Loewner chains associated with the generalized Roper–Suffridge extension operator on some domains ✩

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Received 8 November 2006
Available online 29 April 2007
Submitted by Steven G. Krantz

Abstract

In this paper, we consider the generalized Roper–Suffridge extension operator defined by

\[ \Phi_{n,\beta_2,\gamma_2,...,\beta_n,\gamma_n}(f)(z) = \left( f(z_1), \left( \frac{f(z_1)}{z_1} \right)^{\beta_2} (f'(z_1))^{\gamma_2} z_2, \ldots, \left( \frac{f(z_1)}{z_1} \right)^{\beta_n} (f'(z_1))^{\gamma_n} z_n \right) \]

for \( z = (z_1, z_2, \ldots, z_n) \in \Omega_{p_1,p_2,...,p_n} \), where \( 0 \leq \beta_j \leq 1, 0 \leq \gamma_j \leq 1 - \beta_j, p_j > 1 \), and we choose the branch of the power functions such that \( (f(z_1)/z_1)^{\beta_j}|_{z_1=0} = 1 \) and \( (f'(z_1))^{\gamma_j}|_{z_1=0} = 1, j = 1, 2, \ldots, n \).

We prove that the set \( \Phi_{n,\beta_2,\gamma_2,...,\beta_n,\gamma_n}(S(U)) \) can be embedded in Loewner chains and give the answer to the problem of Liu Taishun. We also obtain that the operator \( \Phi_{n,\beta_2,\gamma_2,...,\beta_n,\gamma_n}(f) \) preserves starlikeness or spirallikeness of type \( \alpha \) on \( \Omega_{p_1,p_2,...,p_n} \) for some suitable constants \( \beta_j, \gamma_j \), where \( S(U) \) is the class of all univalent analytic functions on the unit disc \( U \) in the complex plane \( C \) with \( f(0) = 0 \) and \( f'(0) = 1 \).

Keywords: Loewner chain; Roper–Suffridge extension operator; Biholomorphic starlike mapping; Spirallike of type \( \alpha \)

1. Introduction

Let \( C^n \) be the vector space of \( n \)-complex variables \( z = (z_1, z_2, \ldots, z_n) \) with the usual inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \|_2 = \sqrt{\langle \cdot, \cdot \rangle} \). A domain \( \Omega \subset C^n \) is said to be complete circular if \( z \in \Omega \) implies \( \xi z \in \Omega \) for all \( \xi \in C \) with \( |\xi| \leq 1 \).
A domain $\Omega \subset C^n$ is said to be complete Reinhardt if $(z_1, z_2, \ldots, z_n) \in \Omega$ implies $(\xi_1 z_1, \xi_2 z_2, \ldots, \xi_n z_n) \in \Omega$ for all $\xi_j \in C$ with $|\xi_j| \leq 1$, $j = 1, 2, \ldots, n$. A domain $\Omega \subset C^n$ is said to be starlike if $z \in \Omega$ implies $tz \in \Omega$ for $0 \leq t \leq 1$.

The Minkowski functional of a bounded complete circular domain $\Omega$ in $C^n$ is defined by

$$\rho(z) = \inf \left\{ t > 0, \frac{z}{t} \in \Omega \right\}, \quad z \in C^n.$$ 

If $\Omega$ is a bounded convex circular domain in $C^n$, then $\rho(\cdot)$ is a norm of $C^n$ and $\Omega = \{z \in C^n: \rho(z) < 1\}$ (see [28]).

Assume $p_j > 0$ ($j = 1, 2, \ldots, n$). Let

$$\Omega_{p_1, p_2, \ldots, p_n} = \left\{ (z_1, z_2, \ldots, z_n) \in C^n: \sum_{j=1}^n |z_j|^{p_j} < 1 \right\},$$

then $\Omega_{p_1, p_2, \ldots, p_n}$ is a bounded complete Reinhardt domain in $C^n$, and the Minkowski functional $\rho(z)$ of $\Omega_{p_1, p_2, \ldots, p_n}$ satisfies

$$\sum_{j=1}^n \left| \frac{z_j}{\rho(z)} \right|^{p_j} = 1.$$ (1.2)

If $p_j > 1$ ($j = 1, 2, \ldots, n$), then $\Omega_{p_1, p_2, \ldots, p_n}$ is a bounded convex Reinhardt domain in $C^n$, and the Minkowski functional $\rho(z)$ of $\Omega_{p_1, p_2, \ldots, p_n}$ is a $C^1$ function on $\Omega_{p_1, p_2, \ldots, p_n} \setminus \{0\}$.

Suppose $\Omega \subset C^n$ is a bounded complete circular domain, $H(\Omega)$ denotes the class of all holomorphic mappings $f: \Omega \to C^n$. The first Fréchet derivative and the second Fréchet derivative of a mapping $f \in H(\Omega)$ at point $z$ are denoted by $Df(z)(\cdot)$, $D^2 f(z)(b, \cdot)$, respectively. Their matrix representations are

$$Df(z) = \left( \frac{\partial f_j(z)}{\partial z_k} \right)_{1 \leq j, k \leq n}, \quad D^2 f(z)(b, \cdot) = \left( \sum_{l=1}^n \frac{\partial^2 f_j(z)}{\partial z_k \partial z_l} b_l \right)_{1 \leq j, k \leq n},$$

where $f(z) = (f_1(z), \ldots, f_n(z))$, $b = (b_1, \ldots, b_n) \in C^n$. A mapping $f \in H(\Omega)$ is said to be locally biholomorphic on $\Omega$ if $f$ has a local inverse at each point $z \in \Omega$ or, equivalently, if $\det Df(z) \neq 0$ at each point on $\Omega$.

