then given by

$$r_n(t) = F_n(Q(t)), \quad t \in (0, 1),$$

and the corresponding PL process is given by

$$a_n(t) = n^{1/2}(F_n(t) - t) = a_n(Q(t)). \quad t \in (0, 1).$$

Moreover, we put

$$q_n(t) = \inf\{s : F_n(s) \geq t\} = F(Q_n(t))$$

and

$$b_n(t) = n^{1/2}(q_n(t) - t), \quad t \in (0, 1).$$

The corresponding Bahadur–Kiefer process is denoted by

$$r_n(t) = a_n(t) + b_n(t), \quad t \in (0, 1).$$

We first present a number of lemmas which relate $R_n$ to $r_n$.

**Lemma 1.** Under the conditions of Theorem 1 we have for any $t \in (0, 0)$ that as $n \to \infty$

$$n^{1/4}(R_n(t) - r_n(t)) \overset{p}{\to} 0;$$

$$n^{1/4}(\log \log n)^{-3/4} (R_n(t) - r_n(t)) \to 0 \quad a.s.$$  

*Proof.* We only prove the first statement; the second one is proved in an analogous way. Let $0 < \theta < F(T_C)$. As $R_n - r_n = \beta_n - b_n$, it remains to derive that as $n \to \infty$,

$$n^{1/4}(\beta_n(t) - b_n(t)) \overset{p}{\to} 0, \quad 0 < t < \theta.$$
Remark that
\[ \beta_n(t) = n^{1/2} \frac{f(Q(t))}{f(Q(\theta_{l,n}))} (q_n(t) - t) = \frac{f(Q(t))}{f(Q(\theta_{l,n}))} b_n(t), \]
where \(|\theta_{l,n} - t| \leq n^{-1/2} |b_n(t)|\). As \(n \to \infty\), \(b_n(t) = O_{p}(1)\); hence
\[ n^{1/4} |\beta_n(t) - b_n(t)| \leq n^{1/4} |b_n(t)| \left| \frac{f(Q(t))}{f(Q(\theta_{l,n}))} - 1 \right| = O_{p}(1) \cdot \sup_{t: |t - \varepsilon| \leq n^{-1/2} |b_n(t)|} \frac{|f(Q(t)) - f(Q(\varepsilon))|}{(n^{-1/2} |b_n(t)|)^{1/2}}, \]
which tends to zero in probability by assumption. □

**Lemma 2.** Under the conditions of Theorem 3 we have, as \(n \to \infty\),
\[ n^{1/4} (\log n)^{-1/2} (\log \log n)^{1/4} \|R_n - r_n\|_{0}^{\theta} \to 0 \quad a.s. \]

**Proof.** Let \(\theta < F(T_{0})\). As in the proof of Lemma 1 we find that it suffices to show that
\[ n^{1/4} (\log n)^{-1/2} (\log \log n)^{1/4} \|b_n\|_{0}^{\theta} \left| \frac{f(Q(I))}{f(Q(\theta_{l,n}))} - 1 \right|_{0}^{\theta} \to 0 \quad a.s. \]
First consider the case \(\lim_{x \to 0} f(x) > 0\). From Theorem 5.1 in Aly et al. [1]
\[ \limsup_{n \to \infty} (2 \log \log n)^{-1/2} \|b_n\|_{0}^{\theta} = \|(1 - I) h^{1/2}\|_{0}^{\theta} \quad a.s., \quad (2.1) \]
so that we are finished if we show that under the given conditions
\[ n^{1/4} (\log n)^{-1/2} (\log \log n)^{3/4} \left| \frac{f(Q(I))}{f(Q(\theta_{l,n}))} - 1 \right|_{0}^{\theta} \to 0 \quad a.s. \]
By (1.2) for some \(K_{\theta} \in (0, \infty)\) we have almost surely
\[ \limsup_{n \to \infty} n^{1/4} (\log n)^{-1/2} (\log \log n)^{3/4} \left| \frac{f(Q(I)) - f(Q(\theta_{l,n}))}{f(Q(\theta_{l,n}))} \right|_{0}^{\theta} \leq C K_{\theta} \limsup_{n \to \infty} (\log \log n)^{-1/2} (\log \log n)^{3/4} \left(\|b_n\|_{0}^{\theta}\right)^{1/2}, \]
which equals zero almost surely by application of (2.1).

Now suppose \(\lim_{x \to 0} f(x) = 0\). With the same method as above it is immediate that for any \(0 < \varepsilon < \theta\),
\[ n^{1/4} (\log n)^{-1/2} (\log \log n)^{1/4} \|R_n - r_n\|_{\varepsilon}^{\theta} \to 0 \quad a.s. \quad (2.2) \]
As already mentioned in Aly et al. [1, proof of Theorem 4.3], the proof of (3.3) in Csirogi and Révész [7] can be mimicked to show that for some $C_1 \in (0, \infty)$ and for "small" $\varepsilon > 0$

$$n^{1/4} (\log n)^{-1/2} (\log \log n)^{1/4} \| \beta_n - b_n \|_{\delta(n)} \to 0 \quad \text{a.s.,} \quad (2.3)$$

where $\delta(n) = C_1 n^{-1} \log \log n$, since for "small" $\varepsilon > 0$,

$$\sup_{0.0 < \lambda < \Theta(2\varepsilon)} F(x) \, f'(x) \, (f(x))^{-2} \leq 2a.$$ 

