Bernoulli jets and the zero mean curvature equation

Enrico Valdinoci

Dipartimento di Matematica, Università di Roma Tor Vergata,
Via della Ricerca Scientifica, 1, I-00133 Roma, Italy
Received 28 January 2005; revised 26 August 2005
Available online 27 October 2005

Abstract
We consider an elliptic PDE problem related with fluid mechanics. We show that level sets of
rescaled solutions satisfy the zero mean curvature equation in a suitable weak viscosity sense. In
particular, such level sets cannot be touched from below (above) by a convex (concave) paraboloid
in a suitably small neighborhood.
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MSC: 35J60; 35J70; 35R35; 76M30; 76B10

Keywords: Variational and PDE models for fluid dynamics; $p$-Laplacian operator; Sliding methods; Geometric
and qualitative properties of solutions

“Ma, se il pilota avanza,
rapida si dilegua come parvenza vana…”
(Guido Gozzano)

1. Introduction

Given $p \in (1, +\infty)$, we consider here a problem driven by the $p$-Laplacian operator

$$\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$$

E-mail address: valdinoci@mat.uniroma2.it.
and related with fluid dynamics. Some mean curvature estimates on the level sets of the solutions will be obtained, following a technique recently developed in [14] for smooth phase transition models. The setting in which these mean curvature estimates arise is a weak, quantitative, viscosity sense. Roughly speaking, we will consider a homogeneous $\varepsilon$ rescaling of the solution and prove that the level sets of such rescaled solutions cannot be touched by a curved paraboloid in a neighborhood of order $\sqrt{\varepsilon}$.

As a notation, we will often denote a point $x \in \mathbb{R}^N$ by $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Quantities depending only on $N$, $p$ and $\omega$ will be referred to as (universal) constants. In these framework, we may now give a formal statement of the main result dealt with in this paper:

**Theorem 1.1.** Let $\omega > 0$ and $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ be so that:

\begin{align*}
    u(0) &= 0, \\
    |u| &\leq 1, \\
    \Delta_p u &= 0 \quad \text{for any } x \in \{|u| < 1\}.
\end{align*}

Let us suppose that the following two free boundary growth conditions hold:

- if there is an open ball $B_+ \subseteq \{|u| < 1\}$ touching $\partial\{|u| < 1\}$ at $x_+$, then
  \begin{equation}
  u(x_+ + t_j v_0) \geq 1 - \omega t_j - f_+(t_j),
  \end{equation}
  for an infinitesimal positive sequence $t_j \searrow 0^+$, where $v_0$ is the interior normal of $B_+$ at $x_+$ and $f_+ : (0, 1) \to \mathbb{R}$ is so that
  \begin{equation}
  \lim_{t \to 0^+} \frac{f_+(t)}{t} = 0;
  \end{equation}

- if there is an open ball $B_- \subseteq \{u = -1\}$ touching $\partial\{u < -1\}$ at $x_-$, then
  \begin{equation}
  u(x_- - t_j v_0) \geq -1 + \omega t_j - f_-(t_j),
  \end{equation}
  for an infinitesimal positive sequence $t_j \searrow 0^+$, where $v_0$ is the interior normal of $B_-$ at $x_-$ and $f_- : (0, 1) \to \mathbb{R}$ is so that
  \begin{equation}
  \lim_{t \to 0^+} \frac{f_-(t)}{t} = 0.
  \end{equation}

Assume also that $u$ satisfies the following decay property: there exists a universal $L > 0$ such that:

\begin{equation}
\text{if } \{u = 0\} \cap \{|x'| \leq l\} \subseteq \{x_N \geq -l/100\} \text{ then } u(x) = -1 \quad \text{for any } x \text{ so that } |x'| \leq l/2 \text{ and } x_N \leq -l/10,
\end{equation}
provided \( l \geq L \). Let \( \beta \in (0,1) \) and \( M \in \text{Mat}((N - 1) \times (N - 1)) \) with
\[
\text{tr} M > \beta \|M\| \quad \text{and} \quad \|M\| \leq \beta^{-1}.
\]

Let
\[
\Gamma := \left\{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \text{ s.t. } x_N = \frac{1}{2} x' \cdot M x' \right\}.
\]

Let also \( u_\varepsilon(x) := u(x/\varepsilon) \). Then, there exist a universal \( \beta^* > 0 \) and a function \( \sigma_0 : (0,1) \rightarrow (0,1) \) such that if \( \varepsilon \in (0, \sigma_0(\beta)) \) and \( \beta \in (0, \beta^*) \), then \( \Gamma \) cannot touch \( \{ u_\varepsilon = 0 \} \) by below in \( B_{\beta^{1/2}/\sqrt{\text{tr} M}} \): more explicitly,
\[
\{ u_\varepsilon = 0 \} \cap \left\{ x_N < \frac{1}{2} x' \cdot M x' \right\} \cap \left\{ |x| < \frac{\beta^{1/2} \varepsilon}{\sqrt{\text{tr} M}} \right\} \neq \emptyset.
\]

Using an informal language, one may describe the content of Theorem 1.1 by stating that solutions of the problem considered enjoy a weak viscosity zero mean curvature property, in the sense that the level set \( \{ u_\varepsilon = 0 \} \) cannot be touched from below by a convex paraboloid in a neighborhood of the origin (which gets small with \( \varepsilon \)). And, of course, an analogous statement holds for concave paraboloids touching from above.

It is known (see [4]) that level sets of rescaled minimizers approach a minimal surface in the \( \Gamma \)-convergence setting. Thus, in a way, we may think that Theorem 1.1 says that the level set \( \{ u_\varepsilon = 0 \} \) attains a zero mean curvature property (though in a weak, quantitative, viscosity sense) even “before” converging to a limit surface. The fact that level sets inherit further properties from the minimal surface limit case may be related with the flat regularity of low dimensional level sets first conjectured by De Giorgi (see [9]). Also, Theorem 1.1 here holds for more general solutions than minimizers, differently from the \( \Gamma \)-convergence results in [4], and it provides a geometric and quantitative connection between the problem discussed here and the minimal surfaces.

Let us now briefly discuss some physical motivation behind the model considered here. The problem dealt with in Theorem 1.1 is inspired by ideal fluid jets. For instance, if \( N = p = 2 \), then \( u \) may be seen as the stream function of a fluid, that is, the particles of the fluid move along the level sets \( \{ u = \theta \} \), for \( \theta \in (-1, 1) \). In this sense, (1.3) is just the continuity equation.

In this setting, we remark that the level sets of \( u \), which we study here, have some physical relevance, since the particles of the fluid move along them.

On the free boundary \( \partial \{ |u| < 1 \} \), Bernoulli’s law states that the speed of the fluid (which agrees with \( |\nabla u| \)), must be balanced by the exterior pressure. This is the physical meaning of (1.4) and (1.6), in the sense that these assumptions are just weak versions of the Bernoulli condition “\( |\nabla u| = \omega \) on the free boundary.”
More precisely, the solutions dealt with in Theorem 1.1 are related with the minimizers of a functional widely studied in the literature both from the pure and applied mathematics point of view. Namely, for \( \lambda > 0 \), let us define the following functional on \( W^{1,p}(\Omega) \):

\[
\mathcal{F}_\Omega(u) = \int_{\Omega} \left| \nabla u(x) \right|^p + \lambda \chi_{(-1,1)}(u(x)) \, dx.
\]

We remark that, as a matter of fact, the quantities \( \lambda \) and \( \omega \) will be related by (5.2). Here above and in the sequel, we use the standard notation for the characteristic function of a set \( E \), namely

\[
\chi_E(\xi) = \begin{cases} 1 & \text{if } \xi \in E, \\ 0 & \text{if } \xi \not\in E. \end{cases}
\]

The functional \( \mathcal{F} \) is a model for ideal fluid jets and cavitation problems (see, e.g., [1–3, 13]); roughly speaking, the “kinetic” part \( \left| \nabla u \right|^p \) leads to the PDE equation satisfied by the stream function of the ideal fluid, while the free boundary imposed by the discontinuity of the characteristic function yields the balance with the exterior pressure, according to Bernoulli’s law. We refer to the above cited papers for further discussions upon these facts. Also, similar functionals provide models for flame propagation, combustion and electrochemical processes (see, e.g., [7,8] and references therein).

