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GEOMETRY AND ARITHMETIC CYCLES ATTACHED TO $SL_3(\mathbb{Z})$—I

RONNIE LEE and JOACHIM SCHWERMER

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§0. INTRODUCTION

IN THE study of the arithmetics of modular curves and in the classical theory of automorphic forms on the upper half plane with respect to subgroups of $SL_2(\mathbb{Z})$, modular symbols have served as an indispensable tool linking geometry and arithmetic (e.g. [8], [11]). There is a natural generalization of this notion to the case of an arithmetic subgroup Γ of a reductive algebraic $\mathbb{Q}$-group G as, for example, suggested by Mazur [12]. The construction produces families of relative cycles in the arithmetic quotient X/Γ, where X denotes the symmetric space attached to G. In analogy with the classical case, there then arise various problems concerning these cycles; but they have not yet been solved in any great generality. In particular, one may ask under what conditions one obtains non-vanishing homology classes for X/Γ in this way, or, what is the relation between modular symbols and automorphic forms with respect to Γ, especially those which contribute to the cohomology of Γ. Moreover, can one get some information on special values of L-functions associated to automorphic forms out of modular symbols? (cf. [8], [12]).

Being modest in our aim, we will focus on the case $G = SL_3/\mathbb{Q}$ and a torsion-free subgroup Γ of finite index in $SL_3(\mathbb{Z})$. Then Γ acts properly on the associated symmetric space $X = SO(3)\backslash SL_3(\mathbb{R})$, and the quotient X/Γ is a five-dimensional, non-compact, complete Riemannian manifold of finite volume. Recall that X/Γ may be viewed as the interior of a compact manifold $\bar{X}/\Gamma$ with boundary and that the inclusion is a homotopy equivalence [4].

Given an admissible parabolic $\mathbb{Q}$-subgroup P of $SL_3(\mathbb{R})$ one has the associated modular symbol (cf. §2), which gives rise to relative cycles of degree 2 (3) if P is a Borel subgroup (resp. a maximal parabolic one). One knows that $H_2(\bar{X}/\Gamma, \partial(\bar{X}/\Gamma), \mathbb{C})$ is generated by cycles of the first type [2]. This paper is mainly devoted to the study of the geometry of modular symbols in the second case and their non-trivial contribution to the cohomology of Γ (cf. 5.2, 5.3). This result will be achieved by showing that the modular cycles in question have a positive intersection number with certain associated compact cycles in X/Γ. These so-called hyperbolic cycles are two-dimensional submanifolds $C(b)$ in X/Γ attached to indefinite ternary quadratic forms b over $\mathbb{Q}$, and $C(b)$ is compact if b does not represent zero over $\mathbb{Q}$.

This natural idea to show that the modular cycles are cohomologically non-zero was originally carried out by Millson and Raghunathan [13], [14] in the context of compact arithmetic quotients of, for example, $SO(p, q)$. By analysing the intersection number of hyperbolic cycles with complementary hyperbolic ones they proved that there are uniform arithmetic subgroups, so that the hyperbolic cycles considered in the corresponding locally symmetric space are non-bounding. On the other hand, such hyperbolic cycles also occur naturally as fix point components of involutions (cf. [10], [13], [15]). From the results of Rohlfs [15], the fix point components of an involution on X/Γ can be parametrized by the first non-abelian cohomology set $H^1(\mathfrak{g}, \Gamma)$ attached to $\mathfrak{g} = \{1, \mu\}$ (cf. §4 for definition). Since a modular cycle in X/Γ may also be interpreted as a fix point component of a certain involution, we obtain a good description of the connected components (and their topological nature) of the intersection of a modular cycle with an associated hyperbolic cycle by combining the corresponding parametrizations in terms of non-abelian cohomology sets. It is this algebraic approach we use for the proof of our main result.

In §1 we recall the construction of hyperbolic cycles $C(b)$ in X/Γ. Then we describe in §2 the modular cycles and give a more geometric interpretation in terms of pairs (l, h) of a line l and a plane h in $\mathbb{Q}^3$ in general position. Given such a cycle $C(l, h) = X(l, h)/\Gamma(l, h)$, we then construct for each rational point b_0 in the symmetric domain $X(l, h)$ a hyperbolic space $X(b) \subset X$, which intersects $X(l, h)$ exactly in the point b_0 (§3). Interpreting the corresponding cycles $C(l, h)$ and $C(b) = X(b)/\Gamma(b)$ as fix point components of involutions σ and σ_b on X/Γ (with Γ σ-stable) and using the attached non-abelian cohomology sets, we can parametrize the connected components of $C(b) \cap C(l, h)$ by the relative set $D_\Gamma(\sigma, b) = (\Gamma(b)\backslash\Gamma)^\sigma/\Gamma(l, h)$. The topological nature of a component in the intersection is determined (4.6) by the image of the corresponding element under the natural map

$$\varepsilon: D_\Gamma(\sigma, b) \to D_{\mathbb{R}}(\sigma, b) = (SO(b)(\mathbb{R})\backslash SL_3(\mathbb{R}))^\sigma/M(l,h)$$

of $D_\Gamma(\sigma,b)$ into its "real" analogue. If both $C(l, h)$ and $C(b)$ are oriented, we then give a method to determine the local intersection number attached to non-degenerate intersection points by factorizing ε through an "intermediate" set $D^+_{\mathbb{R}}(\sigma, b)$ (4.8). The degenerate intersection curves in $C(l, h) \cap C(b)$ are discussed in 4.9.

In §5 we deal with the non-trivial contribution of the modular cycles to the cohomology of Γ.

Notation. Besides the usual conventions we fix the following ones:

(1) If M is a set and f: $M \to M$ a map, we denote the fix point set $\{m \in M \mid f(m) = m\}$ of f by Fix (f, M) or by M^f.

(2) With respect to algebraic groups and arithmetic groups we follow [3]. If G is a (Zariski)-connected $\mathbb{Q}$-group, we denote by $G = \mathrm{G}(\mathbb{R})$ the group of real points of G. We put $^\circ\mathrm{G} = \cap \ker \chi^2$, where χ runs through the group $X_{\mathbb{Q}}(\mathrm{G})$ of $\mathbb{Q}$-morphisms from G to GO_1. The group $^\circ\mathrm{G}(\mathbb{R})$ contains each compact subgroup of $\mathrm{G}(\mathbb{R})$ and each arithmetic subgroup of G ([4], 1.2).

§1. HYPERBOLIC CYCLES

1.1. Let $G = SL_3(\mathbb{R})$ be the group of real points of the semi simple algebraic group $\mathrm{G}/\mathbb{Q} = SL_3/\mathbb{Q}$ defined over $\mathbb{Q}$. The associated symmetric space X of maximal compact subgroups of G can be realized as the coset space $X = K_0\backslash G$, where $K_0 = SO(3) \subset G$ denotes the standard maximal compact subgroup of orthogonal real matrices with determinant one. We may also view X as the set of positive definite quadratic forms on $\mathbb{R}^3$ with determinant one. Let Γ be a torsion-free subgroup of finite index in $SL_3(\mathbb{Z})$; it acts properly on X, and the quotient X/Γ is an orientable complete Riemannian manifold of dimension 5 which is non-compact but of finite volume.

There is a natural construction of submanifolds in X/Γ attached to certain quadratic forms. Let b be a symmetric non-degenerate bilinear form on $\mathbb{Q}^3$ and denote by $SO(b)(\mathbb{Q})$ the special orthogonal group of b which consists of the automorphisms of $\mathbb{Q}^3$ (with determinant one) keeping the bilinear form b fixed. There is an embedding $SO(b)(\mathbb{Q}) \to SL_3(\mathbb{Q})$, which also induces one for the groups of real points. One has that $SO(b)(\mathbb{R})$ is a maximal compact subgroup of G if and only if b is definite. For our purpose we suppose therefore that b is indefinite, and, in order to fix it, of signature (2, 1) over $\mathbb{R}$. There is a maximal subgroup K of G such that $K \cap SO(b)(\mathbb{R})$ is maximal compact in $SO(b)(\mathbb{R})$ and we have an embedding of symmetric spaces

$$X(b) = K \cap SO(b)(\mathbb{R}) \backslash SO(b)(\mathbb{R}) \to X = K \backslash G. \tag{1.1.1}$$

With respect to the given group $\Gamma \subset SL_3(\mathbb{Z})$ this gives an immersion of the corresponding arithmetic quotients

$$X(b)/\Gamma \cap SO(b)(\mathbb{R}) \to X/\Gamma. \tag{1.1.2}$$

The manifold $C(b) = X(b)/\Gamma \cap SO(b)(\mathbb{R})$ is of dimension two and will be referred to as a hyperbolic cycle. It is compact if and only if b considered as a quadratic from is anisotropic over $\mathbb{Q}$, i.e. b does not represent zero over $\mathbb{Q}$. (This is due to Siegel, cf. [19] and [20] or more generally [3], §8).