In Aly et al. [1] it is also shown that

$$\| b_n \|_{\delta(n)}^{\delta(n)} = O(n^{-1/2} \log \log n) \quad \text{as} \quad n \to \infty. \quad (2.4)$$

As $f \circ Q$ is regularly varying at zero with positive index $a$, one can construct a non-decreasing function $f_Q$ such that $f_Q \leq f \circ Q$ and $\lim_{x \downarrow 0} f_Q(x)/f(Q(x)) = 1$. (See, e.g., Theorem 1.5.3 in Bingham et al. [4].) Let $U_{(i)} \leq \cdots \leq U_{(n)}$ denote the order statistics of those observations among $U_1, \ldots, U_n$, which are uncensored, i.e., for which $\delta_j = 1$. Assume $\Gamma_n(U_{(i-1)}) < t \leq \Gamma_n(U_{(i)})$. Then for $t \leq U_{(i)}$ and $n$ large enough

$$|\beta_n(t)| \leq n^{1/2} \int_{U_{(i+1)}}^{U_{(i)}} \frac{f(Q(t))}{f(Q(u))} \, du \leq n^{1/2} \int_{U_{(i+1)}}^{U_{(i)}} \frac{f(Q(t))}{f_Q(u)} \, du \leq \sup_{t \in (0, \delta(n))} \frac{f(Q(t))}{f_Q(t)} \cdot n^{1/2} \int_{U_{(i)}}^{U_{(i+1)}} \frac{f_Q(t)}{f_Q(u)} \, du \leq 2b_n(t). \quad (2.5)$$

The case $t > U_{(i)}$ can be handled along the lines of (3.14) in Csirogi and Révész [7], cf. Aly et al. [1, pp. 200–201]. From this remark, (2.5) in combination with (2.4), and (2.4) itself, we have

$$\| \beta_n - b_n \|_{\delta(n)}^{\delta(n)} \leq \| \beta_n \|_{\delta(n)}^{\delta(n)} + \| b_n \|_{\delta(n)}^{\delta(n)} \overset{\text{a.s.}}{=} O(n^{-1/2} (\log n)^{2a}).$$

Combining this with (2.2) and (2.3) completes the proof for the case $\lim_{x \downarrow 0} f(x) = 0$ and hence of the lemma. 

**Lemma 3.** Under the conditions of Theorem 3 we have, as $n \to \infty$,

$$n^{1/4} (\log n)^{-1/2} \| R_n - r_n \|_{\delta(1 - G)}^{\delta(1 - G)} \to 0 \quad \text{a.s.;}$$

$$\left\{ \frac{\| r_n \|_{\delta(1 - G)}^{\delta(1 - G)}}{(\| \beta_n \|_{\delta(n)}^{\delta(n)})^{1/2}} \right\} \left\{ \frac{\| R_n \|_{\delta(1 - G)}^{\delta(1 - G)}}{(\| b_n \|_{(1 - G)}^{\delta(n)})^{1/2}} \right\} \to 0 \quad \text{a.s.} \quad (2.6)$$
Proof. Combination of Lemma 2 and
\[ \lim \inf_{n \to \infty} (\log \log n)^{1/2} \|b_n\|_0^d > 0 \quad \text{a.s.} \quad (2.7) \]
(see Fact 3 below) yields the first statement in (2.6). To prove the second statement it suffices to show that
\[ \left\{ (\|b_n/(1-G)\|_0^d)^{1/2} - (\|\beta_{n,G}\|_0^d)^{1/2} \right\} / (\|\beta_{n,G}\|_0^d)^{1/2} \to 0 \quad \text{a.s.} \]
Using \( x^{1/2} - y^{1/2} = (x - y)/(x^{1/2} + y^{1/2}) \), \( x, y > 0 \), (2.7), and again Lemma 2, the proof reduces to showing
\[ (\log \log n)^{1/2} \|\beta_n - b_n\|_0^d \to 0 \quad \text{a.s.}, \]
which follows from one more application of Lemma 2. \( \Box \)

From Lemma 1 and Lemma 3, it follows that we can confine ourselves to the proofs of Theorems 1–2 and Theorem 3, respectively, in case the \( X_i \)'s are uniformly \((0, 1)\) distributed and the \( Y_i \)'s are distributed according to a \( dG \) (which is now shorthand for \( G \circ Q \)) with support on \((0, 1)\). Observe that \( G \circ Q \) is continuous, since \( F \) is strictly increasing. We also adopt the notation introduced at the beginning of this section.

The remainder of this paper is organized as follows. We begin by recording a number of facts, which are required for the proofs. After that, we give a detailed proof of Theorem 3. Finally, short proofs of Theorems 1 and 2 are presented.

Fact 1 (Burke, Csörgő, and Horváth [5], Major and Rejtő [12]). There exists a two-parameter standard Wiener process \( W \) such that, for any \( \theta \in (0, T_\circ) \),
\[ \|a_n - n^{-1/2}(1-I)W(h(I), n)\|_0^d = O(n^{-1/2}(\log n)^2) \quad \text{a.s.} \quad (2.8) \]
Define a sequence \( \{W_n\}_{n=1}^\infty \) of (one-parameter) standard Wiener processes by
\[ W_n = n^{-1/2}W(I, n), \quad (2.9) \]
write
\[ A_n = (1-I)W_n \circ h, \quad (2.10) \]
and note that for all \( n \in \mathbb{N} : A_n \overset{d}{=} \Lambda, \) with \( \Lambda \) as in Section 1.