Let $N(\Omega)$ denote the class of all locally biholomorphic mappings $f: \Omega \to C^n$ such that $f(0) = 0$, $Df(0) = I$, where $I$ is the unit matrix of $n \times n$. If $f \in N(\Omega)$ is a biholomorphic mapping on $\Omega$ and $f(\cdot)$ is a starlike domain in $C^n$, then we say that $f$ is a biholomorphic starlike mapping on $\Omega$. The class of all biholomorphic starlike mappings on $\Omega$ with $f(0) = 0$, $Df(0) = I$ is denoted by $S^*(\Omega)$. Suppose that $a \in (-\frac{x}{2}, \frac{x}{2})$. If $f \in N(\Omega)$ is a biholomorphic mapping on $\Omega$ and $e^{-\alpha t} f(\cdot) \subset f(\cdot)$ for all $t \geq 0$, then we say that $f$ is a biholomorphic spiralike mapping of type $\alpha$ on $\Omega$. The class of all biholomorphic spiralike mappings of type $\alpha$ on $\Omega$ with $f(0) = 0$, $Df(0) = I$ is denoted by $S_{\alpha}(\Omega)$. Let $S(\Omega)$ be the set of all biholomorphic mappings on $\Omega$ with $f(0) = 0$, $Df(0) = I$. It is obvious that $S_{\alpha}(\Omega) \subset S(\Omega)$ and $S_{0}(\Omega) = S^*(\Omega)$.

Let $\| \cdot \|$ be an arbitrary norm of $C^n$. Suppose that $B = \{z \in C^n: \|z\| < 1\}$ is the unit ball in $(C^n, \| \cdot \|)$ and $U$ is the unit disc in the complex plane $C$. Let $L(C^n, C^m)$ be the space of all continuous linear operators from $C^n$ into $C^m$ with the standard operator norm. For each $z \in C^n \setminus \{0\}$, we set

$$T(z) = \left\{ l_z \in L(C^n, C): \|l_z(z)\| = \|z\|, \|l_z\| = 1 \right\}.$$ 

Then this set is nonempty by the Hahn–Banach theorem (see [28]). Let

$$\mathcal{P} = \left\{ p \in H(U): p(0) = 1, \ Re p(z) > 0, z \in U \right\}, \quad \mathcal{M} = \left\{ p \in H(B): p(0) = 0, \ Dp(0) = I, \ Re \{p(z)\} > 0, z \in B \setminus \{0\}, l_z \in T(z) \right\}.$$ 

If $f, g \in H(B)$, we say that $f$ is subordinate to $g$, and write $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz mapping $v$ (i.e., $v \in H(B)$, $v(0) = 0$, and $\|v(z)\| < 1$, $z \in B$) such that $f(z) = g(v(z))$ for $z \in B$. If $g$ is a biholomorphic mapping on $B$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(B) \subset g(B)$.

We recall that a mapping $F: B \times [0, \infty) \to C^n$ is called a Loewner chain if $F(\cdot,t)$ is biholomorphic on $B$, $F(0,t) = 0$, $DF(0,t) = e^{tI}$ for $t \geq 0$ and

$$F(z,s) < F(z,t), \quad z \in B, \ 0 \leq s \leq t < \infty.$$
Let \( S^1(B) \) denote the subset of \( S(B) \) which can be embedded in Loewner chains, i.e., \( F \in S^1(B) \) if and only if there exists a Loewner chain \( F(z, t) \) such that \( F(z) = F(z, 0) \) for \( z \in B \). It is well known that in the case of one complex variable, \( S^1(U) \equiv S(U) \) (see [25]); but, in \( C^n \) \((n \geq 2)\), \( S^1(B) \neq S(B) \), in fact, the set \( S(B) \) is larger than \( S^1(B) \) (see [5,11]). Hence, generating mappings in \( S^1(B) \) arouse great interest. Pfaltzgraff and Suffridge [24] showed that \( f \in S^2(B) \) if and only if \( f(z, t) = e^t f(z) \), \( z \in B \), \( t \geq 0 \), is a Loewner chain. Hence \( S^2(B) \subset S^1(B) \). Some other subsets of \( S^1(B) \) are given in [5].

In geometric theory of one complex variable, Loewner chains [22] are a very powerful tool to study univalent functions (see [9,25]). In order to investigate biholomorphic mappings of several complex variables, Pfaltzgraff [23] generalized Loewner chains to higher dimensions. Later contributions permitting generalizations to the unit ball of a complex Banach space were made by Poreda [26]. Finally, Graham, Hamada, Kohr et al. perfected this subject and gave various applications, including univalence criteria and characterizations of subclasses of biholomorphic mappings (see [5–14,24], etc.).

In 1995, Roper and Suffridge [27] introduced an extension operator. This operator is defined for normalized locally biholomorphic function \( f \) on the unit disc \( U \) in \( C \) by

\[
F(z) = \Phi_n(f)(z) = \left( f(z_1), \sqrt{f'(z_1)z_0} \right),
\]

where \( z = (z_1, z_0) \) belongs to the unit ball \( B^n \) in \( C^n \), \( z_1 \in U \), \( z_0 = (z_2, \ldots, z_n) \in C^{n-1} \), and we choose the branch of the square root such that \( \sqrt{f'(0)} = 1 \).

Roper and Suffridge [27] proved that: If \( f \in K(U) \), then \( F = \Phi_2(f) \in K(B^2) \), where \( K(\Omega) \) is the class of all biholomorphic convex mappings on \( \Omega \). However, its proof is very complex, Graham and Kohr [7] gave a simplified proof of the theorem of Roper and Suffridge. After that, the other properties of Roper–Suffridge operator were studied by Graham, Hamada, Kohr, Gong and Liu, and others (see [3–6,8–12,14,19–21]). We generalized Roper–Suffridge operator to Banach spaces in [15–17,30,31].