Example. Let b be the ternary form over $\mathbb{Q}$ given by

$$b(x_1, x_2, x_3) = -x_1^2 - x_2^2 + dx_3^2, \quad d \text{ prime} \neq 2$$

From the work of Legendre, one has that b is anisotropic over $\mathbb{Q}$ if and only if -1 is not a square modulo the given prime d, i.e. if d is congruent to 3 modulo 4 (cf. [6], Chap. 6,4.1).

1.2. For a given b as above the real Lie group $SO(b)(\mathbb{R})$ has two connected components. If $\Gamma \cap SO(b)(\mathbb{R})$ is contained in the connected component of the identity of $SO(b)(\mathbb{R})$, then the submanifold $C(b) = X(b)/\Gamma \cap SO(b)(\mathbb{R})$ is orientable. For a given Γ these matters are best discussed in terms of the spinor norm, as in [13], §4. Recall that we can write an element in the orthogonal group $O(q)(k)$ attached to a quadratic from q on a finite dimensional vector space V over a field k [char$(k) \neq 2$] as a finite product $g = \Pi r_{v_i}$ of reflections r_{v_i} in v_i. The spinor norm of g is then given as the well-defined image of g under the homomorphism

$$\theta_k: O(q)(k) \to k^*/(k^*)^2 \tag{1.2.1}$$
$$g \mapsto \Pi_i q(v_i).$$

By restriction we obtain a homomorphism $\theta_k: SO(q)(k) \to k^*/(k^*)^2$, and we put

$$SO(q)^+(k) = \{g \in SO(q)(k) \mid \theta_k(g) = 1\}. \tag{1.2.2}$$

If now b as above is viewed as a quadratic form over $\mathbb{R}$ of signature (2, 1) then the connected component of the identity of $SO(b)(\mathbb{R})$ is just $SO(b)^+(\mathbb{R})$. We have that $g \in SO(b)^+(\mathbb{R})$ if and only if one may write $g = \Pi r_{v_i}$ with $\Pi b(v_i) > 0$.

We now consider a symmetric indefinite integral quadratic form b on $\mathbb{Q}^3$ which is of signature (2, 1) over $\mathbb{R}$. We denote by

$$\Gamma(m) = \{A \in SL_3(\mathbb{Z}) \mid A \equiv \mathrm{Id} \bmod m\} \tag{1.2.3}$$

the full congruence subgroup of level $m, m \geq 3$; it is torsion-free. It is shown in [13], §4 that there exist infinitely many primes q only depending on b such that the elements in $\Gamma(q) \cap SO(b)(\mathbb{R})$ have spinor norm one. This implies that for such $\Gamma(q)$ attached to the given b the hyperbolic cycle $C(b) = X(b)/\Gamma(q) \cap SO(b)(\mathbb{R})$ is orientable.

Remark. There is a very explicit procedure to determine for a given b these infinitely many primes q by giving certain excluding conditions. In particular, q has to be unequal to 2 and the prime divisors of det b. However, we will not need this.

§2. MODULAR SYMBOLS

2.1. Let G be a connected reductive $\mathbb{Q}$-group with $\mathrm{rank}_{\mathbb{Q}} G > 0$ and without non-trivial rational characters defined over $\mathbb{Q}$; the group $G(\mathbb{R})$ of real points of G will be denoted by G. Let P be a parabolic $\mathbb{Q}$-subgroup of G. A closed reductive $\mathbb{Q}$-subgroup M of P is a Levi subgroup if P is the semidirect product $M \ltimes R_u(P)$ of M and its unipotent radical $R_u(P)$; the corresponding decomposition is called a Levi decomposition of P. Recall that two Levi subgroups of P are conjugate by a unique element of $R_u(P)$ defined over $\mathbb{Q}$ (cf. [5], 3.13, 3.14). The corresponding decomposition $P = P(\mathbb{R}) = M \ltimes N$ with $M = M(\mathbb{R})$ and $N = R_u(P)(\mathbb{R})$ of the groups of real points will be called a rational Levi decomposition of the parabolic $\mathbb{Q}$-subgroup P of G.

We fix a maximal compact subgroup K of G and let Γ be an arithmetic subgroup of $G(\mathbb{Q})$. We call a parabolic $\mathbb{Q}$-subgroup P of G with a given rational Levi decomposition *admissible* (with respect to K and Γ) if:

the group $M \cap K$ is a maximal compact subgroup of M; and (2.1.1)
the group $\Gamma \cap M$ is torsion-free and the space $K \cap M\backslash M$ is endowed with an (2.1.2)
orientation such that the action of $\Gamma \cap M$ is orientation-preserving on $K \cap M \backslash M$.

To a given admissible $\mathbb{Q}$-parabolic subgroup P of G and a rational point u in $R_u(P)(\mathbb{Q})$ we associate the space

$$C(M,u) = K \cap M\backslash M/M \cap \Gamma \cap u\Gamma u^{-1}, \tag{2.1.3}$$

which is a connected oriented manifold. For $u = e$ we put $C(M) = C(M, e)$. If $X = K\backslash G$ is the symmetric space associated to G, then Γ acts properly on X, and the quotient X/Γ is a manifold if Γ is torsion-free. Since the natural mapping

$$C(M,u) \to X/\Gamma \tag{2.1.4}$$

induced by the map $M \to G, h \mapsto h\cdot u$, is proper ([1], 2.7) $C(M,u)$ will be viewed as an oriented closed submanifold of X/Γ.

2.2. *The case* $SL_3/\mathbb{Q}$. Let G now be the group $SL_3/\mathbb{Q}$, and let $\mathscr{P}$ be the set of parabolic $\mathbb{Q}$-subgroups of $G = SL_3(\mathbb{R})$. Each element $Q \neq G$ of $\mathscr{P}$ is then conjugate (over $\mathbb{Q}$) either to the minimal parabolic $\mathbb{Q}$-subgroup

$$P_0 = \{(a_{ij}) \in SL_3(\mathbb{R}) \mid a_{ij} = 0,\ i > j\} \tag{2.2.1}$$

or to one of the maximal parabolic $\mathbb{Q}$-subgroups P_1 or its opposite P_2, where

$$P_1 = \left\{\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in G\right\}, \quad P_2 = \left\{\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \in G\right\}. \tag{2.2.2}$$

We fix as a maximal compact subgroup K of G the special orthogonal group $K = SO(3)$, and denote by τ the corresponding Cartan involution on G. Then the Levi decompsotion

$$P_i = M_i \ltimes N_i, \quad i = 0, 1, 2, \tag{2.2.3}$$

where $M_0 = \{\mathrm{diag}(t_i) \in G \mid t_i \in \mathbb{R}\}$ and

$$M_i = \left\{\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \in G\right\}, \quad i = 1, 2 \tag{2.2.4}$$

is defined over $\mathbb{Q}$, and the M_i are τ-stable. Note that $M_i \cap K$ is maximal compact in M_i.

Fix an arithmetic subgroup Γ of $SL_3(\mathbb{Q})$, and assume that P_i $(i = 0, 1, 2)$ is admissible with respect to K and Γ [i.e. (2.1.2) is satisfied]. The space $C(M_0, u)$ corresponding to this rational Levi decomposition of P_0 is then of dimension 2, and this case has been considered in [2]. Owing to the fact that $\Gamma \cap u\Gamma u^{-1} \cap M_i$ is contained as an arithmetic subgroup in ${}^\circ M_i$ (cf. §0 for notation and [9], §2), we see that in the case $i = 1, 2$, $C(M_i, u)$ is the product of a Riemann surface with a line, i.e. it is of dimension 3.