Fact 2 (cf. Shorack [15]). Let \( W_n \) be as above, \( c \in (0, \infty) \) arbitrary, and \( \{k_n\}_{n=1}^\infty \) a sequence of positive numbers such that \( k_n \downarrow, nk_n \uparrow, \log(1/k_n)/\log \log n \to \infty \) and \( \log(1/k_n)/(nk_n) \to 0 \). Then
\[ \limsup_{n \to \infty} \sup_{0 \leq u < c, v \geq 0} \frac{|W_n(u) - W_n(v)|}{(2k_n \log(1/k_n))^{1/2}} \leq 1 \quad \text{a.s.} \quad (2.11) \]
and

$$\limsup_{n \to \infty} \sup_{|u-v| > k_n} \frac{|W_n(u) - W_n(v)|}{(2 |u-v| \log(1/k_n))^{1/2}} \leq 1 \quad \text{a.s.} \quad (2.12)$$

**Fact 3** (Aly, Csörgő, and Horváth [1]). We have almost surely

$$\limsup (\log \log n)^{-1/2} \left( \frac{b_n}{1-G} \right)^\theta_0 = 2^{1/2} \left( \frac{1-I}{1-G} \right)^\theta_0$$

and

$$\pi \delta^{-1/2} (1-\theta) h^{1/2}(\theta) \leq \liminf (\log \log n)^{1/2} \left( \frac{b_n}{1-G} \right)^\theta_0$$

$$\leq \pi \delta^{-1/2} \frac{h^{1/2}(\theta)}{1-G(\theta)}. \quad (2.14)$$

**Proof of Theorem 3.** The proof of the upper bound part is an adaptation of that in Shorack [15], whereas the proof of the lower bound part is based on Deheuvels and Mason [8]. We first show that if $0 < \theta < T_G$,

$$LS := \limsup_{n \to \infty} n^{1/4}(\log n)^{-1/2} \left( r_n \right)^\theta_0 / (\left( b_n / (1-G) \right)^\theta_0)^{1/2} \leq 1 \quad \text{a.s.} \quad (2.15)$$

Note that for any $s \in [0, \theta]$,

$$r_n(s) = a_n(s) - a_n(q_n(s)) + n^{1/2} (\Gamma_n(q_n(s)) - s).$$

In Sander [13] it is shown that

$$n^{1/2} \left\| \Gamma_n \circ q_n - I \right\|^\theta_0 = O(n^{-1/2}) \quad \text{a.s.}$$

Hence, (2.8) and (2.14) entail that

$$LS = \limsup_{n \to \infty} \frac{n^{1/4} \left\| A_n - A_n \circ q_n \right\|^\theta_0}{(\log n / (1-G))^{\theta_0}}^{1/2} \quad \text{a.s.}$$

Let

$$k_n = \pi (1-\theta) h^{1/2}(\theta) / (8 \log \log n)^{1/2},$$

$$I_n = \{(s, t): s \geq 0, 0 \leq t \leq \theta, |h(t) - h(s)| \leq \left\| h \circ q_n - h \right\|^\theta_0,$$

$$\leq (1-t)^2 \left\| h(t) - h(s) \right\| \leq \left(1-I\right)^2 \left(h \circ q_n - h\right)^\theta_0\},$$

$$J_n = \{(s, t) \in I_n: |h(t) - h(s)| \leq k_n\},$$

$$K_n = \{(s, t) \in I_n: |h(t) - h(s)| > k_n\}.$$
Then almost surely

\[
LS \leq \limsup_{n \to \infty} \sup_{(s,t) \in I_n} \frac{|A_n(t) - A_n(s)|}{(\log n \| (q_n - I)/(1 - G)\|_0^0)^{1/2}}
\]

\[
\leq \limsup_{n \to \infty} \sup_{(s,t) \in I_n} \frac{|t - s| |W_n(h(s))|}{(\log n \| (q_n - I)/(1 - G)\|_0^0)^{1/2}}
\]

\[
+ \limsup_{n \to \infty} \sup_{(s,t) \in I_n} \frac{(1 - t) |W_n(h(t)) - W_n(h(s))|}{(\log n \| (q_n - I)/(1 - G)\|_0^0)^{1/2}}
\]

\[
= LS_1 + \limsup_{n \to \infty} \sup_{(s,t) \in I_n} A_n(s, t) = LS_1 + LS_2.
\]

It is well known that for arbitrary \(0 < c < \infty\),

\[
\|W_n\|_0^0 = O((\log \log n)^{1/2}) \quad \text{a.s.} \quad (2.17)
\]

From (2.13) we obtain

\[
\|h \circ q_n - h\|_0^0 \leq \|q_n - I\|_0^0 \|h\|_0^0 \|q \circ (\theta')\|_0^0 \text{ a.s.} \quad = O(n^{-1/2}(\log \log n)^{1/2}). \quad (2.18)
\]