In this setting, we derive from Theorem 1.1 the following result:

**Theorem 1.2.** Let \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \), \( |u| \leq 1 \), be a Class A minimizer of \( \mathcal{F} \), i.e., assume that

\[
\mathcal{F}_\Omega(u) \leq \mathcal{F}_\Omega(u + \phi),
\]

for any \( \phi \in C^\infty_0(\Omega) \), for any bounded domain \( \Omega \). Suppose that \( u(0) = 0 \). Let \( \beta \in (0,1) \) and \( M \in \text{Mat}((N-1) \times (N-1)) \) with

\[
\text{tr} M > \beta \| M \| \quad \text{and} \quad \| M \| \leq \beta^{-1}.
\]

Let \( u_\varepsilon(x) := u(x/\varepsilon) \) and

\[
\Gamma := \left\{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \text{ s.t. } x_N = \frac{1}{2} x' \cdot M x' \right\}.
\]

Then, there exist a universal \( \beta^* > 0 \) and a function \( \sigma_0 : (0,1) \to (0,1) \) such that if \( \varepsilon \in (0, \sigma_0(\beta)) \) and \( \beta \in (0, \beta^*) \), then \( \Gamma \) cannot lie below \( \{ u_\varepsilon = 0 \} \) in \( B_{\beta^*/\sqrt{\text{tr} M}} \) by touching at the origin.

Theorems 1.1 and 1.2 have also the following consequence: if \( \{ u_\varepsilon = 0 \} \) converges uniformly to a hypersurface, then this surface satisfies the zero mean curvature equation in the (standard) viscosity sense. We omit here the details on this, referring the interested reader
to Theorem 2.1 of [15] (see also [13] for conditions under which the uniform convergence of level sets holds).

The proof of the results of this paper is deeply inspired by the geometric construction developed in the masterpiece [14] (see also [15, 17] for related results). Also, Theorems 1.1 and 1.2 will be bridged together via some results in [7, 13]. The results obtained here play also an important rôle in deducing flatness regularity results of De Giorgi type in the Bernoulli jet framework (see [16]).

We organize this paper in the following way. In Section 2, we will construct suitable barriers, in order to estimate the curvatures of the level sets. Barriers are like ships and solutions are like the land where the ships are going to dock. Inspired by the case \( N = 1 \), one suspects that such lands look like flat hills of slope \( \omega \). Thus, a first ship will be constructed with a protruding zero level set (as a little rostrum), in such a way the dock will occur on it. The second ship is a modification of the flat distance function: on the one hand, the distance function is expected to play a rôle, since it encodes curvature information; on the other hand, we need to bend the distance function a bit to get apart from the free boundary. In Section 3, the ships will sail the sea to touch the land: barriers will be slid towards the solution to check the curvatures of its zero level set: a Comparison Principle of [6] will be employed for this. The proof of Theorem 1.1 will then follow at once, by a scaling argument presented in Section 4. The proof of Theorem 1.2 is contained in Section 5. Some elementary properties of the distance to paraboloids are given in full detail in Appendix A. The latter may very well be skipped by expert readers.

2. Useful barriers

Following the ideas in [14, 15], we now construct some barriers in order to trap our solution. Roughly, the crucial idea, which goes back to De Giorgi, is that one-dimensional solutions are the ones which encode much information of the system. We will therefore modify the one-dimensional broken line to get suitable (one-dimensional or rotation) supersolutions. For other heuristic justification of a similar construction, see also [15, Section 5].

The first barrier we introduce is smooth but on the levels 0 and ±1. This will confine touching points on these levels. Actually, a free boundary analysis will avoid touching points to occur at ±1-levels, and this will localize the touching points on the zero level set of the solution.

**Lemma 2.1.** Let \( \kappa, \kappa^* > 0 \) be suitably small and define

\[
\omega_{\pm} := \sqrt{(\omega \pm \kappa^*)^2 \pm 4\kappa} \quad \text{and} \quad a_{\pm} := \frac{\omega_{\pm} - \sqrt{\omega_{\pm}^2 \pm 4\kappa}}{2\kappa}.
\]

Let

\[
g(s) := \begin{cases} 
\omega_+s - \kappa s^2 & \text{if } s \in [0, a_+], \\
\omega_-s - \kappa s^2 & \text{if } s \in [a_-, 0), \\
-1 & \text{if } s \in (-\infty, a_-).
\end{cases}
\]
Let also $y \in \mathbb{R}^N$, $l > 1$ and

$$\Psi^{y,l}(x) := g(|x - y| - l),$$

for any $x \in B_{l+a_+}(y)$. Then, there exists a universal constant $\tilde{c} \in (0, 1)$, so that, if $l \geq 1/(\tilde{c}\kappa)$ and $\kappa, \kappa^* \in (0, \tilde{c})$,

$$\Delta_p \Psi^{y,l}(x) \leq -\tilde{c}\kappa < 0,$$

for any $x$ so that $\Psi^{y,l}(x) \neq 0, \pm 1$.

**Remark 2.2.** Note that $g$ is defined and continuous in $(-\infty, a_+]$, that

$$g(a_\pm) = \pm 1$$

and that $g$ is smooth (with $g' > 0$) except on the level sets $0$ and $-1$. Furthermore,

$$\lim_{\eta \to 0^+} g'(a_- + \eta) = \omega - \kappa^* < \omega < \omega + \kappa^* = \lim_{\eta \to 0^+} g'(a_+ - \eta).$$

Also,

$$a_\pm \sim \pm \frac{1}{\omega}$$

for small positive $\kappa$ and $\kappa^*$. We will freely use these elementary observations in the sequel. Notice also that the rôles of $\kappa$ and $\kappa^*$ with respect to $l$ is quite different: while $\kappa^*$ is $l$-independent, $\kappa$ will be taken of the order of $1/l$.

**Proof of Lemma 2.1.** We take $x$ in the interior of the domain of $\Psi^{y,l}$ such that $\Psi^{y,l}(x) \neq 0, \pm 1$ and we use the short hand notation $t := |x - y| - l$. Since $0 < |g(t)| < 1$, we have that

$$\frac{\omega}{2} \leq g'(t) \leq 2\omega,$$

if $\kappa$ and $\kappa^*$ are small enough. Then, a direct computation of the $p$-Laplacian shows that

$$\Delta_p \Psi^{y,l}(x) = (g'(t))^{p-2}((p - 1)g''(t) + (N - 1)g'(t)/|x - y|).$$

Also, if $\Psi^{y,l}(x) > -1$, then $|x - y| \geq l/2$ by construction. Thus,

$$\Delta_p \Psi^{y,l}(x) \leq (g'(t))^{p-2}\left(-2\kappa(p - 1) + \frac{4\omega(N - 1)}{l}\right),$$

from which the desired result follows. \qed
Lemma 2.3. Let \( \Omega \) be an open domain, let \( u \in W^{1,p}(\Omega) \cap C(\mathbb{R}^N) \) satisfy (1.2)–(1.4) and (1.6) and let \( \Psi^{y,l} \) be as in Lemma 2.1. Let us assume that \( \Psi^{y,l} \geq u \) in their common domain of definition. If \( x^* \in \Omega \) is so that \( \Psi^{y,l}(x^*) = u(x^*) \), then either \( x^* \) is in the interior of \( \{ u = -1 \} \) or \( u(x^*) = 0 \).

**Proof.** Assume that \( u(x^*) \neq 0 \). If also \( |u(x^*)| < 1 \), then \( \Delta_p \Psi^{y,l} < 0 = \Delta_p u \) and \( \nabla \Psi^{y,l} \neq 0 \) in a neighborhood of \( x^* \) (by Lemma 2.1), and a contradiction then follows from the Strong Comparison Principle (see, e.g., Theorem 1.4 in [6] or Theorem 3.2 in [15]). Thus, \( |u(x^*)| = 1 \). If \( x^* \) is in the interior of \( \{ u = 1 \} \), then \( u \equiv 1 \) in a neighborhood \( U \) of \( x^* \); hence, since \( \{ \Psi^{y,l} = 1 \} \) is an \((N-1)\)-dimensional sphere, it would exist \( \hat{x} \in U \) so that \( 1 = u(\hat{x}) > \Psi^{y,l}(\hat{x}) \), contradicting our hypothesis. Therefore, \( x^* \) is either in the interior of \( \{ u = -1 \} \), as claimed, or \( x^* \) lies on the free boundary \( \partial \{ u = \pm 1 \} \). We exclude this possibility by arguing as follows. First, we exclude that \( x^* \in \partial \{ u = 1 \} \). For this, we argue by contradiction and we assume that \( x^* \in \partial \{ u = 1 \} \). Then, from the free boundary growth (1.4), if \( v_0 \in S^{N-1} \) is the interior normal of \( \{ \Psi^{y,l} = 1 \} \) at \( x^* \) (note that \( v_0 \) points towards \( \{ u < 1 \} \), since \( u \leq \Psi^{y,l} \)), we have that

\[
u(x^* + t_j v_0) \geq 1 - \omega t_j - f_+(t_j),
\]

for an infinitesimal positive sequence \( t_j \searrow 0^+ \). On the other hand, recalling Remark 2.2, the fact that \( \Psi^{y,l}(x^*) = 1 \) gives that

\[
\Psi^{y,l}(x^* + t_j v_0) \leq 1 - \left( \omega + \frac{k^*}{2} \right) t_j.
\]

The fact that \( \Psi^{y,l} \geq u \) and the above estimates thus imply that

\[
1 - \omega t_j - f_+(t_j) \leq 1 - \left( \omega + \frac{k^*}{2} \right) t_j,
\]

which easily yields a contradiction with (1.5). This shows that \( x^* \) does not lie on the free boundary \( \partial \{ u = 1 \} \).