There is a different description of the modular cycles $C(M)$. We will concentrate on the case of a maximal parabolic $\mathbb{Q}$-subgroup P of G. Let l be a line and h be a plane in $\mathbb{Q}^3$ which are in general position, i.e. $h \cap l = \{0\}$. Then we put

$$\mathsf{M}(l, h) = \{\varphi \in SL_3(\mathbb{Q}) | \varphi(l) = l, \varphi(h) = h\}. \tag{2.2.5}$$

This subgroup of $SL_3(\mathbb{Q})$ will be called the modular group associated to the pair (l, h) (in general position). If h^0 is the plane spanned by $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$, and l^0 is the line spanned by $e_3 = (0, 0, 1)$, then one has $\mathsf{M}(l^0, h^0) = \mathsf{M}_i$. Since a given pair (l, h) can be transformed by an automorphism of $\mathbb{Q}^3$ of determinant one into the standard pair (l^0, h^0), there is a parabolic $\mathbb{Q}$-subgroup P of G such that $\mathsf{M}(l, h)$ is a rational Levi subgroup of $\mathsf{P}(\mathbb{Q})$. Conversely, a $\mathbb{Q}$-Levi subgroup M of a maximal parabolic $\mathbb{Q}$-subgroup P of G determines a unique pair (l', h') in general position such that $\mathsf{M}(\mathbb{Q}) = \mathsf{M}(l', h')$. To prove the uniqueness, observe that $\mathsf{M}(\mathbb{Q}) = \mathsf{M}(l', h')$ contains a unique non-trivial central element $s_{(l', h')}$ of order two characterized by the fact

$$\sigma_{(l', h')|h'} = -\mathrm{Id}, \quad \sigma_{(l', h')|l'} = \mathrm{Id} \tag{2.2.6}$$

i.e. l' (resp. h') is its positive (resp. negative) eigenspace.

In analysing the modular cycles it is therefore enough to study the pairs (l, h) and their associated modular groups. We have $h \oplus l = \mathbb{Q}^3$, and since an element $\varphi \in \mathsf{M}(l, h)$ fixes l, i.e. the automorphism $\varphi_{|l}$ is given as multiplication by the scalar det $(\varphi_{|h})^{-1}$, we have a natural isomorphism

$$r: \mathsf{M}(l, h) \xrightarrow{\sim} GL_2(h), \quad \varphi \mapsto \varphi_{|h} \tag{2.2.7}$$

of $\mathsf{M}(l, h)$ onto the general linear group of h. We denote by $X(l, h)$ the corresponding symmetric space of positive definite bilinear pairings on $h_{\mathbb{R}} = h \otimes_{\mathbb{Q}} \mathbb{R}$. If Γ is now of finite index in $SL_3(\mathbb{Z})$, we denote by $\Gamma(l, h)$ the intersection $\Gamma \cap \mathsf{M}(l, h)$ and also its image under r; this is then a subgroup of finite index in $GL_2(h_{\mathbb{Z}})$, where $h_{\mathbb{Z}} = h \cap \mathbb{Z}^3$. By a straightforward argument one obtains the following:

PROPOSITION 2.3. *Let Γ be a subgroup of $SL_3(\mathbb{Z})$ of finite index, and let $\mathsf{M}(l, h)$ be the modular group associated to a pair (l, h) in general position. Then the modular cycle $C(l, h) = X(l, h)/\Gamma(l, h)$ is the product of a two-dimensional surface and the positive real line, and $C(l, h)$ is orientable if and only if the composite map*

$$\Gamma(l, h) \xrightarrow{r} GL_2(h_{\mathbb{Z}}) \xrightarrow{\det} \{\pm 1\}$$

is trivial.

COROLLARY 2.4. *If $\Gamma = \Gamma(m) = \{A \in SL_3(\mathbb{Z}) | A \equiv \mathrm{Id} \bmod m\}$ is a congruence subgroup of level m, $m \geq 3$, then the modular cycle $C(l, h)$ is orientable.*

An orientation reversing element γ in $\Gamma(m)(l, h)$ for $X(l, h)$ is simply an element which reverses the orientation of the line l and maps the plane h to itself. Then also $\bar{\gamma}$ sends $\bar{l}$ to $-\bar{l}$ (where the overbar denotes reduction mod m), but this contradicts the fact that $\bar{\gamma} = \mathrm{Id}$.

The following parametrization up to conjugation under $SL_3(\mathbb{Z})$ of the integral pairs $(l_{\mathbb{Z}}, h_{\mathbb{Z}})$, where $l_{\mathbb{Z}} = l \cap \mathbb{Z}^3$, $h_{\mathbb{Z}} = h \cap \mathbb{Z}^3$, is easily obtained.

PROPOSITION 2.5. *Given a pair (l, h) of a line l and a plane h in $\mathbb{Q}^3$ in general position. There exist integers c and d, $0 \le c < d$, $(c, d) = 1$, such that $(l_{\mathbb{Z}}, h_{\mathbb{Z}})$ is $SL_3(\mathbb{Z})$-equivalent to the pair $(\xi_{c,d}, h^0_{\mathbb{Z}})$, where $\xi_{c,d}$ is the line in $\mathbb{Z}^3$ spanned by $(0, c, d)$ and $h^0_{\mathbb{Z}} = h^0 \cap \mathbb{Z}^3$ the plane in $\mathbb{Z}^3$ spanned by $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$. (The number d is the discriminant of the lattice $h_{\mathbb{Z}} \oplus l_{\mathbb{Z}}$ in $\mathbb{Q}^3$.)*

2.6. *Remarks.* (1) This implies that a modular cycle $C(l, h)$ in $X/\Gamma(m)$ is of the form $C(\xi_{c,d}, h^0)$, where $\xi_{c,d}$ is the line spanned by $(0, c, d)$ for some $0 \le c < d$, $(c, d) = 1$ and d is the discriminant of the lattice $h_{\mathbb{Z}} \oplus l_{\mathbb{Z}}$.

(2) By an easy calculation one sees that the fundamental group of the modular cycle $C(l, h)$ in $X/\Gamma(m)$ is isomorphic under the map $r: \mathrm{M}(l, h) \to GL_2(h) \cong GL_2(\mathbb{Q})$ to the congruence subgroup

$$\left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A \in SL_2(\mathbb{Z}) \,\middle|\, \begin{array}{l} \mathrm{A} \equiv \mathrm{Id} \bmod m \\ a_{11} \equiv a_{22} \equiv 1 \bmod d,\ a_{21} \equiv 0 \bmod d \end{array} \right\}$$

of $SL_2(\mathbb{Z})$.

§3. CONSTRUCTION OF HYPERBOLIC CYCLES ATTACHED TO A MODULAR CYCLE

Let $X(l, h) \subset X$ be the three-dimensional symmetric domain associated to a pair (l, h) in general position. To a given rational point b_0 in $X(l, h)$, we construct a two-dimensional hyperbolic space $X(b)$ which intersects $X(l, h)$ exactly in the point b_0. It may be of some help to view both spaces as fix point sets of certain involutions on X.

3.1. Let $X(l, h)$ be the symmetric space of all positive definite symmetric bilinear pairings on $h_{\mathbb{R}} = h \otimes_{\mathbb{Q}} \mathbb{R}$ associated to a given pair (l, h) of a line l and a plane h in $\mathbb{Q}^3$ in general position. As defined in (2.2.6) there is a unique non-trivial central element $\sigma = \sigma_{(l,h)} \in \mathrm{M}(l, h) \subset SL_3(\mathbb{Q})$ of order two attached to (l, h). It operates as an involution (denoted by the same letter) on G by

$$\sigma: G \to G \quad g \mapsto \sigma \cdot g \cdot \sigma^{-1} \tag{3.1.1}$$

and on the space X of positive definite bilinear pairings on $\mathbb{R}^3$ by

$$\sigma: X \to X \quad x \mapsto x \cdot \sigma^{-1}. \tag{3.1.2}$$

We observe that these two actions are compatible with the action of G, i.e. we have

$$\sigma(x \cdot g) = \sigma(x) \cdot \sigma(g), \qquad x \in X, \quad g \in G. \tag{3.1.3}$$

We recall that $X(l, h)$ is embedded into X by extending a positive definite symmetric pairing on $h_{\mathbb{R}}$ to a uniquely determined one on $\mathbb{R}^3 = l_{\mathbb{R}} \oplus h_{\mathbb{R}}$ such that $l_{\mathbb{R}}$ and $h_{\mathbb{R}}$ are orthogonal and the determinant is one. Then it is easy to check that $X(l, h)$ is the fix point set in X under the involution σ, i.e.

$$X(l, h) = \mathrm{Fix}(\sigma, X). \tag{3.1.4}$$

3.2. Corresponding to the fixed maximal compact subgroup $K = SO(3)$ of G there is the Cartan involution τ on G (resp. on X) given (resp. induced) by

$$g \to {}^t g^{-1}, \quad g \in G. \tag{3.2.1}$$

Given an element $b \in SL_3(\mathbb{Q})$, $b = {}^t b$, we define a twisted operation σ_b on G resp. X by

$$\sigma_b(g) = b \cdot \tau(g) \cdot b^{-1}, \quad g \in G \tag{3.2.2}$$

$$\sigma_b(x) = \tau(x) \cdot b^{-1}. \tag{3.2.3}$$

Then the fix point set $\mathrm{Fix}(\sigma_b, X)$ in X under σ_b is exactly the symmetric space $X(b) = K_b \backslash SO(b)(\mathbb{R})$ associated to the special orthogonal group $SO(b)(\mathbb{R})$. For a proof we refer to [15], Proposition 1.6. Note that the hyperbolic submanifold $X(b)$ consists of one point if b considered as a bilinear form is positive definite, and is of dimension 2 if b is indefinite.