Hence from (2.17), (2.18), and (2.14) we have a.s. as \(n \to \infty\),

\[
\sup_{(s,t) \in I_n} \frac{|t - s| |W_n(h(s))|}{(\log n \| (q_n - I)/(1 - G)\|_0^0)^{1/2}} = O(n^{-1/4}(\log n)^{-1/2} (\log \log n)^{5/4}),
\]

implying that \(LS_1 = 0\) a.s. Furthermore,

\[
LS_2 = \limsup_{n \to \infty} \sup_{(s,t) \in I_n} A_n(s, t) + \limsup_{n \to \infty} \sup_{(s,t) \in K_n} A_n(s, t)
\]

\[
= LS_3 + LS_4.
\]

First, by (2.14) we have a.s.

\[
LS_3 \leq \limsup_{n \to \infty} \sup_{(s,t) \in I_n} \frac{(1 - t) |W_n(h(t)) - W_n(h(s))|}{(k_n \log n)^{1/2}}
\]

\[
\leq \limsup_{n \to \infty} \sup_{|u - v| \leq k_n, 0 \leq u \leq h(\theta), v \geq 0} \frac{|W_n(u) - W_n(v)|}{(2k_n \log(1/k_n))^{1/2}}
\]

as \(\log(1/k_n)/\log n \to 1/2\) as \(n \to \infty\). Hence \(LS_3 \leq 1\) a.s., because of (2.11).

Next, since \(G\) is continuous \(\| (q_n - I)/(1 - G)\|_0^0 \sim \| (h \circ q_n - h)(1 - I)^2\|_0^0\)

a.s. as \(n \to \infty\), so that
\[ L S_4 = \limsup_{n \to \infty} \sup_{(\epsilon, t) \in K_n} \frac{(1 - t) |W_n(h(t)) - W_n(h(s))|}{(\log n \| (h \circ q_n - h)(1 - J)^2\|_0)^{1/2}} \]

\[ \leq \limsup_{n \to \infty} \sup_{(\epsilon, t) \in K_n} \frac{|W_n(h(t)) - W_n(h(s))|}{|h(t) - h(s)|^{1/2} (\log n)^{1/2}} \]

\[ \leq \limsup_{n \to \infty} \sup_{\substack{|u - v| > K_n \\ 0 \leq u, v \leq 2h(\theta)}} \frac{|W_n(u) - W_n(v)|}{|u - v|^{1/2} (2 \log(1/k_n))^{1/2}} \quad \text{a.s.} \]

Applying (2.12) yields \(LS_4 \leq 1\) a.s. Hence the proof of (2.15) is completed.

Now it remains to show that if \(0 < \theta < T_G\),

\[ LI := \liminf_{n \to \infty} n^{1/4}(\log n)^{-1/2} \|r_n\|_0^\theta /\|b_n/(1 - G)\|_0^{\theta/2} \geq 1 \quad \text{a.s.} \quad (2.19) \]

Using similar steps as in the upper bound part of this proof we find that if suffices to show that

\[ \liminf_{n \to \infty} n^{1/4}(\log n)^{-1/2} \| (1 - J)(W_n \circ h - W_n \circ h \circ q_n) \|_0^\theta \geq 1 \quad \text{a.s.} \]

Let

\[ h(q_n(t)) = h(t) + n^{-1/2}b_n(t)h'(a(n, t)), \quad (2.20) \]

where \(|a(n, t) - t| \leq n^{-1/2} |b_n(t)|\). Then, with \(h^\prime\) denoting the inverse of \(h\),

\[ LI = \liminf_{n \to \infty} \frac{n^{1/4}}{(\log n)^{1/2}} \quad \times \quad \frac{\| (1 - h^\prime)(W_n - W_n \circ (I + n^{-1/2}(b_n \circ h^\prime)h'(a(n, h^\prime)))) \|_0^{\theta(\theta)}}{\|b_n \circ h^\prime\|_0^{\theta(\theta)}} \quad \text{a.s.} \]

Write for \(v \in [0, n]\),

\[ \psi_{\theta, n}(v) = h^\prime(vh(\theta)/n), \]

\[ \pi_n(v) = 1 - \psi_{\theta, n}(v), \quad (2.21) \]

\[ f_n(v) = (n^{1/2}/h(\theta)) b_n(\psi_{\theta, n}(v)) h'(a(n, \psi_{\theta, n}(v))), \]

and observe that for any standard Wiener process \(W_n\), the process \(\tilde{W}_n\) defined by

\[ \tilde{W}_n(v) = (n/h(\theta))^{1/2} W_n(vh(\theta)/n), \quad v \geq 0, \]
is again a standard Wiener process. So by changing variables 
\( v = (n/h(\theta)) t \), we obtain

\[
LI = \lim_{n \to \infty} \inf \frac{\| \pi_n(\overline{W}_n - \overline{W}_n \circ (I + f_n)) \|^n_0}{(\log n)^{1/2} (\| \pi_n f_n \|^n_0)^{1/2}} \quad \text{a.s.} 
\]

(2.22)

Now to show that the right side of (2.22) is not smaller than one a.s., we 
can make use of the following proposition, which constitutes a generaliza-
tion of Proposition 1 in Deheuvels and Mason [8]. In our proposition we 
will abuse notation by again using sequences of functions \( \{\pi_n\}_{n=1}^\infty \) and 
\( \{f_n\}_{n=1}^\infty \) and a sequence \( \{\overline{W}_n\}_{n=1}^\infty \) of Wiener processes. These sequences 
are defined below and are not related to the above sequences with the same 
names. However, we will apply the Proposition with \( \pi_n, f_n, \) and \( \overline{W}_n \) 
as above. Let \( \{\pi_n\}_{n=1}^\infty \) be a sequence of decreasing functions satisfying:

\((\pi_1)\) there exists some \( c > 0 \), such that

\[
c \leq \| \pi_n \|^n_0 \leq 1 \quad \text{for all} \quad n \geq 1,
\]

\((\pi_2)\) \( \lim \sup n \| \pi_n \|^n_0 < \infty \).