Thus, to complete the proof of the desired claim, we need to exclude that \( x^* \in \partial \{ u = -1 \} \). For this, assume, by contradiction, that \( x^* \in \partial \{ u = -1 \} \). Then, the fact that \( \Psi^{y,l} \geq u \) implies that also \( x^* \in \partial \{ \Psi^{y,l} = -1 \} \). Thus, let \( v_0 \) be the inner normal of \( \partial \{ \Psi^{y,l} = -1 \} \) at \( x^* \). Since \( \Psi^{y,l} \geq u \), it follows that \( v_0 \) points towards \( \{ u = -1 \} \). Thus, the free boundary growth (1.6) yields that

\[
u(x^* - t_j v_0) \geq -1 + \omega t_j - f_-(t_j)
\]

for \( t_j \searrow 0^+ \). Also, recalling Remark 2.2 again, we infer from the fact that \( x^* \in \partial \{ \Psi^{y,l} = -1 \} \) that

\[
\Psi^{y,l}(x^* - t_j v_0) \leq -1 + \left( \omega - \frac{k^*}{2} \right) t_j.
\]
Finally, the fact that $\Psi_{y,l} \geq u$, together with the above estimates, gives that

$$-1 + \omega t_j - f_-(t_j) \leq -1 + \left(\omega - \frac{k^*}{2}\right)t_j,$$

contradicting (1.7) and thus ending the proof of the desired result. \qed

We now introduce another barrier, in order to deal with the distance function and control the curvature of the level sets of the solution.

**Lemma 2.4.** Let $0 < \varepsilon \leq \sigma \leq \delta < 1$, $\xi \in \mathbb{R}^{N-1}$, $M \in \text{Mat}((N-1) \times (N-1))$. Let $\Gamma$ be the hypersurface defined as

$$\Gamma := \left\{ x_N = \frac{\varepsilon}{2} x' \cdot M x' + \sigma \xi \cdot x' \right\}$$

and assume that

$$\text{tr } M \geq \delta, \quad \|M\| \leq \frac{\delta}{3}, \quad |\xi| \leq \frac{\delta}{3}.$$

We define $d_{\Gamma}(x)$ as the signed distance from $x$ to $\Gamma$, with the assumption that $d_{\Gamma}$ is positive above $\Gamma$ with respect to the $e_N$-direction.

Let also $c_1 > 0$ be a suitably small constant. Let us define

$$b_{\pm} := -\omega + \sqrt{\omega^2 + 4c_1 \varepsilon \delta} \quad \text{and} \quad \tilde{g}(s) := \omega s + c_1 \varepsilon \delta s^2, \quad \text{for any } s \in [b_{\pm}, b_{\pm}].$$

Let us also define $\tilde{g}(s) = -1$ for any $s < b_{\pm}$. Then, there exists a function $\sigma_0 : (0, +\infty) \to (0, 1)$ such that, if $\varepsilon \leq \sigma \leq \sigma_0(\delta)$, then

$$\Delta_p \left( \tilde{g}(d_{\Gamma}(x)) \right) < -\tilde{c}_1 \varepsilon \delta < 0$$

at any point $x \in \mathbb{R}^N$ for which $|x'| \leq \sigma/\varepsilon$ and $d_{\Gamma}(x) \in (b_{\pm}, b_{\pm})$, for a suitable small positive constant $\tilde{c}$.

**Remark 2.5.** As usual, we denote by $\{e_1, \ldots, e_N\}$ the standard base of $\mathbb{R}^N$.

Note that $\tilde{g}$ is continuous in $(-\infty, b_{\pm}]$ and smooth (with $\tilde{g}' > 0$) in $(b_{\pm}, b_{\pm})$. Also, $b_{\pm} \sim \pm 1/\omega$ for small positive $c_1$, $\varepsilon$ and $\delta$. What is more, $\tilde{g}(b_{\pm}) = \pm 1$ and

$$\lim_{\eta \to 0^+} \tilde{g}'(b_{\pm} + \eta) = \sqrt{\omega^2 + 4c_1 \varepsilon \delta}.$$

In particular,

$$\lim_{\eta \to 0^+} \tilde{g}'(b_{\pm} - \eta) > \omega > \lim_{\eta \to 0^+} \tilde{g}'(b_{\pm} + \eta).$$
A second order Taylor expansion shows that

\[ b_+ \leq \frac{1}{\omega} - \frac{c_1 \varepsilon \delta}{\omega^3} + \text{const}\left(\frac{c_1 \varepsilon \delta}{\omega^5}\right), \]

thus

\[ b_+ < \frac{1}{\omega} \]

for small positive \( c_1, \delta \) and \( \varepsilon \). These elementary properties of \( \tilde{g} \) will be freely used in what follows.

**Proof of Lemma 2.4.** If \( d_\Gamma(x) \in (b_-, b_+) \),

\[ |d_\Gamma(x)| \leq \frac{2}{\omega}. \quad (2.1) \]

Also, if \( s := d_\Gamma(x) \),

\[ 2\omega \geq \tilde{g}'(s) \geq \frac{\omega}{2}, \quad (2.2) \]

for small \( c_1, \varepsilon \) and \( \delta \).

In an appropriate system of coordinates we have that

\[ D^2 d_\Gamma(x) = \text{diag}\left(\frac{-k_1}{1 - d_\Gamma k_1}, \ldots, \frac{-k_{N-1}}{1 - d_\Gamma k_{N-1}}, 0\right) \in \text{Mat}(N \times N), \quad (2.3) \]

where the \( k_i \)'s are the principal curvatures of \( \Gamma \) at the point where the distance is realized (note, indeed, that (2.1) implies that \( |d_\Gamma(x)| \) is way less than the radius of curvature of \( \Gamma \) and see [11, Section 14.6] for further details on the distance function). By construction,

\[ |k_i| \leq C_1(\delta)\varepsilon, \]

for a suitable \( C_1(\delta) \).

We denote by \( P \) the paraboloid describing \( \Gamma \), i.e.,

\[ P(x') := \frac{\varepsilon}{2} x' \cdot Mx' + \sigma \xi' \cdot x'. \]

If \( |x'| \leq \sigma/\varepsilon \) and \( d_\Gamma(x) \) is realized at \( \zeta \in \Gamma \), then (2.1) implies that \( |\zeta'| \leq 2\sigma/\varepsilon \), thus we may restrict our attention to such a domain. Therefore,

\[ |\nabla P| \leq \text{const}\frac{\sigma}{\delta} \quad (2.4) \]

is a small quantity. Therefore, by the mean curvature equation (see, for instance, [11, Eq. (14.103)]), it follows that
\[\sum_{i=1}^{N-1} k_i = \sum_{i=1}^{N-1} \partial_i \left( \frac{\partial_i P}{\sqrt{1 + |\nabla P|^2}} \right) = \frac{\Delta P}{\sqrt{1 + |\nabla P|^2}} - \frac{(D^2 P \nabla P) \cdot \nabla P}{(1 + |\nabla P|^2)^{3/2}} \geq \frac{1}{2} \Delta P - \text{const}|\nabla P|^2 \|D^2 P\|.
\]

Thus, by using also (2.1), (2.3) and (2.4), we infer that

\[\Delta d_F = \sum_{i=1}^{N-1} \frac{-k_i}{1 - d_F k_i} = -\sum_{i=1}^{N-1} k_i - \sum_{i=1}^{N-1} \frac{d_F k_i^2}{1 - d_F k_i} \leq -\frac{1}{2} \Delta P + C_2(\delta)(|\nabla P|^2 \|D^2 P\| + \varepsilon^2) \leq -\frac{\varepsilon \delta}{2} + C_3(\delta)(\varepsilon \sigma^2 + \varepsilon^2),\]

for suitable \(C_i(\delta)\). In particular,

\[\Delta d_F \leq -\frac{\varepsilon \delta}{4}.
\]

Then, a direct computation on the \(p\)-Laplacian and (2.2) give that

\[\Delta_p(\tilde{g}(d_F(x))) = (\tilde{g}'(d_F(x)))^{p-2}[(p-1)\tilde{g}''(d_F(x)) + \tilde{g}'(d_F(x))\Delta d_F(x)] \leq \left(\frac{\omega}{2}\right)^{p-2} \left[2c_1 \varepsilon \delta (p-1) - \frac{\omega \varepsilon \delta}{8}\right],\]

from which the desired result follows by taking \(c_1\) conveniently small.

\[\square\]

\textbf{Remark 2.6.} The last passage in the proof also gives a good hint on how such a barrier has been constructed: namely, the curvature of \(\Gamma\), which is of order \(-\varepsilon \delta\), is going to compensate the one of \(\tilde{g}\), which is of order \(c_1 \varepsilon \delta\).