If we now choose as in 3.1 a rational point in a given $X(l, h) \subset X$ which is represented by a symmetric positive definite matrix $b_0 \in SL_3(\mathbb{Q})$, then $X(l, h) \subset X$ by definition, and the two involutions σ and σ_{b_0} on X associated to $X(l, h) = \mathrm{Fix}(\sigma, X)$ and to a rational point $b_0 \in X(l, h)$ respectively, commute with each other. Their product

$$\tilde{\sigma} = \sigma \cdot \sigma_{b_0} \tag{3.2.4}$$

is then the involution associated as above to the symmetric pairing b on $\mathbb{R}^3$ defined by $b(u, v) = b_0(u, \sigma(v))$, $u, v \in \mathbb{R}^3$, i.e. we have

$$\tilde{\sigma} = \sigma_b. \tag{3.2.5}$$

We observe that $l_{\mathbb{R}}$ and $h_{\mathbb{R}}$ are orthogonal with respect to b. Since $h_{\mathbb{R}}$ (resp. $l_{\mathbb{R}}$) is the (-1)-eigenspace [resp. $(+1)$-eigenspace] of σ the pairing b is indefinite.

PROPOSITION 3.3. *Let $X(l, h) = \mathrm{Fix}(\sigma, X) \subset X$ be the three-dimensional symmetric domain associated to a pair (l, h) in general position and b_0 a rational point in $X(l, h)$. Then the fix point set $\mathrm{Fix}(\sigma_b, X)$ associated with the involution $\sigma_b = \sigma \circ \sigma_{b_0}$ (given as in 3.2) coincides with the two-dimensional hyperbolic space $X(b)$ (attached to the indefinite $b = b_0 \cdot \sigma$ by 3.2) and intersects $X(l, h)$ transversally exactly in the point b_0, i.e. we have*

$$X(b) \cap X(l, h) = \{b_0\}.$$

The pairing b defined in 3.2 is indefinite, so we have $X(b) = \mathrm{Fix}(\sigma_b, X)$ and it is two-dimensional. By definition the involutions σ and σ_b commute with each other, so $\sigma(X(b)) \subset X(b)$. It follows that $X(b)$ intersects $X(l, h)$ in the points fixed by σ, σ_b and $\sigma_{b_0} = \sigma \circ \sigma_b$, so we get

$$X(b) \cap X(l, h) = \mathrm{Fix}(\sigma_{b_0}, X) = \{b_0\}.$$

§4. INTERSECTION THEORY FOR THE MODULAR AND THE ASSOCIATED HYPERBOLIC CYCLE

First of all, we recall Rohlfs' approach [15] to parametrize the fix point components of an involution μ on X/Γ by the first non-abelian cohomology set $H^1(\mathfrak{g}, \Gamma)$, $\mathfrak{g} = \{1, \mu\}$. This then allows us to describe the components of the intersection of the modular cycle $C(l, h) = X(l, h)/\Gamma(l, h)$ and the hyperbolic cycle $C(b) = X(b)/\Gamma(b)$ in X/Γ for certain Γ. In 4.6 we parametrize the non-degenerate intersection points, which turn out to be isolated intersection points. Under the assumption that both $C(l, h)$ and $C(b)$ are oriented, we give a method to determine the intersection number attached to each of these non-degenerate intersection points (4.8). Finally, we show that the degenerate intersection curves contained in $C(l, h) \cap C(b)$ do not contribute to the intersection number of the two given cycles.

4.1. Let μ be an involution on G induced by a rational automorphism of G, and let

$\Gamma \subset SL_3(\mathbb{Z})$ be a torsion-free subgroup of finite index which is stable under μ. The involutions induced on the associated symmetric space X and on the arithmetic quotient X/Γ are again denoted by μ. Then the connected components of the fix point set $\mathrm{Fix}(\mu, X/\Gamma)$ are parametrized by the non-abelian cohomology set $H^1(\mathfrak{g}, \Gamma)$, where $\mathfrak{g} = \{1, \mu\}$. [To simplify notation we sometimes also write $H^1(\mu, \Gamma)$ instead of $H^1(\mathfrak{g}, \Gamma)$.] For the convenience of the reader we explain this basic fact due to Rohlfs [15].

We recall the definition of $H^1(\mathfrak{g}, \Gamma)$ (cf. [18], I.5.) To $a \in \Gamma$ with $a \cdot \mu(a) = 1$ is associated a cocycle for $H^1(\mathfrak{g}, \Gamma)$ denoted by $(1, a)$ or by a and two cocycles $(1, a)$ and $(1, d)$ are equivalent if there exists $c \in \Gamma$ such that $d = c^{-1} \cdot a \cdot \mu(c)$. If $(1, a)$ is a cocycle for $H^1(\mathfrak{g}, \Gamma)$, we define a new operation μ_a on G and on Γ by $\mu_a(g) = a \cdot \mu(g) \cdot a^{-1}$. The operation induced on X is then given by $\mu_a(x) = \mu(x) \cdot a^{-1}$, $x \in X$. We have $\mu_a(x \cdot g) = \mu_a(x) \cdot \mu_a(g)$ for $x \in X$ and $g \in G$. Therefore we have an operation on X/Γ which coincides with the previous operation μ on X/Γ. Let $\Gamma(a)$ be the set of elements in Γ fixed by μ_a, and $X(a)$ be the fix point set $\mathrm{Fix}(\mu_a, X)$ of the twisted involution μ_a. Then the natural map $\pi_a: X(a)/\Gamma(a) \to X/\Gamma$ is injective and its image

$$F(a) = \mathrm{im}\, \pi_a \cong X(a)/\Gamma(a) \tag{4.1.1}$$

lies in the fix point set $\mathrm{Fix}(\mu, X/\Gamma)$. We observe that $F(a)$ depends only on the cohomology class a in $H^1(\mathfrak{g}, \Gamma)$ represented by the cocycle $a = (1, a)$. Moreover, $F(a)$ is non-empty since μ_a acts as an isometry on X. We will now see that all fix points of μ arise by this construction. Consider a point $\bar{x} \in \mathrm{Fix}(\mu, X/\Gamma)$ represented by $x \in X$. Then there exists a uniquely determined element a in Γ such that $\mu(x) = x \cdot a$. But for the involution μ we have $x = \mu(\mu(x)) = \mu(x \cdot a) = x \cdot a \cdot \mu(a)$. Therefore $a \cdot \mu(a) = 1$, and $(1, a)$ is a cocycle for $H^1(\mathfrak{g}, \Gamma)$. For another representative $y = x \cdot c$, $c \in \Gamma$, of $\bar{x}$ the attached cocycle is $d = c^{-1} \cdot a \cdot \mu(c)$. Therefore every $\bar{x} \in \mathrm{Fix}(\mu, X/\Gamma)$ determines uniquely a class in $H^1(\mathfrak{g}, \Gamma)$, and the fix point set is a disjoint union of the connected non-empty sets $F(a)$, $a \in H^1(\mathfrak{g}, \Gamma)$,

$$\mathrm{Fix}(\mu, X/\Gamma) = \coprod_{a \in H^1(\mathfrak{g}, \Gamma)} F(a). \tag{4.1.2}$$

4.2. As in §3, let (l, h) be a pair of a line l and a plane h in $\mathbb{Q}^3$ in general position. Then the submanifold $X(l, h) \subset X$ will be viewed as the fix point set $\mathrm{Fix}(\sigma, X)$ of the involution $\sigma = \sigma_{(l, h)}$ on X. We assume that the given torsion-free subgroup $\Gamma \subset SL_3(\mathbb{Z})$ is fixed under σ. Then the connected components of the fix point set of the involution σ induced on X/Γ are parametrized by the non-abelian cohomology set $H^1(\sigma, \Gamma)$ with $\sigma = \{1, \sigma\}$, i.e.

$$\mathrm{Fix}(\sigma, X/\Gamma) = \coprod_{\gamma \in H^1(\sigma, \Gamma)} F(\gamma). \tag{4.2.1}$$

Using the general description of the $F(\gamma)$ in 4.1 and 3.1, one sees that the associated modular cycle $C(l, h)$ coincides with the connected component $F(\gamma_0)$ corresponding to the base point $(1, e)$ in $H^1(\sigma, \Gamma)$ (e = trivial element).