For any \( \gamma > 1, a > 1, \eta > 0, \nu \geq 1 \) we denote by \( \mathcal{F}_\gamma(\gamma, a, \eta, \nu) \) the subclass of 
all sequences \( \{f_n\}_{n=1}^\infty \) of real-valued functions defined on \( [0, \infty) \) such that

\((F_1)\) for all \( n \geq 3, \)

\[
\gamma^{-1}n^{1/2}/\log^2 n \leq \| \pi_n f_n \|^n_0 \leq \gamma n^{1/2} \log^2 n,
\]

\((F_2)\) for all \( n \geq \nu, \)

\[
M_n(\pi_n f_n) := \max \{ \inf_{s \in I_n} \pi_n^2(s) f_n(s), \inf_{s \in I_n} (-\pi_n^2(s) f_n(s)) \} \geq a^{-1} \| \pi_n f_n \|^n_0,
\]

for some closed interval \( I_n \subset [0, n] \) of length \( \eta n^{-\log \log n^2} \),

\((F_3)\) for all \( n \geq 1, 0 \leq s + f_n(s) \) for \( s \in [0, n] \).

Let \( \mathcal{F}_\pi = \cap_{a>1} (\bigcup_{\gamma>0} \bigcup_{\eta>0} \bigcup_{\nu \geq 1} \mathcal{F}_\gamma(\gamma, a, \eta, \nu)) \). (Here \( \gamma, a, \eta \) are assumed 
to be rational and \( \nu \) an integer.)

**Proposition.** With the above notation and \( \{\overline{W}_n\}_{n=1}^\infty \) being any sequence 
of standard Wiener process on \( [0, \infty) \) sitting on a joint probability space, we 
have with probability one for all sequences \( \{f_n\}_{n=1}^\infty \in \mathcal{F}_\pi, \)

\[
\lim \inf_{n \to \infty} R_n(\pi_n, f_n) \geq 1,
\]

where \( R_n(\pi_n, f_n) = \left\| \pi_n f_n \right\|^n_0 \log n \right\|^{1/2} \| \pi_n(\overline{W}_n \circ (I + f_n) - \overline{W}_n) \|^n_0.\)
Proof. Choose any \( \{f_n\}_{n=1}^{\infty} \in \mathcal{F}_n(\gamma, a, \eta, v) \), where \( \gamma > 1, a > 1, \eta > 0 \) are rationals and \( v \) is a positive integer. Define

\[
h_n(k) = \gamma^{-1} a^k n^{1/2}/\log^2 n
\]

for \( k = -3, -2, -1, 0, \ldots, k(n) := \lceil \log_a (c^{-2} \gamma^2 \log^4 n) \rceil + 1 \)

and

\[
I_n(m) = \lceil mn \exp(\log(n)^2), (m+1) \delta n \exp(\log(n)^2) \rceil
\]

for \( m = 0, 1, \ldots, m(n) := \lceil \delta^{-1} \exp((\log(n)^2) \rceil + 1, \)

with \( \delta = \eta/6 \). Let \( I_n = [\lambda_n, \rho_n] \). By (F,1) and (π1), for all \( n \geq 3 \) we can find an \( 0 < l \leq k(n) \) such that

\[
h_n(l_n - 1) \leq \|\pi_n^2 f_n\|_0^2 / \pi_n^2(\rho_n) \leq h_n(l_n). \tag{2.23}
\]

Hence by (F,2), for all \( n \geq \max(v, 3) \),

\[
\pi_n^2(\rho_n) h_n(l_n - 2) \leq a^{-1} \|\pi_n^2 f_n\|_0^2 \leq M_n(\pi_n^2 f_n) \leq \|\pi_n^2 f_n\|_0^2 \leq \pi_n^2(\rho_n) h_n(l_n). \tag{2.24}
\]

Now

\[
R_n(\pi_n, f_n) \geq \sup_{s \in I_n} \left\{ \sup_{s \in I_n} |W_n(s) + f_n(s) - W_n(s)| / ((\log(n) h_n(l_n)))^{1/2} =: A_n. \tag{2.25}
\]

Furthermore by (π1) and (π2) there exist \( K > 0, v_1 > 1 \) such that for \( n \geq v_1 \),

\[
\pi_n(\lambda_n) / \pi_n(\rho_n) = 1 + \left\{ \pi_n(\lambda_n) - \pi_n(\rho_n) \right\} / \pi_n(\rho_n)
\leq 1 + |c^{-1}(\rho_n - \lambda_n) \pi_n(\theta_n)|
\leq 1 + K\eta e^{-(\log\log n)^2}/n \leq a^{1/2}, \tag{2.26}
\]

with \( \lambda_n \leq \theta_n \leq \rho_n \). Also we may choose an \( 1 \leq m \leq m(n) \) such that

\[
I_n(m - 1) \cup I_n(m) \subset I_n. \tag{2.27}
\]