In this paper, we shall discuss some properties of the generalized Roper–Suffridge extension operator defined by

\[
\Phi_{\beta_1, \beta_2, \gamma_1, \ldots, \beta_n, \gamma_n}(f)(z) = \left( f(z_1), \left( \frac{f(z_1)}{z_1} \right)^{\beta_2} (f'(z_1))^{\gamma_2} z_2, \ldots, \left( \frac{f(z_1)}{z_1} \right)^{\beta_n} (f'(z_1))^{\gamma_n} z_n \right)
\]

for \( z = (z_1, z_2, \ldots, z_n) \in \Omega_{p_1, p_2, \ldots, p_n} \), where \( 0 \leq \beta_j \leq 1, 0 \leq \gamma_j \leq 1 - \beta_j, p_j > 1 \), and we choose the branch of the power functions such that \( \left( \frac{f(z_1)}{z_1} \right)^{\beta_j} \big| z_1 = 0 = 1 \) and \( (f'(z_1))^{\gamma_j} \big| z_1 = 0 = 1 \), \( j = 1, 2, \ldots, n \), \( \Omega_{p_1, p_2, \ldots, p_n} = \{ (z_1, z_2, \ldots, z_n) \in C^n : \sum_{j=1}^{n} |z_j|^{p_j} < 1 \} \).

Recently, Liu and Xu [14] proved the following result.

**Theorem A.** Suppose that \( n \geq 2 \), \( p_1 = 2 \), \( p_j > 1 \), \( \beta_j \in [0,1] \), \( \gamma_j \in [0, \frac{1}{p_j}] \) with \( \beta_j + \gamma_j \leq 1 \), \( j = 2, 3, \ldots, n \). Then \( \Phi_{\beta_1, \beta_2, \gamma_1, \ldots, \beta_n, \gamma_n}(S(U)) \) can be embedded in Loewner chains, where \( \Omega_{2, p_2, \ldots, p_n} \) is defined by (1.1), and \( \Phi_{\beta_1, \beta_2, \gamma_1, \ldots, \beta_n, \gamma_n}(f)(z) \) is defined by (1.4) for \( f \in S(U) \) and \( \Phi_{2, p_2, \ldots, p_n} \).

But, they did not know whether the operator \( \Phi_{\beta_1, \beta_2, \gamma_1, \ldots, \beta_n, \gamma_n}(f)(z) \) can be embedded in a Loewner chain on the more general domain \( \Omega_{p_1, p_2, \ldots, p_n} \). The purpose of this paper is to give the answer to the above problem [14]. Thus we are able to replace the exponent 2 for the main variable \( z_1 \) by a general exponent \( p_1 > 1 \).

**2. Main results**

In order to obtain our main results, we need the following lemmas.

**Lemma 2.1.** (See [25].) A family of functions \( \{ f(z,t) \}_{t \geq 0} \) with \( f(0, t) = 0 \), \( f'(0, t) = e^t \), is a Loewner chain if and only if the following conditions hold:

(i) There exist \( r \in (0,1) \) and a constant \( M \geq 0 \) such that \( f(\cdot, t) \) is holomorphic on \( D_r \) for each \( t \geq 0 \), where \( D_r = \{ z \in C : |z| < r \} \), locally absolutely continuous in \( t \geq 0 \) locally uniformly with respect to \( z \in D_r \), and \( |f(z, t)| \leq Me^t, |z| \leq r, t \geq 0 \).
(ii) There exists a function $p(z, t)$ such that $p(\cdot, t) \in \mathcal{P}$ for each $t \geq 0$, $p(z, \cdot)$ is measurable on $[0, +\infty)$ for each $z \in U$, and for all $z \in D_r$, $\frac{\partial f}{\partial t}(z, t) = zf'(z)p(z, t)$, a.e. $t \geq 0$.

(iii) For each $t \geq 0$, $f(\cdot, t)$ is the analytic continuation of $f(\cdot, t)|_{D_r}$ to $U$, and furthermore this analytic continuation exists under the assumptions (i) and (ii).

**Lemma 2.2.** (See [26].) Let $f(z, t) = e^t z + \cdots$ be a mapping from $B \times [0, +\infty)$ into $C^n$ such that

(a) $f(\cdot, t) \in H(B)$ for each $t \geq 0$;
(b) $f(z, t)$ is a locally absolutely continuous function of $t \in [0, +\infty)$ locally uniformly with respect to $z \in B$.

Let $h : B \times [0, +\infty) \to C^n$ satisfy the following conditions:

(i) $h(\cdot, t) \in \mathcal{M}$, $t \geq 0$;
(ii) for each $z \in B$, $h(z, t)$ is a measurable function of $t \in [0, +\infty)$.

Suppose that
\[
\frac{\partial f}{\partial t}(z, t) = Df(z, t)h(z, t), \quad \text{a.e. } t \geq 0,
\]
and for all $z \in B$, and suppose there exists a nonnegative sequence $\{t_m\}$, increasing to $+\infty$, such that
\[
\lim_{m \to +\infty} e^{-t_m} f(z, t_m) = G(z)
\]
locally uniformly on $B$. Then $f(z, t)$ is a Loewner chain on $B$ and the following inequalities hold:
\[
\frac{\|z\|}{(1 + \|z\|)^2} \leq \|e^{-t} f(z, t)\| \leq \frac{\|z\|}{(1 - \|z\|)^2}, \quad z \in B, \quad 0 \leq t < +\infty.
\]

In particular, if $f(z) = f(z, 0)$, then
\[
\frac{\|z\|}{(1 + \|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1 - \|z\|)^2}, \quad z \in B.
\]

**Lemma 2.3.** Suppose that $\Omega \subset C^n$ is a bounded complete Reinhardt domain, its Minkowski functional $\rho(z)$ is a $C^1$ function except for a lower dimensional manifold $\Omega_0$ in $\overline{\Omega}$. Then we have the following properties of Minkowski functional $\rho(z)$:

1. $\frac{\partial \rho(z)}{\partial z_j} z_j \geq 0$ for $z = (z_1, z_2, \ldots, z_n) \in \Omega \setminus \Omega_0$ and $j = 1, 2, \ldots, n$.
2. $\rho(z) = 2 \sum_{j=1}^n \frac{\partial \rho(z)}{\partial z_j} z_j$ for $z = (z_1, z_2, \ldots, z_n) \in \Omega \setminus \Omega_0$.