Now we choose a rational point b_0 in $X(l, h)$. Attached to this by 3.3, there is a hyperbolic manifold $X(b)$ which may be viewed as the fix point set in X of the involution $\sigma_b = \sigma \circ \sigma_{b_0}$.

If Γ is also invariant under σ_b, then as above one has

$$\mathrm{Fix}(\sigma_b, X/\Gamma) = \coprod_{\delta \in H^1(\sigma_b, \Gamma)} F(\delta) \tag{4.2.2}$$

and the hyperbolic cycle $C(b) = X(b)/\Gamma(b)$ (cf. §1) coincides with the connected component $F(\delta_0)$ corresponding to the base point $(1, e)$ in $H^1(\sigma_b, \Gamma)$. In general, the projection of $X(b)$ gives a well-defined cycle $C(b) = X(b)/\Gamma(b)$ in X/Γ. Since $X(b)$ is stable under the involution σ, its projection $C(b)$ is also invariant under the involution σ on X/Γ.

PROPOSITION 4.3. *Given a group $\Gamma \subset SL_3(\mathbb{Z})$ of finite index which is stable under the involution $\sigma = \sigma_{(l,h)}$ attached to $X(l,h)$, the connected components of the intersection of $C(l,h) = X(l,h)/\Gamma(l,h)$ with the hyperbolic cycle $C(b) = X(b)/\Gamma(b)$ associated with the choice of a rational point $b_0 \in X(l,h)$ (by 3.2. and 3.3) are in one-to-one correspondence with the set*

$$(\Gamma(b)\backslash\Gamma)^{\sigma}/\Gamma(l,h).$$

Since σ commutes with σ_b, it acts on Fix $(\sigma_b, X/\Gamma)$ and permutes its components. But the point $b_0 \in C(b)$ is fixed by σ. Hence σ acts on $C(b) = X(b)/\Gamma(b)$. Clearly, we have Fix $(\sigma, C(b)) = C(b) \cap$ Fix $(\sigma, X/\Gamma)$, and its connected components are parametrized by $H^1(\sigma, \Gamma(b))$. The inclusion Fix $(\sigma, C(b)) \to$ Fix $(\sigma, X/\Gamma)$ corresponds to the map $i: H^1(\sigma, \Gamma(b)) \to H^1(\sigma, \Gamma)$ induced by the inclusion $\Gamma(b) \to \Gamma$. Since the modular cycle $C(l,h)$ corresponds to the base point $(1, e)$ in $H^1(\sigma, \Gamma)$, the elements in $H^1(\sigma, \Gamma(b))$ mapped onto $(1, e)$ parametrize the connected components of $C(b) \cap C(l,h)$. In order to determine $i^{-1}((1,e))$, we consider the exact sequence of pointed cohomology sets (cf. [18], Proposition 36)

$$1 \to H^0(\sigma, \Gamma(b)) \to H^0(\sigma, \Gamma) \to H^0(\sigma, \Gamma(b)\backslash\Gamma) \xrightarrow{\partial} H^1(\sigma, \Gamma(b)) \xrightarrow{i} H^1(\sigma, \Gamma) \tag{4.3.1}$$

associated to the fibration

$$1 \to \Gamma(b) \to \Gamma \to \Gamma(b)\backslash\Gamma.$$

We have $H^0(\sigma, \Gamma) = \Gamma^{\sigma} = \Gamma(l,h)$ respectively $H^0(\sigma, \Gamma(b)\backslash\Gamma) = (\Gamma(b)\backslash\Gamma)^{\sigma}$, and the inverse image of $(1, e)$ under i can be identified with the orbit space of $(\Gamma(b)\backslash\Gamma)^{\sigma}$ under the action of $\Gamma(l,h)$ (cf. [18], Corollary 1 to Proposition 36).

Given a point x in the intersection $C(b) \cap C(l,h)$ we will say that the cycles $C(b)$ and $C(l,h)$ *intersect each other transversally at* x if the tangent spaces $T_{C(b),x}$ and $T_{C(l,h),x}$ of the two cycles in x span the entire tangent space $T_{X/\Gamma,x}$ of X/Γ in x. Otherwise we will refer to x as a *degenerate intersection point*. We have the following:

LEMMA 4.4. *The two cycles $C(b)$ and $C(l,h)$ intersect each other transversally at a point x in X/Γ if and only if x is an isolated point in the intersection $C(b) \cap C(l,h)$.*

The proof of this lemma is clear because $C(b) \cap C(l,h)$ is totally geodesic.

4.5. As in the proof of Proposition 4.3 there is an exact sequence of pointed cohomology sets

$$H^0(\sigma, SL_3(\mathbb{R})) \to H^0(\sigma, SO(b)\backslash SL_3(\mathbb{R})) \to H^1(\sigma, SO(b)) \xrightarrow{j} H^1(\sigma, SL_3(\mathbb{R})) \tag{4.5.1}$$

associated to the fibration $1 \to SO(b) \to SL_3(\mathbb{R}) \to SO(b)\backslash SL_3(\mathbb{R})$ and again we have that

$$j^{-1}((1,e)) = (SO(b)\backslash SL_3(\mathbb{R}))^{\sigma}/M(l,h). \tag{4.5.2}$$

To simplify notations we will put

$$D_{\Gamma}(\sigma, b) = (\Gamma(b)\backslash\Gamma)^{\sigma}/\Gamma(l,h) \tag{4.5.3}$$

$$D_{\mathbb{R}}(\sigma, b) = (SO(b)\backslash SL_3(\mathbb{R}))^{\sigma}/M(l,h). \tag{4.5.4}$$

There is a natural mapping

$$\varepsilon: D_{\Gamma}(\sigma, b) \to D_{\mathbb{R}}(\sigma, b) \tag{4.5.5}$$

induced by the inclusion $\Gamma \to SL_3(\mathbb{R})$. This follows from the compatibility of the two sequences (4.5.1) and (4.3.1). The following result tells us that this map controls whether or not an element in $D_{\Gamma}(\sigma, b)$ represents an isolated intersection point in $C(b) \cap C(l,h)$.

THEOREM 4.6. (i) *The space $D_{\mathbb{R}}(\sigma, b)$ consists of two elements. They are denoted by e and f, where e lies in the image of the trivial coset $SO(b)\cdot I$.* (ii) *Let $\varepsilon: D_\Gamma(\sigma, b) \to D_{\mathbb{R}}(\sigma, b)$ be the natural map induced by the inclusion $\Gamma \subset SL_3(\mathbb{R})$. Then an element $[z]$ in $D_\Gamma(\sigma, b)$ represents an isolated intersection point in $C(b) \cap C(l, h)$ if and only if $\varepsilon([z]) = e$.*

For the proof we recall that $b_{|h_{\mathbb{R}}}$ is negative definite and $b_{|l_{\mathbb{R}}}$ is positive definite. Hence there exists a basis of $h_{\mathbb{R}}$ and $l_{\mathbb{R}}$ such that b is of the form diag $(-1, -1, 1)$ with respect to this basis. Using [15], Remark 1.4, we have $H^1(\sigma, SO(b)(\mathbb{R})) \simeq H^1(\sigma, O(b_{|h_{\mathbb{R}}}))$, and the involution σ operates trivially on $O(b_{|h_{\mathbb{R}}})$. Therefore the cohomology set $H^1(\sigma, O(b_{|h_{\mathbb{R}}}))$ can be identified with the set of conjugacy classes of involutions in $O(b_{|h_{\mathbb{R}}})$. There are three such conjugacy classes, represented by $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$, and we obtain the representatives $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$ for the classes of $H^1(\sigma, SO(b)(\mathbb{R}))$. In determining $D_{\mathbb{R}}(\sigma, b)$, the first one has to be excluded because it is a positive definite matrix which is inequivalent to b. This proves our claim in (i).