In analogy with Lemma 2.3 we point out the following result for the barrier \(\tilde{g} \circ d_F\) constructed above. Though the proof is similar in spirit to the one of Lemma 2.3, we provide full details of it for the reader’s convenience.

\textbf{Lemma 2.7.} Let \(\Omega\) be an open domain, let \(u \in W^{1,p}(\Omega) \cap C(\mathbb{R}^N)\) satisfy (1.2)–(1.4) and (1.6) and let \(\Gamma\) and \(\tilde{g}\) be as in Lemma 2.4. Let us assume that \(\tilde{g} \circ d_F \geq u\) in their common domain of definition (which is supposed to be nonempty). Then, if \(x^* \in \Omega\) is so that \(\tilde{g}(d_F(x^*)) = u(x^*)\), then \(x^*\) lies in the interior of \(\{u = -1\}\).

\textbf{Proof.} Let us first observe that \(x^*\) cannot lie on the free boundary \(\partial\{u = \pm 1\}\). First, we show that \(x^* \notin \partial\{u = -1\}\). We argue by contradiction, assuming that \(x^* \in \partial\{u = -1\}\). Then, since \(\tilde{g} \circ d_F \geq u\), we have that also \(x^* \in \partial\{\tilde{g} \circ d_F = -1\}\). Thus, let \(v_0\) be the normal of \(\partial\{\tilde{g} \circ d_F = -1\}\) at \(x^*\) pointing towards \(\{\tilde{g} \circ d_F = -1\}\). The fact that \(\tilde{g} \circ d_F \geq u\) also
implies that \( v_0 \) points towards \( \{ u = -1 \} \). Thus, the fact that \( x^* \in \partial \{ u = -1 \} \) and the free boundary growth (1.6) yield that
\[
u_0(x_\ell - t_j v_0) \geq -1 + \omega t_j - f_-(t_j),
\]
with \( t_j \downarrow 0^+ \). Also, recalling Remark 2.5, the fact that \( x^* \in \partial \{ \tilde{g} \circ d_\Gamma = -1 \} \) gives that
\[
\tilde{g}(d_\Gamma(x^* - t_j v_0)) \leq -1 + \sqrt{\omega^2 - 2c_1 \varepsilon \delta t_j}.
\]
Using again that \( \tilde{g} \circ d_\Gamma \geq u \), one thus gets that
\[
-1 + \omega t_j - f_-(t_j) \leq -1 + \sqrt{\omega^2 - 2c_1 \varepsilon \delta t_j}.
\]
The latter estimate and (1.7) lead to a contradiction, thus \( x^* \notin \partial \{ u = -1 \} \).

We now show that \( x^* \notin \partial \{ u = 1 \} \). To see this, let us argue by contradiction and let us assume that \( x^* \in \partial \{ u = 1 \} \). By construction, \( x^* \) also belongs to the \((N-1)\)-dimensional surface
\[
\Sigma = \{ x \in \mathbb{R}^N \mid d_\Gamma(x) = b_+ \}.
\]
Let \( v_0 \in S^{N-1} \) be the interior normal of \( \Sigma \) at \( x^* \). Note that \( v_0 \) points towards \( \{|u| < 1\} \), since \( u \leq \tilde{g} \circ d_\Gamma \). Then, by the free boundary growth (1.4),
\[
u_0(x^* + t_j v_0) \geq 1 - \omega t_j - f_+(t_j),
\]
for an infinitesimal positive sequence \( t_j \). Also, by construction, \( u \leq \tilde{g} \circ d_\Gamma \) and thus
\[
\tilde{g}(d_\Gamma(x^* + t_j v_0)) \geq 1 - \omega t_j - f_+(t_j).
\]
By the elementary properties described in Remark 2.5, we also have that
\[
\tilde{g}(d_\Gamma(x^* + t_j v_0)) \leq 1 - \sqrt{\omega^2 + 2c_1 \varepsilon \delta t_j}
\]
if \( j \) is large enough. These estimates give that
\[
1 - \omega t_j - f_+(t_j) \leq 1 - \sqrt{\omega^2 + 2c_1 \varepsilon \delta t_j},
\]
which easily provides a contradiction with (1.5) for \( j \) large.

We thus have that \( x^* \) does not lie on the free boundary \( \partial \{ u = \pm 1 \} \). Furthermore, \( x^* \) cannot lie in the interior of \( \{ u = 1 \} \). To see this, let us assume, by contradiction, that \( x^* \in U \subset \{ u = 1 \} \), for a suitable open set \( U \). Then, since \( \Sigma \), as defined here above, is an \((N-1)\)-dimensional surface, there would exist \( x_0 \in U \) such that \( \tilde{g}(d_\Gamma(x_0)) < 1 \). Therefore,
\[
\tilde{g}(d_\Gamma(x_0)) < 1 = u(x_0),
\]
in contradiction with our assumptions.
We have thus proved that either $x^*$ is in the interior of $\{u = -1\}$ or $u(x^*) \in (-1, 1)$. The latter possibility, however, cannot hold, due to the Strong Comparison Principle (see, e.g., Theorem 1.4 in [6] or Theorem 3.2 in [15]).  

3. Sliding methods

We now use the barriers introduced in Section 2 to deduce an estimate on the curvature of the paraboloids which may touch our solution. In particular, we will show that a zero mean curvature property is attained by the level sets of our solution, though in a weak viscosity sense. Next result will play a crucial rôle in the proof of Theorem 1.1, which indeed will follow via a natural rescaling.

**Theorem 3.1.** Let

$$
\Omega := \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid |x'| < l, |x_N| < l\} \subset \mathbb{R}^N.
$$

Let $l, \theta, \delta > 0$ and $M_1 \in \text{Mat}((N-1) \times (N-1))$. Let $u$ be as in Theorem 1.1. Assume that $u(x) < 0$ for any $x = (x', x_N) \in \Omega$ so that

$$
x_N < \frac{\theta}{2l^2} x' \cdot M_1 x' + \frac{\theta}{l} \xi \cdot x'.
$$

Then, there exist a universal constant $\delta_0 > 0$ and a function $\sigma : (0, 1) \to (0, 1)$ so that, if

$$
\delta \in (0, \delta_0], \quad \delta \leq \theta, \quad \frac{\theta}{l} \in \left(0, \sigma(\delta)\right], \quad \|M_1\| \leq \frac{1}{\delta} \quad \text{and} \quad |\xi| \leq \frac{1}{\delta},
$$

then

$$
\text{tr} M_1 \leq \delta.
$$

**Proof.** The proof is similar to the one of Lemma 3.2.2 in [14] and Lemma 6.6 in [15]. However, several quantitative estimates here differ from similar ones in [14,15]. The main reason for such difference is that the smooth transitions considered in [14,15] lead to exponentially decaying barriers, while the barriers constructed here have “something like” a linear decay. Thence, due to the technicalities involved, we provide full details for the reader’s convenience.

We will apply Lemmas 2.1 and 2.4 by choosing

$$
\kappa := \frac{1}{cl} \quad \text{and} \quad \varepsilon := \frac{\theta}{2l^2}.
$$