**Proof.** (1) First we use the same method as in [2] to prove that $\frac{\partial \rho(z)}{\partial z_j} z_j$ is a real number for $z = (z_1, z_2, \ldots, z_n) \in \Omega \setminus \Omega_0$, where $j = 1, 2, \ldots, n$.

In fact, fix $j$ ($1 \leq j \leq n$) and $z = (z_1, z_2, \ldots, z_n) \in \Omega \setminus \Omega_0$, since $\Omega$ is a bounded complete Reinhardt domain, then we have
\[
(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n) \in \Omega \iff (z_1, \ldots, z_{j-1}, z_j e^{i\theta}, z_{j+1}, \ldots, z_n) \in \Omega, \quad -\pi < \theta < \pi.
\]

By the definition of $\rho(z)$, we obtain
\[
\rho(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n) = \rho(z_1, \ldots, z_{j-1}, z_j e^{i\theta}, z_{j+1}, \ldots, z_n), \quad -\pi < \theta < \pi.
\]

It follows that
\[
0 = \frac{d}{d\theta} \rho(z_1, \ldots, z_{j-1}, z_j e^{i\theta}, z_{j+1}, \ldots, z_n) \bigg|_{\theta=0} = \frac{\partial \rho(z)}{\partial z_j} z_j i - \frac{\partial \rho(z)}{\partial z_j} z_j i.
\]

Hence we have $\frac{\partial \rho(z)}{\partial z_j} z_j = \frac{\partial \rho(z)}{\partial z_j} z_j = (\frac{\partial \rho(z)}{\partial z_j} z_j)$, i.e., $\frac{\partial \rho(z)}{\partial z_j} z_j$ is a real number.
Next, we prove that \( \frac{\partial \rho(z)}{\partial z_j} z_j \geq 0 \) for \( z = (z_1, z_2, \ldots, z_n) \in \Omega \setminus \Omega_0 \), where \( j = 1, 2, \ldots, n \).

Suppose that \( z = (z_1, z_2, \ldots, z_n) \in \Omega \setminus \Omega_0 \). Since \( \Omega \) is a bounded complete Reinhardt domain, then we have \( (z_1, \ldots, z_{j-1}, \lambda z_j, z_{j+1}, \ldots, z_n) \in \Omega \) for \( 0 < \lambda \leq 1 \). By the definition of \( \rho(z) \), we obtain
\[
\rho(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n) \geq \rho(z_1, \ldots, z_{j-1}, \lambda z_j, z_{j+1}, \ldots, z_n), \quad 0 < \lambda \leq 1.
\]

It follows that
\[
\frac{d}{d\lambda} \rho(z_1, \ldots, \lambda z_j, \ldots, z_n) \bigg|_{\lambda=1} = \lim_{\lambda \to 1^-} \frac{\rho(z_1, \ldots, \lambda z_j, \ldots, z_n) - \rho(z_1, \ldots, z_j, \ldots, z_n)}{\lambda - 1} \geq 0.
\]

Hence we get
\[
0 \leq \frac{d}{d\lambda} \rho(z_1, \ldots, \lambda z_j, \ldots, z_n) \bigg|_{\lambda=1} = \frac{\partial \rho(z)}{\partial z_j} z_j + \frac{\partial \rho(z)}{\partial \overline{z}_j} \overline{z}_j = 2 \frac{\partial \rho(z)}{\partial z_j} z_j,
\]
where \( j = 1, 2, \ldots, n \).

(2) Since \( \Omega \) is a bounded complete Reinhardt domain, by the definition of \( \rho(z) \), we have \( \rho(tz) = t \rho(z) \) for \( t \in [0, 1] \) (some properties see [18]). Hence for every \( z \in \Omega \setminus \Omega_0 \), we obtain
\[
\rho(z) = \frac{d}{dt} \rho(tz) \bigg|_{t=1} = \sum_{j=1}^{n} \left( \frac{\partial \rho(z)}{\partial z_j} z_j + \frac{\partial \rho(z)}{\partial \overline{z}_j} \overline{z}_j \right) = 2 \sum_{j=1}^{n} \frac{\partial \rho(z)}{\partial z_j} z_j.
\]

This completes the proof. \( \square \)

**Lemma 2.4.**

(1) Suppose that \( g \in H(U) \) satisfies \( g(U) \subset U \). Then
\[
|g'(\xi)| \leq \frac{1 - |g(\xi)|^2}{1 - |\xi|^2} \quad (2.1)
\]
for all \( \xi \in U \).

(2) Suppose that \( p \in H(U) \) satisfies \( \text{Re} \, p(\xi) > 0 \) for \( \xi \in U \) with \( p(0) = 1 \). Then
\[
\text{Re} \left[ p(\xi) + \xi p'(\xi) \right] \geq \frac{1 - \frac{1}{2} |\xi|^2}{1 - |\xi|^2} \text{Re} \, p(\xi) \quad (2.2)
\]
for all \( \xi \in U \).