Suppose first that \( M_n(\pi_n^2 f_n) = \inf_{s \in I_n} (\pi_n^2(s) f_n(s)) \). Then by (2.23), (2.24), and (2.26) we have for all \( n \geq v_1 \) and \( s \in I_n \),

\[
h_n(l_n - 3) \leq h_n(l_n - 2) \pi_n^2(\rho_n) / \pi_n^2(\lambda_n)
\leq M_n(\pi_n^2 f_n) / \pi_n^2(\lambda_n) \leq f_n(s)
\leq \|\pi_n^2 f_n\|_0^2 / \pi_n^2(\rho_n) \leq h_n(l_n). \tag{2.28}
\]
So for $s \in I_n$ and $n$ large enough,

$$|f_n(s) - h_n(l_n)| \leq (1 - a^{-3}) h_n(l_n)$$

and, thus,

$$\sup_{s \in I_n} |\tilde{W}_n(s + f_n(s)) - \tilde{W}_n(s)| \geq \sup_{s \in I_n} |\tilde{W}_n(s + h_n(l_n)) - \tilde{W}_n(s)|$$

$$- \sup_{0 \leq s \leq 2n} \sup_{0 \leq t \leq \epsilon h_n(l_n)} |\tilde{W}_n(s + t) - \tilde{W}_n(s)|,$$

where $\tau = 1 - a^{-3}$. Hence in this case,

$$A_n \geq A_n(a, \gamma, \delta) - D_n(a, \gamma, \tau),$$

where

$$A_n(a, \gamma, \delta) = \min_{-3 \leq k \leq k(n)} \min_{0 \leq s \leq 2n} \max_{0 \leq t \leq \epsilon h_n(l_n)} \sup_{s \in I_n} |\tilde{W}_n(s + h_n(k)) - \tilde{W}_n(s)|/(h_n(k) \log n)^{1/2}$$

and

$$D_n(a, \gamma, \tau) = \max_{-3 \leq k \leq k(n)} \sup_{0 \leq s \leq 2n} \sup_{0 \leq t \leq \epsilon h_n(l_n)} |\tilde{W}_n(s + t) - \tilde{W}_n(s)|/(h_n(k) \log n)^{1/2}.$$

Next, suppose $M_n(\pi_n^2 f_n) = \inf_{s \in I_n} (-\pi_n^2(s) f_n(s))$. Then similarly as in the preceding case one shows that for $s \in I_n$ and $n$ large enough,

$$|f_n(s) + h_n(l_n - 3)| \leq \tau h_n(l_n)$$

and $0 \leq s + f_n(s) \leq s - h_n(l_n - 3)$. Thus,

$$\sup_{u \in I_n} |\tilde{W}_n(u + f_n(u)) - \tilde{W}_n(u)| \geq \sup_{u \in I_n} |\tilde{W}_n(u - h_n(l_n - 3)) - \tilde{W}_n(u)|$$

$$- \sup_{0 \leq s \leq 2n} \sup_{0 \leq t \leq \epsilon h_n(l_n)} |\tilde{W}_n(s + t) - \tilde{W}_n(s)|.$$

Note that there exists $v_2 \geq \max(v, 3)$ such that for all $n \geq v_2$,

$$h_n(l_n) \leq h_n(k(n)) \leq \frac{1}{2} \delta_n \exp(- (\log \log n)^2). \quad (2.29)$$

Hence we have \{u = s + h_n(l_n - 3) : s \in I_n(m - 1)\} $\subset I_n(m - 1) \cup I_n(m) \subset I_n$, so that in the present case,

$$A_n \geq (h_n(l_n) \log n)^{-1/2} \left\{ \sup_{s \in I_n(m - 1)} |\tilde{W}_n(s + h_n(l_n - 3)) - \tilde{W}_n(s)| \right.$$ 

$$- \sup_{0 \leq s \leq 2n} \sup_{0 \leq t \leq \epsilon h_n(l_n)} |\tilde{W}_n(s + t) - \tilde{W}_n(s)| \right\}$$

$$\geq (h_n(l_n - 3)/h_n(l_n))^{1/2} (A_n(a, \gamma, \delta) - D_n(a, \gamma, \tau)).$$
Hence, in both cases possible we have
\[ A_n \geq a^{-3/2} A_n(a, \gamma, \delta) - D_n(a, \gamma, \tau). \]

Now from (slight modifications of) Lemmas 1 and 2 in Deheuvels and Mason [8] we obtain with probability one uniformly over all sequences \( \{f_n\}_{n=1}^\infty \in \mathcal{F}_n(\gamma, a, \eta, \nu) \),
\[ \liminf_{n \to \infty} R_n(\pi_n, f_n) \geq a^{-3/2} - 2(1 - a^{-1})^{1/2}. \quad (2.30) \]

Observing that the right side of (2.30) can be chosen arbitrary close to one for a suitable choice of \( a > 1 \) completes the proof. \( \square \)