(3.1)
Note that, by our assumptions, \( l \geq \delta / \sigma ( \delta ) \) may and will be assumed to be a large quantity. In particular, we may and do assume \( \kappa \) and \( \kappa^* \) to be suitably small with respect to \( \delta \). Let also
\[
l_+ := \frac{l}{4} + a_+,
\]
so that, by Lemma 2.1, the barrier \( \Psi^{y,l/4} \) is defined in \( B_{l_+}(y) \). Define also
\[
\Gamma_1 := \left\{ x = (x', x_N) \in \mathbb{R}^N \text{ s.t. } x_N = \frac{\theta}{2l^2} x' \cdot M_1 x' + \frac{\theta}{l} \xi \cdot x' \right\}.
\]
Let us make some elementary observations upon the above paraboloid. First of all, by construction, \( u \) is negative below \( \Gamma_1 \) in \( \Omega \). What is more, the principal curvatures of \( \Gamma_1 \) are bounded by \( \text{const} \sigma(\delta)/(l \delta) \): thus, if \( \sigma(\delta)/\delta \) is sufficiently small, then, given any \( \zeta \in \Gamma_1 \), there exists a ball of radius \( l/4 \) which touches \( \Gamma_1 \) from below at \( \zeta \). Given \( \zeta \in \Gamma_1 \), let \( \nu_\zeta \) be the normal direction of \( \Gamma_1 \) at \( \zeta \) pointing downwards. Let
\[
\mathcal{K} := \left\{ |x'| \leq \frac{l}{4} \right\} \cap \left\{ d_{\Gamma_1}(x) \in \left[ -\frac{l}{8}, a_+ \right] \right\}.
\]
We now claim that
\[
u(x) \leq g \left( d_{\Gamma_1}(x) \right)
\]
for any \( x \in \mathcal{K} \). To prove (3.4), first notice that, by construction, the zero level set of \( u \) is above \( \Gamma_1 \), hence above the hyperplane \( \{ x_N = -l/100 \} \); thus, from (1.8),
\[
u(x) < \Psi^{-(l/2)e_N,l/4}(x)
\]
for any \( x \) in their common domain of definition, provided
\[
\Psi^{-(l/2)e_N,l/4}(x) \neq -1.
\]
Then, for a given \( \zeta \in \Gamma_1 \) we define
\[
x_0 = x_0(\zeta) := \zeta + (l/4) \nu_\zeta,
\]
where we have denoted by \( \nu_\zeta \) the normal direction of \( \Gamma_1 \) at \( \zeta \) pointing downwards. In particular, from the above observation, it follows that
\[
B_{l/4}(x_0) \text{ touches } \Gamma_1 \text{ from below at } \zeta.
\]
Further, by construction,
\[
x_{0,N} \geq -|\xi_N| - \frac{l}{4} \geq -\frac{\text{const} \theta}{\delta} - \frac{l}{4} \geq -\frac{3l}{8}.
\]
We now slide the surface $\Psi^{-\left(l/2\right)eN,l/4}$ in the direction of the vector
\[ v = v(\zeta) := x_0 + \left(l/2\right)e_N, \] (3.8)
that is, we will consider the surface $\Psi^t := \Psi^{-\left(l/2\right)eN+tv,l/4}$ for $t > 0$. Note that
\[ v_N > 0, \] (3.9)
due to (3.7). We will show that
\[ \Psi^t(\tilde{x}) > u(\tilde{x}) \] (3.10)
for any $t \in [0,1)$ and any $\tilde{x}$ in their common domain of definition, provided $\Psi^t(\tilde{x}) \neq -1$. Indeed, as a consequence of Lemma 2.1, we have that $\Psi^t$ and $u$ cannot touch each other on the free boundary $\partial\{u = -1\}$. In the light of this observation, we may take $t \in [0,1)$ as the first time (if any) on which $\Psi^t$ touches $u$ at a point in $\{\Psi^t \neq -1\}$. We now show that
\[ \{\Psi^t < 0\} \text{ lies in } \{u < 0\} \] (3.11)
for any $t \in [0,1)$ (see Fig. 1). To prove this, by our assumptions, it is enough to prove that
\[ \{\Psi^t < 0\} \text{ lies below } \Gamma_1. \] (3.12)
The latter is confirmed via the following argument.

By the definitions in Lemma 2.1
\[ \{\Psi^t < 0\} = B_{l/4} \left(-\frac{l}{2}e_N + tv\right). \]
Thus, if \( z \in \{ \Psi^t > 0 \} \) and \( t \in [0, 1) \),

\[
z_N < -\frac{l}{2} + v_N + \frac{l}{4} = x_{0,N} + \frac{l}{4},
\]

thanks to (3.8) and (3.9). From this and (3.6), we conclude that \( z \) is below \( \Gamma_1 \), thus proving (3.12) and thence (3.11).

Thanks to these considerations, we have that \( u \) cannot be equal to \( \Psi^t \) and touching points between \( u \) and \( \Psi^t \) cannot occur on \( \{ \Psi^t = 0 \} \), if \( t \in [0, 1) \). On the other hand, Lemma 2.3 says that touching points cannot occur anywhere else. This proves (3.10).

We are now in the position to complete the proof of (3.4), by arguing as follows.

We deduce from (3.10) that \( \Psi^1(\tilde{x}) \geq u(\tilde{x}) \) for any \( \tilde{x} \) in the common domain of definition of \( \Psi^1 \) and \( u \), that is, for any \( \tilde{x} \in B_{l_+}(x_0) \). Take now any \( x \in K \) and let \( \zeta \) realize \( d_{\Gamma^1}(x) \). Let also \( x_0 \) be as in (3.5), so that \( x \in B_{l_+}(x_0) \): then,

\[
g(d_{\Gamma^1}(x)) = g(|x - x_0| - l/4) = \Psi^{x_0,l/4}(x) = \Psi^1(x) \geq u(x).
\]

This proves (3.4).

We now complete the proof of the desired result arguing by contradiction and supposing that \( \text{tr} M_1 > \delta \). We define

\[
\Gamma_2 := \left\{ x = (x', x_N) \in \mathbb{R}^N \text{ s.t. } x_N = \frac{\theta}{2l^2} x' \cdot M_1 x' + \frac{\theta}{l} \xi \cdot x' - \frac{\varepsilon \delta}{2(N - 1)} |x'|^2 \right\},
\]

where \( \varepsilon \) has been introduced in (3.1). By Lemma 2.4, we get that \( \tilde{g} \circ d_{\Gamma_2} \) is strictly \( p \)-superharmonic. We note that the quantities \( M, \sigma \) and \( \xi \) in Lemma 2.4 correspond here to \( 2M_1 - \delta/(N - 1) \), \( \theta/(2l) \) and \( 2\xi \), respectively.

Furthermore, by the definitions of \( \Gamma_1 \) and \( \Gamma_2 \), if \( |x'| = l/4 \) and \( |d_{\Gamma_2}(x)| \leq l/8 \), then

\[
d_{\Gamma_2}(x) \geq d_{\Gamma_1}(x) + c(\delta) \tag{3.13}
\]

for a suitable \( c(\delta) \in (0, 1) \) (see Fig. 2).

---

Fig. 2. The distance to the paraboloids \( \Gamma_1 \) and \( \Gamma_2 \).
Though quite elementary, we provide a full detail proof of (3.13) in Appendix A. The expert reader may certainly ignore Appendix A.

We now take $\kappa$ and $\kappa^*$ in Lemma 2.1 to be positive and suitably small (possibly in dependence of $\delta$) in such a way that

$$|\omega_{\pm} - \omega| \leq \frac{c(\delta)\omega^2}{4}. \tag{3.14}$$

We remark that $l \sim 1/\kappa$, thus the fact that $\kappa$ is small in dependence of $\delta$ is warranted by the fact that $1/l$ is small in dependence of $\delta$, by (3.1).

For further use, recalling Remark 2.2, we will also assume that

$$a_+ \geq \frac{1}{\omega} - \frac{c(\delta)}{2}.$$

We now recall Remark 2.5 and note that the latter assumption and (3.13) thus imply the following estimate: if $\hat{x}$ is so that $d_{\Gamma_2}(\hat{x}) \geq -l/8$, $d_{\Gamma_1}(\hat{x}) \geq a_+$ and $|\hat{x}'| = l/4$, then

$$d_{\Gamma_2}(\hat{x}) \geq a_+ + c(\delta) \geq \frac{1}{\omega} + \frac{c(\delta)}{2} > b_+, \tag{3.15}$$

which will be of later use.

The choice in (3.14) implies that, if $s$ is in the domain of $g$,

$$g(s) \leq \omega s + \frac{c(\delta)\omega^2|s|}{4} - \kappa s^2 \leq \omega s + \frac{c(\delta)\omega}{2}.$$

Thus, in the light of (3.13), we have that

$$g(d_{\Gamma_1}(x)) \leq \omega d_{\Gamma_1}(x) + \frac{c(\delta)\omega}{2} \leq \omega d_{\Gamma_2}(x) - \frac{c(\delta)\omega}{2} \leq \tilde{g}(d_{\Gamma_2}(x)) - \frac{c(\delta)\omega}{2}$$

for any $x$ for which the above functions are defined, so that $|x'| = l/4$.

We point out that if $x \in K$, then, by (3.3), we have that

$$d_{\Gamma_1}(x) \leq a_+$$

and so $d_{\Gamma_1}(x)$ is in the domain of $g$. From this observation, (3.4) and (3.16), we gather that

$$u(x) < \tilde{g}(d_{\Gamma_2}(x)), \tag{3.17}$$

for any $x \in K$ so that $|x'| = l/4$ and $d_{\Gamma_2}(x)$ is in the domain of $\tilde{g}$.