Suppose that under the mapping $\varepsilon: D_\Gamma(\sigma, b) \to D_{\mathbb{R}}(\sigma, b)$ the element $[z]$ is sent to e in $D_{\mathbb{R}}(\sigma, b)$. Then it follows from the discussion at the end of 3.2 that the corresponding fix point set in $X(b)$ is an isolated fix point. On the other hand, if the image lies in the other coset f, then the fix point set is an image of $O(1, 1)$ in $X(b)$. Hence the corresponding intersection component in $C(b) \cap C(l, h)$ is of degenerate type.

4.7. We now assume that the cycle $C(l, h)$ and the cycle $C(b)$ attached to a rational base point b_0 in $X(l, h)$ are oriented. At every non-degenerate intersection point there is then a well-defined orientation number of value ± 1. We will now provide an effective method for determining this sign.

We denote by $SO(b)^+(\mathbb{R})$ the subgroup of $SO(b)(\mathbb{R})$ consisting of elements of spinor norm one (cf. §1). For the modular group $\mathrm{M}(l, h)$ we put $M^+(l, h) = \{\varphi \in \mathrm{M}(l, h)(\mathbb{R}) \mid \det(\varphi|h) > 0\}$ (cf. 2.2). Recall that a necessary and sufficient condition for $C(b)$ [resp. $C(l, h)$] to be orientable is that $\Gamma(b)$ [resp. $\Gamma(l, h)$] is mapped under $\Gamma \subset SL_3(\mathbb{R})$ into $SO(B)^+(\mathbb{R})$ resp. $M^+(l, h)$. We form

$$D^+_{\mathbb{R}}(\sigma, b) = [SO(b)^+(\mathbb{R})\backslash SL_3(\mathbb{R})]^\sigma / M^+(l, h). \tag{4.7.1}$$

Then the map $\varepsilon: D_\Gamma(\sigma, b) \to D_{\mathbb{R}}(\sigma, b)$ factors through $D^+_{\mathbb{R}}(\sigma, b)$ and we get

$$\varepsilon: D_\Gamma(\sigma, b) \xrightarrow{\varepsilon_+} D^+_{\mathbb{R}}(\sigma, b) \xrightarrow{p} D_{\mathbb{R}}(\sigma, b) \tag{4.7.2}$$

where ε_+ is induced by the inclusion $\Gamma \subset SL_3(\mathbb{R})$ and p denotes the natural projection. In order to state the next result we recall that $D_{\mathbb{R}}(\sigma, b)$ consists of two elements e and f, where e corresponds to the trivial coset [cf. 4.6 (i)].

THEOREM 4.8. (i) *There are exactly two elements e_+ resp. e_- in the inverse image $p^{-1}(e)$ of e in $D_{\mathbb{R}}(\sigma, b)$ under the map $p: D^+_{\mathbb{R}}(\sigma, b) \to D_{\mathbb{R}}(\sigma, b)$.* (ii) *An element $[z]$ in $D_\Gamma(\sigma, b)$ represents a non-degenerate intersection point in $C(b) \cap C(l, h)$ with intersection number $+1$ if and only if $\varepsilon_+([z]) = e_+$ under the map $\varepsilon_+: D_\Gamma(\sigma, b) \to D^+_{\mathbb{R}}(\sigma, b)$.*

Proof. First of all, there is a fibration

$$\begin{array}{ccc} F := SO(b)^+(\mathbb{R})\backslash SO(b)(\mathbb{R}) & \longrightarrow & SO(b)^+(\mathbb{R})\backslash SL_3(\mathbb{R}) =: E \\ \downarrow & & \downarrow \pi \\ * & \longrightarrow & SO(b)(\mathbb{R})\backslash SL_3(\mathbb{R}) =: B \end{array} \tag{4.8.1}$$

of homogeneous spaces. Note that the fibre consists of two elements $\{\pm 1\}$, so this fibration is in fact a regular covering. Since $SO(b)^+(\mathbb{R})\backslash SL_3(\mathbb{R})$ is simply connected, it is the universal covering space over $SO(b)(\mathbb{R})\backslash SL_3(\mathbb{R})$, and the quotient $SO(b)^+(\mathbb{R})\backslash SO(b)(\mathbb{R})$ is isomorphic to the fundamental group of the base. The action of the involution σ preserves this covering, and since σ induces the identity on the fundamental group there is a fibration for the fix point set of σ.

$$\begin{array}{ccc} F & \rightarrow & E^\sigma \\ \downarrow & & \downarrow \\ * & \rightarrow & B^\sigma \end{array} . \tag{4.8.2}$$

In particular, the inverse image over every point in the base consists of two elements. As in the proof of 4.6 (i), we can now assume that (l, h) is the standard pair (l^0, h^0) and $SO(b)(\mathbb{R})$ is the standard orthogonal group $SO(2, 1)(\mathbb{R})$. On the covering space E^σ there is a natural action of $M(l^0, h^0) \cong GL_2(\mathbb{R})$, and π is equivariant with respect to this action. On the base B^σ, the action has two orbits corresponding to the two elements e and f in $D_{\mathbb{R}}(\sigma, b)$. In particular, the orbit passing through the coset $SO(2, 1)(\mathbb{R})\cdot I$ gives rise to the element e, and we observe that the isotropy subgroup at $SO(2, 1)(\mathbb{R})$ is $O(2)(\mathbb{R}) = SO(2, 1)(\mathbb{R}) \cap GL_2(\mathbb{R})$. Hence this orbit can be identified with $O_2(\mathbb{R})\backslash GL_2(\mathbb{R})$. On the covering space there is a similar orbit $SO(2, 1)^+(\mathbb{R})\cdot I\cdot GL_2(\mathbb{R})$ passing through the identity coset, but this time the isotropy subgroup is $SO_2(\mathbb{R}) = SO(2, 1)^+(\mathbb{R}) \cap GL_2(\mathbb{R})$. Hence the orbit is of the form $SO_2(\mathbb{R})\backslash GL_2(\mathbb{R})$. Now it is easy to see that $SO_2(\mathbb{R})\backslash GL_2(\mathbb{R})$ coincides with the inverse image $\pi^{-1}(O_2(\mathbb{R})\backslash GL_2(\mathbb{R}))$. We then observe that under the action of $M^+(l, h) = SL_2(\mathbb{R})$ on $(SO(2, 1)^+(\mathbb{R})\backslash SL_3(\mathbb{R}))^\sigma$ the subspace $SO_2(\mathbb{R})\backslash GL_2(\mathbb{R})$ is invariant and consists of two orbits. This implies that there are exactly two elements in $D^+_{\mathbb{R}}(\sigma, b)$ which project under p onto e. The element lying in the image of the identity coset $SO(b)^+(\mathbb{R})\cdot I\cdot M^+(l, h)$ will be denoted by e_+, the other one by e_-. This proves (i).

In order to prove (ii) we have to analyse the interaction between $D_\Gamma(\sigma, b)$ or $D_{\mathbb{R}}(\sigma, b)$ and the corresponding fix point components. A cocycle $(1, \gamma)$ in the cohomology set $H^1(\sigma, \Gamma)$ is by definition an element $\gamma \in \Gamma$ with $\gamma\cdot\sigma(\gamma) = \gamma\cdot\sigma\cdot\gamma\cdot\sigma^{-1} = 1$ or equivalently $(\gamma\sigma)^2 = 1$, and two cocycles $\gamma, \bar{\gamma}$ represent the same cohomology class if and only if there exists a β in Γ such that $\gamma = \beta^{-1}\bar{\gamma}\sigma(\beta)$, or equivalently $\gamma\sigma = \beta^{-1}\cdot\bar{\gamma}\cdot\sigma\cdot\beta$ i.e. $\gamma\cdot\sigma$ and $\bar{\gamma}\cdot\sigma$ are conjugate by Γ. This implies that the non-abelian cohomology set can be identified with the set of elements of order two of the form $\gamma\cdot\sigma$, $\gamma\in\Gamma$, in the group $\langle\sigma, \Gamma\rangle$ modulo the equivalence relation given by conjugation with elements in Γ. We denote this relation by "$\sim$", then the correspondence is of the form

$$(1, \gamma) \in H^1(\sigma, \Gamma) \leftrightarrow \{\gamma\sigma\} \tag{4.8.1}$$

where $\{\gamma\sigma\}$ denotes the equivalence class of the order two elements $\gamma\sigma$ in $\langle\sigma, \Gamma\rangle$.