**Proof.** We only prove the part (2) of Lemma 2.4 because the part (1) of Lemma 2.4 is just Schwarz–Pick’s Lemma (see [1, p. 132]).

Since \( p \in H(U) \) satisfies \( \text{Re} \, p(\xi) > 0 \) for \( \xi \in U \) with \( p(0) = 1 \), then we have \( p(\xi) < \frac{1 + \xi}{1 - \xi} \). Hence there exists a function \( w \in H(U) \) with \( |w(\xi)| \leq |\xi| \) for \( \xi \in U \) such that \( p(\xi) = \frac{1 + w(\xi)}{1 - w(\xi)} \) for \( \xi \in U \). Using the part (1) of Lemma 2.4, we obtain
\[
|\xi p'(\xi)| = \left| \frac{2 \xi w'(\xi)}{(1 - w(\xi))^2} \right| \leq \frac{2 |\xi| (1 - |w(\xi)|^2)}{(1 - |\xi|^2)(1 - |w(\xi)|^2)} = \frac{2 |\xi|}{1 - |\xi|^2} \text{Re} \, p(\xi).
\]

It follows that inequality (2.2) holds. This completes the proof. \( \square \)

**Lemma 2.5.** Let \( p > 0 \) and
\[
a = a(p) = \begin{cases} 1, & \text{if } 0 < p \leq 2, \\ \frac{1}{(\sqrt{2} + 1)^{p-1}} \frac{1}{(\sqrt{2}+1)^p}, & \text{if } p > 2. \end{cases} \quad (2.3)
\]

Then we have
\[
2 - apt^{p-2} + (ap - 2)t^p \leq 0 \quad (2.4)
\]
for \( t \in [\sqrt{2} - 1, 1] \).
**Proof.** Case 1. When $0 < p \leq 2$, let $\phi(t) = 2 - pt^{p-2} + (p - 2)t^p$. Then

$$
\phi'(t) = p(p - 2)t^{p-3}(t^2 - 1) \geq 0
$$

for $t \in [\sqrt{2} - 1, 1]$. Notice that $a = 1$, we get

$$
2 - apr^{p-2} + (ap - 2)t^p = \phi(t) \leq \phi(1) = 0
$$

for $t \in [\sqrt{2} - 1, 1]$.

Case 2. When $p > 2$. From (2.3), some simple computations yield that $a = a(p) > a(2) = 1$ for $p > 2$, and

$$
ap - 2 = \frac{(\sqrt{2} + 1)^p - 1}{\sqrt{2} + 1} - 2 > \frac{(\sqrt{2} + 1)^2 - 1}{\sqrt{2} + 1} - 2 = 0. \tag{2.5}
$$

Let $\psi(t) = 2 - apr^{p-2} + (ap - 2)t^p$ for $t \in [\sqrt{2} - 1, 1]$. Then we have

$$
\psi'(t) = -ap(p - 2)t^{p-3} + p(ap - 2)t^{p-1}
$$

$$
= p(ap - 2)t^{p-3}\left(t^2 - \frac{ap - 2a}{ap - 2}\right)
$$

$$
= p(ap - 2)t^{p-3}\left(t - \frac{ap - 2a}{ap - 2}\right)\left(t + \sqrt{\frac{ap - 2a}{ap - 2}}\right).
$$

Notice that $ap - 2 > 0$, $0 < \sqrt{\frac{ap - 2a}{ap - 2}} < 1$ for $p > 2$ and $\psi(1) = \psi(\sqrt{2} - 1) = 0$, we obtain

$$
2 - apr^{p-2} + (ap - 2)t^p \leq \max_{\sqrt{2} - 1 \leq t \leq 1} \psi(t) = \max\{\psi(\sqrt{2} - 1), \psi(1)\} = 0
$$

for $t \in [\sqrt{2} - 1, 1]$. This completes the proof. \hfill \Box

**Lemma 2.6.** Let $p_1 > 1$, $p_j > 1$, $\gamma_j \in [0, \frac{1}{ap_j}]$ $(j = 2, 3, \ldots, n)$, where $a = a(p_1)$ is defined by (2.3). Suppose that $\rho(z)$ is the Minkowski functional of $\Omega_{p_1, p_2, \ldots, p_n}$ defined by (1.1). Then we have

$$
\frac{\partial \rho(z)}{\partial z_1}z_1 + \frac{1 - 2|z_1| - |z_1|^2}{1 - |z_1|^2} \sum_{j=2}^{n} \gamma_j \frac{\partial \rho(z)}{\partial z_j} z_j \geq 0 \tag{2.6}
$$

for $z = (z_1, z_2, \ldots, z_n) \in \Omega_{p_1, p_2, \ldots, p_n} \setminus \{0\}$.

**Proof.** From Lemma 1.1 in [29], we have

$$
\frac{\partial \rho(z)}{\partial z_j} = \frac{p_j z_j |z_j|^{p_j - 1}}{2p(z) \sum_{k=1}^{n} p_k |z_k|^{p_k} - \frac{z_j}{p(z)} |z_j|^{p_j}}, \quad j = 1, 2, \ldots, n. \tag{2.7}
$$

for $z = (z_1, z_2, \ldots, z_n) \in \Omega_{p_1, p_2, \ldots, p_n} \setminus \{0\}$. This implies that

$$
\frac{\partial \rho(z)}{\partial z_j} z_j \geq 0, \quad j = 1, 2, \ldots, n. \tag{2.8}
$$

for $z = (z_1, z_2, \ldots, z_n) \in \Omega_{p_1, p_2, \ldots, p_n} \setminus \{0\}$.