With these estimates, we are now ready to deduce the contradiction that will finish the proof of the desired result. To this end, we define

$$K^* := \left\{ |x'| \leq \frac{l}{4} \right\} \cap \left\{ d_{\Gamma_1}(x) \geq -\frac{l}{8} \right\}.$$
Note that, by (3.3), $\mathcal{K}^* \supset \mathcal{K}$. Recalling Lemma 2.7, we slide $\tilde{g} \circ d\Gamma_2$ from $-\infty$ in the $e_N$-direction until we touch $u$ in the closed domain $\mathcal{K}^*$ at some point, say $x^*$, with $u(x^*) > -1$. Note that this touching must indeed occur sooner or later, since $u(0) = 0$, due to (1.1).

In order to deal with this touching point, we consider, for $t \in \mathbb{R}$,

$$g^t(x) := \tilde{g}(d\Gamma_2(x - te_N))$$

and we increase $t$ from $-\infty$. By Lemma 2.7, we have that the first touching points between $g^t$ and $u$ at a level greater than $-1$ must occur on $\partial \mathcal{K}^*$. Note that, by definition, $\partial \mathcal{K}^*$ is composed by two parts: the “side,” given by the cylinder $|x'| = l/4$ and the “bottom,”

$$\{d\Gamma_1(x) = -l/8\}.$$

We now show that no first touching points between $g^t$ and $u$ at a level greater than $-1$ may occur on $\partial \mathcal{K}^*$ and this will give the desired contradiction.

By the definition of $\Gamma_1$, one sees that the bottom of $\mathcal{K}^*$ lies in $\{x_N \leq -l/9\}$, thus, by (1.8), $u = -1$ on the bottom of $\mathcal{K}^*$. This excludes that first touching points between $g^t$ and $u$ at a level greater than $-1$ may occur on the bottom of $\partial \mathcal{K}^*$.

But these touching points cannot occur on the side of $\partial \mathcal{K}^*$ either, thanks to the following argument. We assume, by contradiction that there exists a first touching point $x^*$ between $g^t$ and $u$ lying on the side of $\partial \mathcal{K}^*$ (that is, $|(x^*)'| = l/4$). There are two cases: either $d\Gamma_1(x^*) \leq a_+$ or the converse. Let us first assume that $d\Gamma_1(x^*) \leq a_+$. Then, by (3.3), $x^* \in \mathcal{K}$. Observe also that the fact that $u(0) = 0$ implies $t \leq 0$; thus, an elementary observation gives that

$$d\Gamma_2(x^* - te_N) \geq d\Gamma_2(x^*).$$

But then, since $\tilde{g}$ is nondecreasing, we deduce from the fact that $x^* \in \mathcal{K}$ and (3.17) that

$$\tilde{g}(d\Gamma_2(x^* - te_N)) = g^t(x^*) = u(x^*) < \tilde{g}(d\Gamma_2(x^*)) \leq \tilde{g}(d\Gamma_2(x^* - te_N)).$$

This contradiction shows that only the second case may occur, i.e., $d\Gamma_1(x^*) > a_+$. But even this last case cannot hold. Indeed,

$$-1 < u(x^*) = g^t(x^*) = \tilde{g}(d\Gamma_2(x^* - te_N))$$

and so

$$d\Gamma_2(x^* - te_N) \geq b_- \geq -\frac{l}{8}. \quad (3.18)$$

In addition, since $t \leq 0$,

$$d\Gamma_1(x^* - te_N) \geq d\Gamma_1(x^*) > a_. \quad (3.19)$$
Also, by our assumptions,

\[ |(x^* - te_N)'| = |(x^*)'| = \frac{l}{4}. \quad (3.20) \]

Consequently, from (3.18)–(3.20) and (3.15), we get that

\[ d_{\Gamma_2}(x^* - te_N) > b_+ , \]

hence \( x^* \) would not lie in the domain of \( g^\prime \).
This contradiction concludes the proof of Theorem 3.1. \( \square \)

4. Proof of Theorem 1.1

The proof of Theorem 1.1 can be now completed by arguing as follows.
We will apply Theorem 3.1 by making use of the following choice of parameters:

\[ l := \frac{\beta}{\sqrt{\varepsilon \text{ tr } M}}, \quad \delta := \theta := \beta^2, \quad M_1 := \frac{1}{\text{ tr } M} M, \quad \xi := 0. \]

If, by contradiction, the claim of Theorem 1.1 were false, by scaling back and using the above parameters, we would have that \( \Gamma_1 \) touches the zero level set of \( u \) by below, where

\[ \Gamma_1 = \left\{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \text{ s.t. } x_N = \frac{\theta}{2l^2} x' \cdot M_1 x' + \frac{\theta}{l} \xi \cdot x' \right\}. \]

By Theorem 3.1, we gather that \( 1 > \delta \geq \text{ tr } M_1 = 1 \), which is the contradiction that proves Theorem 1.1. \( \square \)

5. Proof of Theorem 1.2

The proof of Theorem 1.2 will be accomplished once we show that either the function \( u \) in Theorem 1.2 or the function \( \tilde{u} := -u \) satisfies the assumptions of Theorem 1.1. Note, indeed, that \( \tilde{u} \) is a class A minimizer of \( F \) too. Namely, we have to prove the continuity of \( u \), (1.3), (1.4), (1.6) and (1.8) for either \( u \) or \( \tilde{u} \).

First of all, \( u \) is uniformly continuous by the celebrated result in [10]. More precisely, it is uniformly Lipschitz continuous: see [13, Theorem 2.1].

The proof of (1.3) is standard and we omit it.

In order to proof (1.4) we argue as follows. First observe that the continuity of \( u \) implies that the free boundary \( \partial\{u = -1\} \) is uniformly separated from the free boundary \( \partial\{u = 1\} \). Therefore, by elementary observations (see, e.g., Lemma 3.3 in [13]), there exists a universal \( \rho > 0 \) so that, if

\[ x_\pm \in \partial\{u = \pm 1\} = \partial\{\tilde{u} = \mp 1\}, \]
then $1 \mp u = 1 \pm \tilde{u}$ is a class A minimizer for the functional

$$\tilde{F}(w) = \int_{B_\rho(x_\pm)} |\nabla w(x)|^p + \lambda \chi(0, +\infty)(w(x)) \, dx.$$ 

Then, by [7, Theorem 7.1], if $x \in \mathbb{R}^N$ and $r > 0$ are so that $B_r(x) \subset B_\rho(x_\pm)$ and $B_r(x) \cap \partial \{\pm u < 1\} \neq \emptyset$, we have that

$$\sup_{B_r(x)} |\nabla u| \leq \omega + Cr^\alpha, \quad (5.1)$$

for suitable universal positive $C$ and $\alpha$ and

$$\omega := \left(\frac{\lambda}{p - 1}\right)^{1/p}. \quad (5.2)$$

In particular, if there is a ball $B \subseteq \{|u| < 1\}$ touching $\partial \{\pm u = 1\}$ at $x_\pm$, we define $v_0$ as the interior normal of $B$ at $x_\mp$, we fix small positive $s < t$ and $\eta$ and take $x := x_\pm + t v_0$, $y := x_\pm + s v_0$ and $r := t + \eta$. With this choice, both $y$ and $x_\pm$ are in $B_r(x)$, so (5.1) yields

$$|u(x) - u(y)| \leq (\omega + Cr^\alpha)|x - y| = (\omega + C(t + \eta)^\alpha)(t - s).$$

Sending $s$ and $\eta$ to zero, the continuity of $u$ implies that

$$1 \mp u(x_\pm + t v_0) \leq |u(x_\pm + t v_0) \mp 1| \leq \omega t + Ct^{1+\alpha},$$

thus

$$u(x_+ + tv_0) \geq 1 - \omega t - Ct^{1+\alpha} \quad \text{and} \quad \tilde{u}(x_- + tv_0) \geq 1 - \omega t - Ct^{1+\alpha}.$$

This proves (1.4) for both $u$ and $\tilde{u}$.