Let us now finish the proof of Theorem 3. Observe that (\( \pi_1 \)) and (\( \pi_2 \)) are easily checked for \( \pi_n \) as defined in (2.21). So it suffices to verify the conditions (F,1), (F,2), and (F3) in case
\[ f_n(s) = (n^{1/2}/h(\theta)) b_n(\psi_{0,n}(s)) h'(a(n, \psi_{0,n}(s))) \]
\[ = s \cdot (n/h(\theta)) h(q_n(\psi_{0,n}(s))). \]

where the last equality follows from (2.20). So (F3) is immediate. To check (F,1) remark that for any \( 0 < \theta < T \),\( \quad(0 < u < \theta) \)

Using condition (\( \pi_1 \)) and Fact 3 we see that \( \{f_n\}_{n=1}^\infty \) satisfies (F,1) almost surely for \( \gamma \) large enough.

Finally, we show that \( \{f_n\}_{n=1}^\infty \) satisfies (F,2) almost surely. Let \( \kappa_n = \eta n \exp(- (\log \log n)^2) \). For any \( s, t \in [0, n] \),
\[ |\pi_n^2(t) f_n(t) - \pi_n^2(s) f_n(s)| \leq |(\pi_n^2(t) - \pi_n^2(s)) f_n(t)| + |\pi_n^2(s)(f_n(t) - f_n(t))| \]
\[ =: d_1(s, t) + d_2(s, t). \]

First,
\[ d_1(s, t) \leq 2 \|\pi_n\|_0^\gamma \|f_n\|_0^\gamma |\pi_n(t) - \pi_n(s)| \]
\[ \leq 2 c^{-2} \|\pi_n^2 f_n\|_0^\gamma \|\pi_n^2 \|_0^\gamma |t - s|, \]

where for the last inequality (\( \pi_1 \)) is used twice. Hence, uniformly over all intervals \( I_n \) of length \( \kappa_n \), we have a.s. as \( n \to \infty \) that
\[ \sup_{s, t \in I_n} d_1(s, t) = O((\kappa_n/n) \|\pi_n^2 f_n\|_0^\gamma) = o(\|\pi_n^2 f_n\|_0^\gamma). \]
Next by (π1) and standard manipulations
\[
d_2(s, t) \leq |f_n(t) - f_n(s)|
\]
\[
\leq \left(\frac{n^{1/2}}{h(\theta)}\right) \|b_n\|_0^\theta \left|h'(a(n, \psi_{n, \theta}(t))) - h'(a(n, \psi_{n, \theta}(s)))\right|
\]
\[
+ \left(\frac{n^{1/2}}{h(\theta)}\right) \|h^\theta \cdot q_n(\theta)\| \|b_n(\psi_{n, \theta}(t)) - b_n(\psi_{n, \theta}(s))\|
\]
\[
= d_5(s, t) + d_4(s, t).
\]
As \(h' \geq 1\) on \([0, \theta]\), it follows that \(\psi_{n, \theta}\) is a Lipschitz function: for all \(n \geq 1\) and all \(s, t \in [0, n]\),
\[
|\psi_{n, \theta}(t) - \psi_{n, \theta}(s)| \leq \left(h(\theta)/n\right)|t - s|.
\] (2.31)
Moreover, \(h'\) is uniformly continuous on \([0, \theta]\) for \(0 < \theta < T_G\), since \(G\) is assumed to be continuous. Hence, also using (2.13), we have uniformly over all intervals \(I_n\) of length \(\kappa_n\) that a.s. as \(n \to \infty\),
\[
\sup_{s, t \in I_n} d_5(s, t) = o(n^{1/2} \|b_n\|_0^\theta) = o(\|\pi_n f_n\|_0^n),
\]
where the last "equality" follows from the fact that \(\|\pi_n f_n\|_0^n \geq c n^{1/2} \|b_n\|_0^\theta / h(\theta)\).

Observe that for any \(s, t \in [0, n]\),
\[
d_4(s, t) \leq \left(\frac{n^{1/2}}{h(\theta)}\right) \|h^\theta \cdot q_n(\theta)\| \|r_n\|_0^\theta
\]
\[
+ n^{1/2} \|h^\theta \cdot q_n(\theta)\| \|a_n(\psi_{n, \theta}(t)) - a_n(\psi_{n, \theta}(s))\|
\]
\[
= d_5 + d_6(s, t).
\]
From (2.13), the upper bound part of this proof and the fact that \(h'\) is bounded from above on \([0, \theta]\) for \(0 < \theta < T_G\), we see that a.s. as \(n \to \infty\),
\[
d_5 = O\left(n^{1/4} (\log n)^{1/2} (\log \log n)^{1/4}\right) = o(\|\pi_n f_n\|_0^n),
\]
where for the last "equality" (Fₚ₁₁) is applied. From (2.31) we get with the help of Schäfer [14, Corollary 3.2] or Aly et al. [1, Theorem 2.1] that a.s. as \(n \to \infty\),
\[
\sup_{I_n: |I_n| = \kappa_n, s, t \in I_n} \sup_{I_n \subset [0, n]} d_6(s, t)
\]
\[
\leq n^{1/2} \|h^\theta \cdot q_n(\theta)\| \sup_{J_n: |J_n| = h(\theta) \kappa_n/n, u, v \in J_n} \sup_{J_n \subset [0, \theta]} |a_n(u) - a_n(v)|
\]
\[
= O\left(n^{1/2} (\log n)^{1/2} (\kappa_n/n)^{1/2}\right).
\]
As \((\kappa_n \log n)^{1/2} = o(\|\pi_n^2 f_n\|_0^n)\), we can conclude that uniformly over all intervals \(I_n = [0, n]\) of length \(\kappa_n\) we have a.s. as \(n \to \infty\),
\[
\sup_{s, t \in I_n} |\pi_n^2(s) f_n(s) - \pi_n^2(t) f_n(t)| = o(\|\pi_n^2 f_n\|_0^n).
\]
Hence \((F_n 2)\) holds almost surely \((a > 1\) arbitrary), finishing the proof. 