Now we split into two cases to prove that the inequality (2.6) holds.

Case 1. If $0 \leq |z_1| \leq \sqrt{2} - 1$, then we have $\frac{2|z_1|}{1 - |z_1|^2} \leq 1$. Noting that $\gamma_j \geq 0$ $(j = 2, \ldots, n)$, from (2.8), we obtain

$$
\frac{\partial \rho(z)}{\partial z_1}z_1 + \frac{1 - 2|z_1| - |z_1|^2}{1 - |z_1|^2} \sum_{j=2}^{n} \gamma_j \frac{\partial \rho(z)}{\partial z_j} z_j = \frac{\partial \rho(z)}{\partial z_1}z_1 + \left(1 - \frac{2|z_1|}{1 - |z_1|^2}\right) \sum_{j=2}^{n} \gamma_j \frac{\partial \rho(z)}{\partial z_j} z_j \geq 0
$$

for $z = (z_1, z_2, \ldots, z_n) \in \Omega_{p_1, p_2, \ldots, p_n} \setminus \{0\}$. 

Case 2. If $\sqrt{2} - 1 \leq |z_1| < 1$, we let $w_j = \frac{z_j}{\rho(z)}$, $j = 1, 2, \ldots, n$, $A = \sum_{k=1}^{n} p_k |w_k|^p_k$. By the definition of $\rho(z)$, we have $\rho(z) \leq 1$ for $z \in \Omega_{p_1, p_2, \ldots, p_n}$. Hence we have $\sqrt{2} - 1 \leq |z_1| \leq |w_1| < 1$ and $1 - 2|w_1| - |w_1|^2 \leq 0$. From (2.7), (2.8), Lemma 2.5, noting that $\sum_{j=1}^{n} |w_j|^p_j = 1$ and $\gamma_j \geq 0$ ($j = 2, \ldots, n$), we obtain

$$\frac{\partial \rho(z)}{\partial z_1} z_1 + \frac{1 - 2|z_1| - |z_1|^2}{1 - |z_1|^2} \sum_{j=2}^{n} \gamma_j \frac{\partial \rho(z)}{\partial z_j} z_j$$

$$\geq \frac{\partial \rho(z)}{\partial z_1} z_1 + \left(1 - \frac{2|z_1|}{1 - |z_1|^2}\right) \sum_{j=2}^{n} \gamma_j \frac{\partial \rho(z)}{\partial z_j} z_j$$

$$= \frac{\rho(z)}{2A} \left( p_1 |w_1|^p_1 + \frac{1 - 2|w_1| - |w_1|^2}{1 - |w_1|^2} \sum_{j=2}^{n} \gamma_j p_j |w_j|^p_j \right)$$

$$\geq \frac{\rho(z)}{2A} \left( p_1 |w_1|^p_1 + \frac{1 - 2|w_1| - |w_1|^2}{1 - |w_1|^2} \sum_{j=2}^{n} \gamma_j p_j |w_j|^p_j \right)$$

$$= \frac{\rho(z)}{2 \sqrt{a} (1 - |w_1|^2)} \left( ap_1 |w_1|^p_1 (1 - |w_1|^2) + (1 - 2|w_1| - |w_1|^2) \right)$$

$$= \frac{\rho(z)}{2 \sqrt{a} (1 - |w_1|^2)} \left( |w_1|^2 \left[ 2 - ap_1 |w_1|^p_1 - (ap_1 - 2) |w_1|^p_1 \right] + (1 - |w_1|)^2 (1 - |w_1|^p_1) \right) \geq 0$$

for $z = (z_1, z_2, \ldots, z_n) \in \Omega_{p_1, p_2, \ldots, p_n} \setminus \{0\}$. This completes the proof. \qed

Lemma 2.7. (See [13].) Suppose $f \in S(B)$, $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $a = \tan \alpha$, then $f$ is a spirallike mapping of type $\alpha$ if and only if

$$f(z, t) = e^{1-i\alpha t} f(e^{i\alpha t} z), \quad z \in B, \ t \geq 0,$$

is a Loewner chain. In particular, $f$ is a starlike mapping if and only if $f(z, t) = e^t f(z)$ is a Loewner chain.

Lemma 2.8. Let $p_j \geq 1$ ($j = 1, 2, \ldots, n$). Suppose that $\rho(z)$ is the Minkowski functional of $\Omega_{p_1, p_2, \ldots, p_n}$. Then there exist two positive numbers $A$ and $B$ such that

$$A \|z\|_2 \leq \rho(z) \leq B \|z\|_2, \quad z \in C^n,$$

where $\|z\|_2 = (\sum_{j=1}^{n} |z_j|^2)^{1/2}$ for $z = (z_1, z_2, \ldots, z_n) \in C^n$.

Proof. Let $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, 0, \ldots, 0, 1)$. For every $z = (z_1, z_2, \ldots, z_n) \in C^n$, we have $z = \sum_{j=1}^{n} z_j e_j$. Since $p_j \geq 1$ ($j = 1, 2, \ldots, n$), then $\rho(\cdot)$ is a norm of $C^n$. Hence we obtain

$$\rho(z) \leq \sum_{j=1}^{n} \rho(z_j e_j) = \sum_{j=1}^{n} |z_j| \rho(e_j)$$

$$\leq \left( \sum_{j=1}^{n} |z_j|^2 \right)^{1/2} \left( \sum_{j=1}^{n} [\rho(e_j)]^2 \right)^{1/2} = B \|z\|_2$$

for $z \in C^n$, where $B = (\sum_{j=1}^{n} [\rho(e_j)]^2)^{1/2}$.