We now prove (1.6) by arguing in the following way. With no loss of generality, we may assume that $x_- = 0$ and $v_0 = e_N$. We define $U(x) := 1 + u(x)$ and

$$U_\rho(x) := \frac{U(\rho x)}{\rho}.$$

Then, exploiting (5.1)–(5.5) of [7], we have that, for $\rho \to 0^+$,

$$U_\rho \text{ converges in } C^\alpha_{\text{loc}} \text{ to a suitable } U_0, \quad (5.3)$$

for any $\alpha \in (0, 1)$, while

$$\partial \{U_\rho > 0\} \text{ converges locally in the Hausdorff distance to } \partial \{U_0 > 0\}. \quad (5.4)$$
Using (5.3) and our assumption on $B_-$ in (1.6), it follows that

$$U_0(x) = 0 \text{ if } x_N > 0. \quad (5.5)$$

By (5.4), we have that, given any $\sigma > 0$, there exists $\rho(\sigma)$ so that, for $0 < \rho \leq \rho(\sigma)$, $\partial \{U_\rho > 0\} \cap B_1$ lies in a $\sigma$-neighborhood of $\partial \{U_0 > 0\}$ and thus, by (5.5), in $\{x_N \leq \sigma\}$. By scaling back, we thus deduce that, if $0 < \rho \leq \rho(\sigma)$, then

$$\partial \{U > 0\} \cap B_\rho \subseteq \{x_N \leq \sigma \rho\}$$

and therefore

$$U(x) = 0 \text{ if } x \in B_\rho \text{ and } x_N \geq \sigma \rho. \quad (5.6)$$

In the setting of [7, Definition 6.1], (5.6) means that $U \in F(\sigma, 1; \infty)$ in $B_\rho$, provided that $0 < \rho \leq \rho(\sigma)$. Let now $\beta$, $\sigma_0$ and $\tau_0$ be as in [7, Theorem 9.1] and take

$$\sigma := \sigma_0 \quad \text{and} \quad \rho := \min\{\rho_0, \tau_0 \sigma_0^{2/\mu}, \rho(\sigma_0)\}. \quad (5.7)$$

Then, by Theorem 9.1 of [7], we have that $\partial \{u = -1\} \cap B_{\rho/4}(x_-)$ is a $C^{1,\alpha}$ graph in the $v_0$-direction, for some universal $\alpha \in (0, 1)$. Thence, we will write $\{u = -1\}$ as the graph

$$x_N = \gamma(x')$$

near $x_-$, for a suitable $\gamma \in C^{1,\alpha}$. From this and [12, Theorem 1], we conclude that, in a neighborhood of $x_-$, $u$ is $C^1$ up to the free boundary. In addition, by [7, Lemma 5.4], we have that

$$\lim_{x \to x_-} \frac{\nabla u(x)}{\nabla u(x)^{\leq -1}} = \omega. \quad (5.8)$$

We now consider the “odd extension” $u^*$ of $u$ across the free boundary. Namely, we define

$$T_\pm(x) := (x', x_N \pm \gamma(x')),$$

which, by construction is a $C^1$ map near $x_-$. Also,

$$v(x) := u(T_+(x)) + 1$$

is a $C^1$ map in $\{x_N \leq 0\}$ near $x_-$, and $v(x', 0) = 0$. Thus, the function

$$v^*(x) := \begin{cases} v(x) & \text{if } x_N \leq 0, \\
-v(x', -x_N) & \text{if } x_N > 0 \end{cases}$$

is $C^1$ near $x_-$. Consequently, the function

$$u^*(x) := v^*(T_-(x)) - 1$$
is $C^1$ near $x_-$ and $u^*(x) = u(x)$ if $x_N < \gamma(x)$, i.e., if $u(x) > -1$. Therefore,

$$|\nabla u^*(x_-)| = \omega \neq 0,$$

due to (5.7). Consequently,

$$-\nu_0 = \frac{\nabla u^*(x_-)}{|\nabla u^*(x_-)|} = \frac{1}{\omega} \lim_{\tau \to 0^+} \nabla u(x_- - \tau \nu_0).$$

(5.8)

Hence, for any small $t \geq 0$,

$$u(x_- - t\nu_0) = -1 - \int_0^t \nabla u(x_- - \tau \nu_0) \cdot \nu_0 \, d\tau$$

$$\geq -1 + \omega t - \left| \int_0^t \left( \nabla u(x_- - \tau \nu_0) + \omega \nu_0 \right) \cdot \nu_0 \, d\tau \right|,$$

which, together with (5.8), gives (1.6) for $u$. The proof of (1.6) for $\tilde{u}$ is analogous, so (1.6) holds for both $u$ and $\tilde{u}$.

The proof of (1.8) is by contradiction. Assume that $\{u = 0\} \cap \{|x'| \leq l\}$ is above the hyperplane $\{x_N = -l/100\}$ and that there exists $x \in \{|u| < 1\}$ so that $|x'| \leq l/2$ and $x_N \leq -l/10$. Then, by the Density Estimate in [13, Theorem 2.2] (see also [5] for analogous results in the case $p = 2$),

$$\mathcal{L}^N (B_C(x) \cap \{|u| = 1\}) > 0 \quad \text{and} \quad \mathcal{L}^N (B_C(x) \cap \{|u| = -1\}) > 0,$$

where $\mathcal{L}^N$ is the Lebesgue measure and $C$ is a suitable positive constant. In particular, $B_C(x)$ contains points $x_{\pm}$ so that $u(x_{\pm}) = \pm 1$. Thus, by the continuity of $u$, there must be a point $x_*$ on the segment joining $x_+$ and $x_-$ so that $u(x_*) = 0$. By construction, $x_* \in B_C(x)$, thus

$$|x_*'| \leq |x'| + C \leq \frac{l}{2} + C < l \quad \text{and} \quad x_{*,N} \leq x_N + C \leq -\frac{l}{10} + C < -\frac{l}{100},$$

if $l$ is large enough. This contradicts the assumption that $\{u = 0\} \cap \{|x'| \leq l\}$ is above the hyperplane $\{x_N = -l/100\}$. Thus $|u(x)| = 1$ for any $x$ so that $|x'| \leq l/2$ and $x_N \leq -l/8$. This proves that (1.8) is fulfilled by either $u$ or $\tilde{u}$.

Then, Theorem 1.2 follows by applying Theorem 1.1 to either $u$ or $\tilde{u}$ and by noticing that $\{\tilde{u}_\varepsilon = 0\} = \{u_\varepsilon = 0\}$. □
Acknowledgments

I am indebted to Arshak Petrosyan for having pointed out to me some results from the literature, which have been exploited here. This paper has been presented in the occasion of a graduate course at Tor Vergata: I thank the participants for their valuable feedback. This research has been supported by MIUR Variational Methods and Nonlinear Differential Equations.

Appendix A. Detailed proof of (3.13)

Fix \(x\) so that \(|x'| = l/4\) and \(|d_{\Gamma_2}(x)| \leq l/8\). Let \(x_i \in \Gamma_i\) be realizing \(d_{\Gamma_i}(x)\), for \(i = 1, 2\).

Then,

\[|x_2 - x| = |d_{\Gamma_2}(x)| \leq \frac{l}{8}.
\]

In particular, \(|x'_2 - x'| \leq l/8\), so

\[|x'_2| \geq |x'| - |x'_2 - x'| \geq \frac{l}{4} - \frac{l}{8} = \frac{l}{8} \tag{A.1}
\]

and

\[|x'_2| \leq |x'| + |x'_2 - x'| \leq \frac{l}{4} + \frac{l}{8} \leq \frac{l}{2}. \tag{A.2}
\]

For \(\tilde{x} \in \mathbb{R}^N\), we define

\[P(\tilde{x}') := \frac{\theta}{2l^2} \tilde{x} \cdot M_1 \tilde{x}' + \frac{\theta}{l} \xi \cdot \tilde{x}' \quad \text{and} \quad Q(\tilde{x}') := P(\tilde{x}') - \frac{\theta \delta}{4l^2(N-1)} |\tilde{x}'|^2,
\]

so that \(\Gamma_1 = \{\tilde{x}_N = P(\tilde{x}')\}\) and \(\Gamma_2 = \{\tilde{x}_N = Q(\tilde{x}')\}\). By means of (A.2), we have that

\[|\nabla Q(x'_2)| \leq 1. \tag{A.3}
\]

Thanks to the continuity of the distance function, in the proof of (3.13), we may and do assume that the point \(x\) does not belong to \(\Gamma_1 \cup \Gamma_2\) (one then recovers the case \(x \in \Gamma_1 \cup \Gamma_2\) by a limit process). Thus, we may define

\[v := \frac{d_{\Gamma_2}(x)}{|d_{\Gamma_2}(x)|} \frac{x - x_2}{|x - x_2|} \in \mathbb{S}^{N-1}.
\]

We claim that

\[v_N \geq c_* \tag{A.4}
\]

for some constant \(c_* \in (0, 1)\). For proving this, let us first observe that, by our sign convention on the distance function, \(v_N \geq 0\). In addition, by the minimization property of \(x_2\),
we have that $v$ is orthogonal to $\Gamma_2$ at $x_2$. Then, by means of (A.3), it follows that

$$v_N = \frac{1}{\sqrt{1 + |\nabla Q(x'_2)|^2}} \geq \frac{1}{\sqrt{2}},$$

which proves (A.4). Let now

$$f(t) := x_{2,N} + t v_N - P(x'_2 + t v').$$

Note that $f(0) \leq 0$, since $x_2 \in \Gamma_2$. More precisely, if

$$t_- := c \theta \delta \quad \text{and} \quad t_+ := C \theta \delta \leq l,$$

where $1/c$ and $C$ are suitably large constants, using (A.1) and (A.2), it follows that

$$f(t_-) \leq P(x'_2) - P(x'_2 + t_- v) - \text{const} \theta \delta \leq \sup_{B_{2l}} |\nabla P| t_- - \text{const} \theta \delta
\leq \left( \text{const} \frac{\sigma(\delta)}{\delta} - \text{const} \right) \theta \delta \leq 0$$

and, by (A.2) and (A.4), we have that

$$f(t_+) \geq t_+ v_N + P(x'_2) - P(x'_2 + t_+ v') - \text{const} \theta \delta \geq c_* t_+ - \sup_{B_{2l}} |\nabla P| t_+ - \text{const} \theta \delta
\geq t_+ \left( c_* - \text{const} \frac{\sigma(\delta)}{\delta} \right) - \text{const} \theta \delta \geq \frac{c_* t_+}{2} - \text{const} \theta \delta \geq 0.$$ 

In particular, there exists $t_1 \in [t_-, t_+]$ so that $f(t_1) = 0$. Consequently, if

$$y_1 := x_2 + t_1 v$$

then $y_{1,N} = P(y'_1)$, i.e., $y_1 \in \Gamma_1$. Note that, by construction, $x$, $y_1$ and $x_2$ are collinear.