**Proof of Theorem 1.** The derivation of the limit finite dimensional distributions of \(r_n\) follows the lines of the proof of Theorem 3 in Beirlant et al. [3]. We only sketch the proof.

First with the help of approximation results (cf. Fact 1) one shows that it is possible to construct a sequence \(\{W_n\}_{n=1}^{\infty}\) of standard Wiener processes extended to \((-\infty, \infty)\), in such a way that for \(0 < s < \theta\), as \(n \to \infty\),
\[
n^{1/4} \left| r_n(s) - (1 - s) \left\{ W_n(h(s)) - W_n \left( h(s) - \frac{n^{-1/2} W_n(h(s))}{(1 - s)(1 - G(s))} \right) \right\} \right| = o_p(1).
\]
For any choice of \(k \geq 1\) and \(0 < s_1 < \cdots < s_k < \theta\) fixed, let
\[
W_n^{(i)}(x_i) = n^{-1/4} \left\{ W_n(h(s_i) + n^{-1/2} x_i) - W_n(h(s_i)) \right\}, \quad x_i \in \mathbb{R}, \quad i = 1, \ldots, k,
\]
and let
\[
V_n := (W_n - h)/((1 - I)(1 - G)).
\]
Using Lemma 2.2 in Beirlant et al. [3] one shows that as \(n \to \infty\),
\[
(W_n^{(1)}, ..., W_n^{(k)}, V_n) \xrightarrow{d} (W^{(1)}, ..., W^{(k)}, V),
\]
where \(W^{(1)}, ..., W^{(k)}\) are independent two-sided Wiener processes independent of \(V = d V_n\). To this end one only needs to check that
\[
n^{1/4} \right\{ \text{Cov} \left[ W_n(h(s) + n^{-1/2} x), - W_n(h(t))/((1 - t)(1 - G(t))) \right] \}
- \text{Cov} \left[ W_n(h(s)), - W_n(h(t))/((1 - t)(1 - G(t))) \right] \to 0 \quad \text{as} \quad n \to \infty
\]
for any \(s \in (0, \theta)\) and \(x \in \mathbb{R}\). From this weak convergence result one deduces that as \(n \to \infty\),
\[
(1 - s_1) W_n^{(1)}(V_n(s_1)), ..., (1 - s_k) W_n^{(k)}(V_n(s_k)) \xrightarrow{d} (1 - s_1) W^{(1)}(V(s_1)), ..., (1 - s_k) W^{(k)}(V(s_k)). \tag{2.32}
\]
Since the right side of (2.32) is equal in distribution to
\[
((1 - s_1) W^{(1)}(\bar{A}_G(s_1)/(1 - s_1)^2), ..., (1 - s_k) W^{(k)}(\bar{A}_G(s_k)/(1 - s_k)^2))
\]
\[
= (Z_1 |\bar{A}_G(s_1)|^{1/2}, ..., Z_k |\bar{A}_G(s_k)|^{1/2}), \tag{d}
\]
the result follows.
Proof of Theorem 2. This proof can be given along similar lines as that of the upper bound part of Theorem 3. However, it is simpler because no supremum (0 ≤ s ≤ θ) and no denominator \((\|\beta_n\|_\theta)^{1/2}\) is involved. Here follows a short proof.

Writing \(l_n = n^{1/4}/(\log \log n)^{3/4}\) we have for arbitrary \(ε > 0\), almost surely, the following string of (in)equalities:

\[
\limsup_{n \to \infty} l_n |r_n(s)| = \limsup_{n \to \infty} l_n |A_n(s) - A_n(q(s))| \\
\leq \limsup_{n \to \infty} l_n \sup_{t : |h(s) - h(t)| \leq |h(s) - h(q(s))|} |(1 - s) W_n(h(s)) - (1 - t) W_n(h(t))| \\
\leq \limsup_{n \to \infty} l_n (1 - s) \sup_{t : |h(s) - h(t)| \leq (1 + ε) 2h(s) \log \log n)^{1/2} n^{-1/2} (1 - G(s))^{-1/2} L(s).
\]

It is easily shown that

\[
L(s) \leq (1 + ε)^{1/2} 2^{3/4} h^{1/4}(s)(1 - s)^{1/2} (1 - G(s))^{-1/2} \quad \text{a.s.}
\]

Noting that \(ε > 0\) is arbitrary, yields the desired result. ■

Note added in proof. After completion of our paper, Paul Deheuvels informed us that he and Ming Gu did research on this subject too.

REFERENCES