On the other hand, since $\Omega_{p_1, p_2, \ldots, p_n}$ is a bounded complete Reinhardt domain, then we have $(0, \ldots, 0, z_j, 0, \ldots, 0) \in \Omega_{p_1, p_2, \ldots, p_n}$ when $(z_1, z_2, \ldots, z_n) \in \Omega_{p_1, p_2, \ldots, p_n}$. By the definition of $\rho(z)$, we have $\rho(z_1, z_2, \ldots, z_n) \geq \rho(0, \ldots, 0, z_j, 0, \ldots, 0) = |z_j| \rho(e_j)$ for $j = 1, 2, \ldots, n$. Hence we obtain
Lemma 2.9. Let $p_j \geq 1$ $(j = 1, 2, \ldots, n)$. Suppose that $\rho(z)$ is the Minkowski functional of $\Omega_{p_1, p_2, \ldots, p_n}$, the norm of $C^n$ is $\| \cdot \| = \rho(\cdot)$. If $\rho(z)$ is differentiable at $z = (z_1, z_2, \ldots, z_n) \in \Omega_{p_1, p_2, \ldots, p_n} \setminus \{0\}$, then we have
\[
T(z) = \{2\langle z, \frac{\partial \rho(z)}{\partial z} \rangle \}, \text{where } \frac{\partial \rho(z)}{\partial z} = (\frac{\partial \rho(z)}{\partial z_1}, \frac{\partial \rho(z)}{\partial z_2}, \ldots, \frac{\partial \rho(z)}{\partial z_n}).
\]

Proof. Suppose that $\rho(z)$ is differentiable at $z = (z_1, z_2, \ldots, z_n) \in \Omega_{p_1, p_2, \ldots, p_n} \setminus \{0\}$, the norm of $C^n$ is $\| \cdot \| = \rho(\cdot)$. Then we have
\[
\Omega_{p_1, p_2, \ldots, p_n} = \{ z \in C^n : \| z \| = \rho(z) < 1 \}.
\]
We let $l_z(\cdot) = 2\langle \cdot, \frac{\partial \rho(z)}{\partial z} \rangle$ and $\Omega_z = \{ w \in C^n : \rho(w) < \rho(z) \}$, then $\Omega_z$ is a convex domain in $C^n$, and $\frac{\partial \rho(z)}{\partial z}$ is the outer normal vector of $\partial \Omega_z$ at $z$. From Lemma 2.3, we have
\[
l_z(z) = 2\langle z, \frac{\partial \rho(z)}{\partial z} \rangle = \rho(z) = \| z \|.
\]
Hence $\| l_z \| \geq 1$.

For every $w \in C^n \setminus \{0\}$, we have $\frac{w}{\rho(w)} \rho(z) \in \partial \Omega_z$. Hence we obtain
\[
\text{Re}\left( z - \frac{w}{\rho(w)} \rho(z), \frac{\partial \rho(z)}{\partial z} \right) \geq 0.
\]
Noting that $2\langle z, \frac{\partial \rho(z)}{\partial z} \rangle = \rho(z)$, we get
\[
2\text{Re}\left( w, \frac{\partial \rho(z)}{\partial z} \right) \leq \rho(w). \tag{2.10}
\]

Set $\theta = \text{arg}\langle w, \frac{\partial \rho(z)}{\partial z} \rangle$ for $\langle w, \frac{\partial \rho(z)}{\partial z} \rangle \neq 0$ and $\theta = 0$ for $\langle w, \frac{\partial \rho(z)}{\partial z} \rangle = 0$. Using (2.10) and the fact that $\rho(e^{-i\theta} w) = \rho(w)$, we have
\[
2\text{Re}\left( e^{-i\theta} w, \frac{\partial \rho(z)}{\partial z} \right) = 2\left| \langle w, \frac{\partial \rho(z)}{\partial z} \rangle \right| \leq \rho(w).
\]
It follows that $|l_z(w)| = 2|\langle w, \frac{\partial \rho(z)}{\partial z} \rangle| \leq \rho(w) = \| w \|$. This implies $\| l_z \| \leq 1$. Hence we have $\| l_z \| = 1$ and $l_z \in T(z)$.

Conversely, suppose that $\rho(z)$ is differentiable at $z = (z_1, z_2, \ldots, z_n) \in \Omega_{p_1, p_2, \ldots, p_n} \setminus \{0\}$. If $l_z(\cdot) \in T(z)$, then $l_z(\cdot)$ is a bounded linear functional for the norm $\rho(\cdot)$ of $C^n$. From Lemma 2.8, $l_z(\cdot)$ also is a bounded linear functional for the norm $\| \cdot \|$ of $C^n$. By Riesz representation theorem (see [28, p. 142]), there exists a vector $y(z) \in C^n$ such that $l_z(\cdot) = \langle \cdot, y(z) \rangle$. Hence the vector $y(z)$ is the normal of the plane
\[
Q_z = \{ w \in C^n : \text{Re}\langle w - z, y(z) \rangle = 0 \} = \{ w \in C^n : \text{Re}l_z(w) = \rho(z) \}.
\]
For every $w = (w_1, w_2, \ldots, w_n) \in Q_z$, we have $\text{Re}l_z(w - z) = \text{Re}l_z(w) - \text{Re}l_z(z) = \rho(z) - \rho(z) = 0$. It follows that
\[
\rho(z) = \text{Re}l_z(z) + t \text{Re}l_z(w - z) = \text{Re}l_z(z + t(w - z)) \leq \| l_z \| \| z + t(w - z) \| = \rho(z + t(w - z)), \quad t \in R.
\]
This implies that $\rho(z) = \min_{t \in R} \rho(z + t(w - z))$. It follows that
\[
0 = \left. \frac{d}{dt} \rho(z + t(w - z)) \right|_{t=0} = \sum_{j=1}^{n} \frac{\partial \rho(z)}{\partial z_j} (w_j - z_j) + \sum_{j=1}^{n} \frac{\partial \rho(z)}{\partial z_j} (w_j - z_j)
\[
\begin{align*}
&= 2 \text{Re} \left\{ \sum_{j=1}^{n} \frac{\partial \rho(z)}{\partial z_j} (w_j - z_j) \right\} \\
&= 2 \text{Re} \left\{ w - z, \frac{\partial \rho(z)}{\partial z} \right\}, \quad w \in Q_z.
\end{align*}
\]
Hence \( \frac{\partial \rho(z)}{\partial z} \) also is a normal vector of the plane \( Q_z \). This implies that \( \gamma(z) = \lambda \frac{\partial \rho(z)}{\partial z} \) for some \( \lambda \in R \setminus \{0\} \). Using the fact that \( l_z(z) = \langle z, \gamma(z) \rangle = \|z\| = \rho(z) \) and \( 2(z, \frac{\partial \rho(z)}{\partial z}) = \rho(z) \), we get \( \lambda = 2 \), which is \( l_z(\cdot) = \langle \cdot, 2 \frac{\partial \rho(z)}{\partial z} \rangle = 2\langle \cdot, \frac{\partial \rho(z)}{\partial z} \rangle \). This completes the proof. \( \square \)