For the proof of (3.13), we now distinguish two cases: either $d_{\Gamma_1}(x) \geq 0$ or $d_{\Gamma_1}(x) \leq 0$. Let us first assume that $d_{\Gamma_1}(x) \geq 0$. In this case, a direct inspection yields that

$$d_{\Gamma_2}(x) = |x - y_1| + |y_1 - x_2|. \quad \text{(A.5)}$$

Since $y_1 \in \Gamma_1$, it follows that

$$d_{\Gamma_1}(x) = |d_{\Gamma_1}(x)| \leq |x - y_1|$$

and so, from (A.5), that

$$d_{\Gamma_2}(x) \geq d_{\Gamma_1}(x) + |y_1 - x_2| = d_{\Gamma_1}(x) + t_1 \geq d_{\Gamma_1}(x) + t_- = d_{\Gamma_1}(x) + \text{const} \theta \delta,$$

which proves (3.13) when $d_{\Gamma_1}(x) \geq 0$. 

Let us now prove (3.13) when \( d_{\Gamma_1}(x) \leq 0 \). We argue in a similar way. First, with no loss of generality, we may and do assume that 
\[ d_{\Gamma_1}(x) \geq -l/7, \]
otherwise 
\[ d_{\Gamma_2}(x) - d_{\Gamma_1}(x) \geq -\frac{l}{8} + \frac{l}{7} \]
which gives (3.13) and we are done. In the light of this assumption,
\[ |x_1 - x| = |d_{\Gamma_1}(x)| \leq \frac{l}{7} \]
and so
\[ |x'| \in \left[ \frac{l}{4} - \frac{l}{7}, \frac{l}{4} + \frac{l}{7} \right]. \]  
(A.6)

In particular,
\[ |\nabla P(x'_1)| \leq 1. \]  
(A.7)

We now define
\[ \mu := \frac{x_1 - x}{|x_1 - x|} \in S^{N-1}. \]

Since we are in the case \( d_{\Gamma_1}(x) \leq 0 \), we have that \( \mu_N \geq 0 \). More precisely, the minimization property of \( x_1 \) implies that \( \mu \) is perpendicular to \( \Gamma_1 \) at \( x_1 \) and so, by means of (A.7),
\[ \mu_N = \frac{1}{\sqrt{1 + |\nabla P(x'_1)|^2}} \geq \frac{1}{\sqrt{2}}, \]  
(A.8)

Let now
\[ g(t) := x_{1,N} - t \mu_N - Q(x'_1 - t \mu). \]

From (A.6),
\[ g(t_-) \geq -\sup_{B_{2l}} |\nabla P| t_- - t_- + \text{const} \theta \delta \geq -\left( \text{const} \frac{\sigma(\delta)}{\delta} + c \right) \theta \delta + \text{const} \theta \delta \geq 0, \]
if \( c \) is small enough. Analogously, from (A.6) and (A.8),
\[ g(t_+) \leq \sup_{B_{2l}} |\nabla P| t_+ - \frac{t_+}{\sqrt{2}} + \text{const} \theta \delta \leq \left( \text{const} \frac{\sigma(\delta)}{\delta} - \frac{1}{\sqrt{2}} \right) C \theta \delta + \text{const} \theta \delta \leq 0, \]
if $C$ is large enough. The continuity of $g$ thus implies that there exists $t_2 \in [t_-, t_+]$ so that, if

$$y_2 := x_1 - t_2 \mu,$$

then $y_2 \in \Gamma_2$. Also, by construction, $x$, $x_1$ and $y_2$ are collinear.

We now distinguish two subcases: either $d_{\Gamma_2}(x) \leq 0$ or $d_{\Gamma_2}(x) \geq 0$. In the first subcases, our assumptions give that

$$d_{\Gamma_2}(x) - d_{\Gamma_1}(x) = |d_{\Gamma_1}(x)| - |d_{\Gamma_2}(x)| \geq |x - x_1| - |x - y_2|$$

$$= |x_1 - y_2| = t_2 \geq t_- \geq \text{const} \theta \delta,$$

which yields (3.13).

The proof of (3.13) will then be finished once we take into account the (sub)case in which

$$d_{\Gamma_1}(x) \leq 0 \leq d_{\Gamma_2}(x), \quad (A.9)$$

i.e., the case in which $x$ is trapped between $\Gamma_1$ and $\Gamma_2$. Thus, we now focus on this last possibility. We claim that, in this setting,

$$|x - x_1| \geq \frac{1}{100} |x - y_1|. \quad (A.10)$$

In order to prove this, let $\angle (\cdot, \cdot) \in [0, \pi)$ denote the Euclidean angle between two directions and let us consider the triangle of vertices $x, x_1$ and $y_1$. Since $x_1 - x$ is orthogonal to $\Gamma_1$ at $x_1$, (A.6) implies that

$$\angle (x_1 - x, e_N) \leq \frac{\sigma(\delta)}{\delta} \leq \frac{\pi}{100}.$$ 

Analogously, since $x - x_2$ is orthogonal to $\Gamma_2$ at $x_2$, (A.2) gives that

$$\angle (y_1 - x, e_N) = \angle (x - x_2, e_N) \leq \frac{\sigma(\delta)}{\delta} \leq \frac{\pi}{100}. $$

By these estimates, we infer that

$$\angle (x_1 - x, y_1 - x) \leq \frac{\pi}{50}.$$ 

Also, from (A.2),

$$|y'_1| \leq |x'_2| + t_+ < l.$$ 

This, (A.6) and the fact that both $x_1$ and $y_1$ belong to $\Gamma_1$ imply that

$$|x_{1,N} - y_{1,N}| \leq \frac{\text{const} \sigma(\delta)}{\delta} |x'_1 - y'_1| = \frac{\text{const} \sigma(\delta)}{\delta} |x_{1,N} - y_{1,N}| \cdot |\tan(\angle (y_1 - x_1, e_N))|. $$
This implies that
\[ |\angle(y_1 - x_1, e_N) - \frac{\pi}{2}| \leq \frac{\pi}{100}. \]

From these considerations, some Euclidean geometry on the triangle $xx_1y_1$ gives that
\[ |\angle(y_1 - x_1, x - x_1) - \frac{\pi}{2}| = |\frac{\pi}{2} - \angle(x_1 - x, e_N) - \angle(y_1 - x_1, e_N)| \leq \frac{\pi}{50}. \]

Analogously,
\[ |\angle(x - y_1, x_1 - y_1) - \frac{\pi}{2}| = |\frac{\pi}{2} - \angle(y_1 - x_1, x - x_1) - \angle(x_1 - x, y_1 - x)| \leq \frac{\pi}{25}. \]

Further, by elementary trigonometry,
\[ \frac{|x - y_1|}{\sin(\angle(y_1 - x_1, x - x_1))} = \frac{|x - x_1|}{\sin(\angle(x - y_1, x_1 - y_1))}. \]

The latter three estimates prove (A.10).

Then, by (A.9), (A.10) and the collinearity of $x$, $x_2$ and $y_1$,
\[ d_{\Gamma_2}(x) - d_{\Gamma_1}(x) = |d_{\Gamma_1}(x)| + |d_{\Gamma_2}(x)| = |x - x_1| + |x - x_2| \geq \frac{1}{100} |x - y_1| + |x - x_2| \geq \frac{1}{100} (|x - y_1| + |x - x_2|) = \frac{1}{100} |y_1 - x_2| = \frac{t_1}{100} \geq \frac{t_0}{100} \geq \text{const } \theta \delta. \]

This completes the proof of (3.13) in this last (sub)case. \[ \square \]

References