Let $(1, \gamma)$ be a cocycle representing a class in $H^1(\sigma, \Gamma)$; the corresponding element $\gamma\cdot\sigma$ induces an involution $\gamma\cdot\sigma$ on X and as before we put

$$X(\gamma\cdot\sigma) = \mathrm{Fix}(\gamma\cdot\sigma, X) \tag{4.8.2}$$

for the fix point set of this involution. Letting $(1, \gamma')$ run over the cocycles cohomologous to $(1, \gamma)$ we obtain a family of submanifolds $X(\gamma'\cdot\sigma)$, $\gamma'\cdot\sigma \sim \gamma\cdot\sigma$. We denote their union by

$$X(\{\gamma\cdot\sigma\}) = \bigcup_{\gamma'\cdot\sigma\sim\gamma\cdot\sigma} X(\gamma'\cdot\sigma) = \bigcup_{z\in\Gamma} X(z^{-1}\gamma\sigma z).$$

An element ω in Γ maps $X(\gamma'\sigma)$ into $X(\omega^{-1}\gamma'\sigma\omega)$, so it operates on $X(\{\gamma\sigma\})$. Since Γ is torsion-free, one sees by a straightforward argument (cf. [10], proof of Proposition 7.5) that

$X(\{\gamma\sigma\})$ is a disjoint union,

$$X(\{\gamma\sigma\}) = \coprod_{z\in\Gamma} X(z^{-1}\gamma\sigma z) \tag{4.8.3}$$

and its orbit space $X(\{\gamma\sigma\})/\Gamma$ under the action of Γ coincides with the connected component $F(\gamma)$ of Fix $(\sigma, X/\Gamma)$ corresponding to the cohomology class in $H^1(\sigma, \Gamma)$ represented by the cycle $(1, \gamma)$, i.e. we have [cf. (4.1.2)]

$$X(\{\gamma\sigma\})/\Gamma = F(\gamma). \tag{4.8.4}$$

Of course, this procedure applies also to $H^1(\sigma, \Gamma(b))$.

By assumption, the cycle $C(l, h) = X(\{\gamma_0 \cdot \sigma)/\Gamma = F(\gamma_0)$, where $(1, \gamma_0)$ represents the base point in $H^1(\sigma, \Gamma)$, is oriented, and by pulling back this orientation we obtain one on each $X(\gamma' \cdot \sigma)$. Moreover, the action of Γ on $X(\{\gamma_0\sigma\})$ preserves all these orientations. Suppose we have an element $[z]$ in $D_\Gamma(\sigma, b)$ which is sent to e_+ *in* $D^+_{\mathbb{R}}(\sigma, b)$ under the map ε_+. Then we may write $z = \alpha \cdot \beta$ with $\alpha \in SO(b)^+(\mathbb{R})$ and $\beta \in M^+(l, h)$, and we have $z\sigma z^{-1} = \alpha\sigma \cdot \alpha^{-1}$, i.e. the component $X(z\sigma z^{-1})$ is sent to $X(\sigma)$ by α. On the other hand, α is in $SO(b)(\mathbb{R})$, so it keeps $X(b)$ fixed and the intersection point of $X(b)$ and $X(z\sigma z^{-1})$ is sent to the intersection point of $X(b)$ and $X(\sigma)$. In fact, as an element in $SO(b)^+(\mathbb{R})$, α preserves the orientation of $X(b)$. Hence, we only have to verify that α preserves the orientation of $X(z\sigma z^{-1})$ and $X(\sigma)$, because this implies that the intersection number $X(b) \cdot X(z\sigma z^{-1})$ is the same as the intersection number of $X(b) \cdot X(\sigma)$, which is equal to one. For our claim, we observe that since β is in $M^+(l, h)$ the induced mapping $X(\sigma) \to X(\sigma)$ preserves the orientation, and $z \in \Gamma$ preserves the orientations as a map $X(z\sigma z^{-1}) \to X(\sigma)$ as pointed out above; therefore $\alpha = z \cdot \beta^{-1}$ is orientation-preserving.

By a similar argument one verifies that the intersection number is -1 whenever $[z]$ is sent to e_-. We leave the details to the reader.

In dealing with the degenerate intersection points in the intersection of the modular cycle $C(l, h)$ and the hyperbolic cycle $C(b)$, we have the following result concerning their contribution to the intersection number.

PROPOSITION 4.9. *The degenerate intersection curves contained in the intersection $C(b) \cap C(l, h)$ do not contribute to the intersection number of the two given cycles $C(l, h)$ and $C(b)$.*

This follows from the following lemma which says, roughly speaking, that we can pull the manifold $C(b)$ away from $C(l, h)$ along the degenerate intersection circles S^1. Hence, by treating these manifolds as if they were disjoint from each other along these circles, we see that the degenerate intersection points do not affect the intersection number of $C(b)$ and $C(l, h)$.

LEMMA 4.10. *Let X and Y be two oriented submanifolds in an oriented manifold Z of complementary codimensions, i.e.* $\dim Z = \dim X + \dim Y$. *Suppose the intersection $X \cap Y$ of X and Y contains a smoothly embedded circle S^1. Then there exists a neighbourhood U of S^1 in Z with $S^1 = X \cap Y \cap U$ and an isotopy h: $X \times [0, 1] \to Z$ such that:*

(1) *h_0: $X \to Z$ is the inclusion $X \to Z$.*
(2) *Let $V = U \cap X$ be the neighbourhood of S^1 in X; then, on the complement of V, the isotopy h: $(X - V) \times [0, 1] \to Z$ is the trivial isotopy keeping every point fixed.*
(3) *$h_1(V)$ lies in the neighbourhood U and is disjoint from Y.*

Let $N_{S^1}(X)$, $N_{S^1}(Y)$ and $N_{S^1}(Z)$ be the normal bundles of S^1 in X, Y and Z. Then by a dimension argument the sum of $N_{S^1}(X)$ and $N_{S^1}(Y)$ generate a sub-bundle in $N_{S^1}(Z)$ of codimension one; we have

$$N_{S^1}(Z) = N_{S^1}(X) \oplus N_{S^1}(Y) \oplus \eta$$

where η is a one-dimensional line bundle. Since X, Y and Z are orientable manifolds, the first Stiefel–Whitney class of $N_{S^1}(X)$, $N_{S^1}(Y)$ and $N_{S^1}(Z)$ is trivial. By the addition formula for the first Stiefel–Whitney classes, this is also true for η, i.e. $w_1(\eta) = 0$. It follows that $N_{S^1}(X)$, $N_{S^1}(Y)$ and η are trivial line bundles, so their total space is the product of S^1 with the real line $\mathbb{R}$. Let U be a tubular neighbourhood of S^1 in Z. As $N_{S^1}(Z)$ is the sum of these trivial line bundles, the total space of $N_{S^1}(Z)$ has a natural structure of a product of S^1 with $\mathbb{R}^3$ such that $S^1 \times \mathbb{R} \times (1, 0, 0) \approx U \cap X$ and $S^1 \times \mathbb{R} \times (0, 1, 0) \approx U \cap Y$. With this explicit description of U it is easy to define an ε-neighbourhood of $S^1 \times (0, 0, 0)$ in U. Inside this ε-neighbourhood we can define an isotopy h which moves $S^1 \times \mathbb{R} \times (1, 0, 0)$ away from the subspace $S^1 \times \mathbb{R} \times (0, 1, 0)$, and outside this ε-neighbourhood h is the trivial isotopy fixing every element. Extend this isotopy $X \cap U \to Z$ to the rest of the space by letting it be trivial on the complement $X - U$. This proves our claim.

§5. NON-VANISHING ARITHMETIC CYCLES

The techniques developed so far to control the intersection of a modular cycle $C(l, h)$ and an associated hyperbolic cycle $C(b)$ will be used to show that both represent non-trivial homology classes with respect to certain $\Gamma \subset SL_3(\mathbb{Z})$ of finite index.

5.1. We consider a pair (l, h) of a line l and a plane h in $\mathbb{Q}^3$ in general position such that $l_{\mathbb{Z}} \oplus h_{\mathbb{Z}}$ is of discriminant one in $\mathbb{Q}^3$. Since (l, h) is $SL_3(\mathbb{Z})$-equivalent to the standard pair (l^0, h^0) by 2.5, we may restrict our discussion to this case. In the modular symmetric domain $X(l^0, h^0) \subset X$ we choose as a rational base point b_0, the positive definite pairing given by the integral matrix

$$b_0 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & d \end{pmatrix}, \quad d \text{ prime} > 2.$$

The two-dimensional hyperbolic space $X(b)$ associated to these data in §3 is then the one attached to the indefinite symmetric pairing of signature (2, 1) given by the matrix

$$b = \begin{pmatrix} -1 & & \\ & -1 & \\ & & d \end{pmatrix}.$$

If Γ is now an arbitrary torsion-free subgroup Γ of $SL_3(\mathbb{Z})$ of finite index, then the corresponding hyperbolic cycle $X(b)/\Gamma \cap SO(b)(\mathbb{R})$ in X/Γ is a compact manifold if and only if the quadratic form b is anisotropic over $\mathbb{Q}$. The example in 1.1 then implies that $X(b)/\Gamma \cap SO(b)(\mathbb{R})$ is compact if and only if -1 is not a square mod d, i.e. $d \equiv 3 \bmod 4$.

In particular we will consider full congruence subgroups $\Gamma(q) = \{A \in SL_3(\mathbb{Z}) \mid A \equiv I \bmod q\}$ with $q \in \mathbb{N}$, $q \geq 3$. We know from 2.4 that for such a group $\Gamma(q)$ the modular cycle $C(l^0, h^0; \Gamma(q)) = X(l^0, h^0)/\Gamma(q) \cap M(l^0, h^0)$ is orientable. On the other hand, it follows from the discussion in §1 (see [13], p. 121) that for *a fixed d* with $d \equiv 3 \bmod 4$ there exists an *infinite* set J_d of natural numbers (depending only on d) such that for a given $m \in J_d$ the compact hyperbolic cycle $C(b; \Gamma(m)) = X(b)/\Gamma(m) \cap SO(b)(\mathbb{R})$ exists and is orientable. We fix in such a case an orientation on $C(b; \Gamma(m))$ such that at the base point b_0 mod $\Gamma(m)$ the local intersection number of $C(b; \Gamma(m))$ and $C(l^0, h^0; \Gamma(m))$ is one. We then have the following:

THEOREM 5.2. *Let $X(l^0, h^0) \subset X = K\backslash SL_3(\mathbb{R})$ be the modular symmetric domain corresponding to the standard pair* (l^0, h^0) (cf. 2.5). *Choose a rational base point b_0 in $X(l^0, h^0)$ represented by*

$$b_0 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & d \end{pmatrix} \text{ with } d \equiv 3 \bmod 4, \quad d \text{ a prime} > 2.$$

Then there exists an infinite set J_d of natural numbers (depending only on d) such that for $m \in J_d$ the hyperbolic cycle $C(b; \Gamma(m))$ corresponding to the indefinite symmetric matrix

$$b = \begin{pmatrix} -1 & & \\ & -1 & \\ & & d \end{pmatrix}$$

is a compact oriented two-dimensional submanifold in $X/\Gamma(m)$. If one has for an element m of J_d that $d \not\equiv 1 \bmod m$ then the algebraic intersection number of $C(b; \Gamma(m))$ and $C(l^0, h^0; \Gamma(m))$ in $X/\Gamma(m)$ is a positive number.

COROLLARY. *Under the above assumptions with $m \in J_d$, $d \not\equiv 1 \bmod m$, the hyperbolic cycle $C(b; \Gamma(m))$ represents a non-trivial homology class in $H_2(X/\Gamma(m); \mathbb{C})$ and the modular cycle $C(l^0, h^0; \Gamma(m))$ represents a non-trivial relative homology class in $H_3(\bar{X}/\Gamma(m), \partial(\bar{X}/\Gamma(m)); \mathbb{C})$, where $\bar{X}/\Gamma(m)$ denotes the Borel–Serre compactification of $\bar{X}/\Gamma(m)$ (cf. [4]).*

All assertions except the one on the intersection number follow from the discussion above. We now consider, for a fixed d with $d \equiv 3 \bmod 4$ and an $m \in J_d$ such that $C(b; \Gamma(m))$ is orientable and $d \not\equiv 1 \bmod m$, the space $D_\Gamma(\sigma, b) = (\Gamma(b)\backslash\Gamma)^\sigma/\Gamma(l^0, h^0)$ where we have put $\Gamma = \Gamma(m)$. The coset space $\Gamma(b)\backslash\Gamma$ admits a natural interpretation as the space of 3-by-3 integral symmetric matrices $S = (S_{ij})$, $1 \le i, j \le 3$, $S \equiv I \bmod m$ and S is Γ-equivalent to b. The action of σ on such a matrix S changes the entries S_{13}, S_{23} to their negatives, so $(\Gamma(b)\backslash\Gamma)^\sigma$ consists of those S of the form

$$S = \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{33} \end{pmatrix} =: \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{S_0}} & 0 \\ & & 0 \\ 0 & 0 & S_{33} \end{pmatrix}$$

On the other hand, $\Gamma(l^0, h^0)$ is isomorphic to the full congruence subgroup $\Gamma_2(m)$ of $SL_2(\mathbb{Z})$, and it acts on S by

$$S \to \begin{pmatrix} \gamma S_0 \gamma^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & S_{33} \end{pmatrix}.$$

Observe that $\det S = d$, so $S_{33} = d/\det S_0$. Hence we can think of $D_\Gamma(\sigma, b)$ as the space of matrices S_0 as above, up to $\Gamma_2(m)$-equivalence. We have the following possibilities:

(1) $S_{33} = d$, $\quad \det S_0 = 1$
(2) $S_{33} = -d$, $\quad \det S_0 = -1$
(3) $S_{33} = 1$, $\quad \det S_0 = d$
(4) $S_{33} = -1$, $\quad \det S_0 = -d$.

In cases (2) and (4), the matrix S_0 is indefinite. Hence the corresponding S in $D_\Gamma(\sigma, b)$ is mapped under ε onto f in $D_\mathbb{R}(\sigma, b)$, so they give degenerate intersection components. Case (3) cannot occur because there is a matrix A in $\Gamma(m)$ such that

$$A \begin{pmatrix} -1 & & \\ & -1 & \\ & & d \end{pmatrix} A^t = S$$

and this implies that $d \equiv 1 \bmod m$, contradicting our assumption. In the remaining case (1), we still have to consider two alternatives, namely S_0 is positive or negative definite. But the first one would lead to a positive definite matrix S which would not be equivalent to b. So we only have to consider negative definite matrices S_0 with $\det S_0 = 1$. By reduction theory, any such binary form is equivalent under $SL_2(\mathbb{Z})$ to the standard form (cf. [7]), i.e. there is a $\delta \in SL_2(\mathbb{Z})$ such that

$$S_0 = {}^t\delta\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}\delta.$$

This means that the corresponding S is represented by the coset $\Gamma(b)\cdot\delta$ and mapped under $\varepsilon_+ : D_\Gamma(\sigma, b) \to D_{\mathbb{R}}^+(\sigma, b)$ to the $SL_2(\mathbb{R})$-orbit of the identity coset $SO(b)^+(\mathbb{R})\cdot I$ in $D_{\mathbb{R}}^+(\sigma, b)$. From Theorem 4.8 the corresponding intersection point of $C(b; \Gamma)$ and $C(l^0, h^0; \Gamma)$ contributes $+1$ to the intersection number.

5.3. *Remark*. By Poincare duality such modular cycles also give rise to cohomology classes in $H^2(\bar{X}/\Gamma; \mathbb{C})$. By analytic methods (in [16], [17], 9.11) one has a direct sum decomposition of the de Rham cohomology $H(\bar{X}/\Gamma; \mathbb{C}) = H_{\text{cusp}}(\bar{X}/\Gamma, \mathbb{C}) \oplus H_{\text{Eis}}(\bar{X}/\Gamma; \mathbb{C})$ into the cusp cohomology and a subspace $H_{\text{Eis}}(\bar{X}/\Gamma; \mathbb{C})$ generated by classes which can be represented by a regular value of an Eisenstein series or a residue of such. In general, the duals of modular cycles have a component at infinity, but by forming linear combinations of modular cycles we have examples of groups $\Gamma(m)$ where one gets a non-trivial contribution to the cusp cohomology $H^2_{\text{cusp}}(\bar{X}/\Gamma(m), \mathbb{C})$. In view of the results in [10] this application motivates this work; it involves different techniques, e.g. Hecke operators, and it will be the subject of a sequel to this paper.

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Department of Mathematics
Yale University
New Haven, CT 06520
U.S.A.

Mathematisches Institut
d. Universität
Wegelerstr. 10
5300 *Bonn* 1, *F.R.G.*