**Theorem 2.1.** Suppose that \( \rho(z) \) is the Minkowski functional of \( \Omega_{p_1,p_2,\ldots,n} \) and \( n \geq 2 \), \( p_1 > 1 \), \( p_j > 1 \), \( \beta_j + \gamma_j \leq 1 \), \( \beta_j \in [0,1], \gamma_j \in [0, \frac{1}{ap_j}], j = 2, 3, \ldots, n \), where \( a = a(p_1) \) defined by (2.3), and \( \Omega_{p_1,p_2,\ldots,n} \) is defined by (1.1). Let \( f \) be univalent in the unit disk \( U \). Then the mapping
\[
\Phi_{n,\beta_2,\gamma_2,\ldots,\beta_n,\gamma_n}(f)(z) = \left( f(z_1), \left( \frac{f(z_1)}{z_1} \right)^{\beta_2} (f'(z_1))^{\gamma_2}, \ldots, \left( \frac{f(z_1)}{z_1} \right)^{\beta_n} (f'(z_1))^{\gamma_n} z_n \right)
\]
can be embedded in a Loewner chain on the unit ball \( B \) of \( C^n \) for the norm \( \| \cdot \| = \rho(\cdot) \), where \( z = (z_1, z_2, \ldots, z_n) \in \Omega_{p_1,p_2,\ldots,n}, \) the branch of the power functions are chosen such that \( (\frac{f(z_1)}{z_1})^{\beta_j} |_{z_1=0} = 1 \) and \( (f'(z_1))^{\gamma_j} |_{z_1=0} = 1 \), \( j = 2, \ldots, n \).

**Proof.** For every \( f \in S(U) \), let \( F_\Phi = \Phi_{n,\beta_2,\gamma_2,\ldots,\beta_n,\gamma_n}(f) \). Since \( f \in S(U) \equiv S^1(U) \), then there exists a Loewner chain \( f(z_1,t) \) on \( U \) such that \( f(z_1) = f(z_1,0) \) for \( z_1 \in U \). From Lemma 2.1, we obtain that the following conditions hold:

(a) \( f(\cdot, t) \) is holomorphic on \( U \) for each \( t \geq 0 \), locally absolutely continuous in \( t \geq 0 \) locally uniformly with respect to \( z_1 \in U \), and for each \( r \in (0, 1) \), there exists a positive constant \( M = M(r) \) such that
\[
|f(z_1,t)| \leq Me^t, \quad |z_1| \leq r, \quad t \geq 0.
\]
(b) There exists a function \( p(z_1,t) \) such that \( p(\cdot, t) \in P \) for each \( t \geq 0 \), \( p(z_1, \cdot) \) is measurable on \( [0, +\infty) \) for each \( z_1 \in U \), and
\[
\frac{\partial f}{\partial t} (z_1,t) = z_1 f'(z_1,t) p(z_1,t), \quad \text{a.e. } t \geq 0,
\]
for all \( z_1 \in U \).

Since \( p_j \geq 1 \) for \( j = 1, 2, \ldots, n \), then \( \rho(\cdot) \) is a norm of \( C^n \) and
\[
\Omega_{p_1,p_2,\ldots,n} = \{ z \in C^n : \|z\| < 1 \} = B.
\]
We define the mapping \( F_\Phi(z,t) \) as
\[
F_\Phi(z,t) = \left( f(z_1,t), e^{(1-\beta_2-\gamma_2)t} \left( \frac{f(z_1,t)}{z_1} \right)^{\beta_2} (f'(z_1,t))^{\gamma_2}, \ldots, e^{(1-\beta_n-\gamma_n)t} \left( \frac{f(z_1,t)}{z_1} \right)^{\beta_n} (f'(z_1,t))^{\gamma_n} z_n \right)
\]
for \( z = (z_1, z_2, \ldots, z_n) \in \Omega_{p_1,p_2,\ldots,n} \) and \( t \geq 0 \).

Now we show that \( F_\Phi(z,t) \) is a Loewner chain on \( B \).

In fact, by a simple computation, we obtain that \( F_\Phi(\cdot, t) \in H(B) \), \( F_\Phi(0, t) = 0 \) and \( DF_\Phi(0, t) = e^t I \) for \( t \geq 0 \), and \( F_\Phi(z,t) \) satisfies the conditions (b) of Lemma 2.2. Computing the derivatives \( \frac{\partial F_\Phi}{\partial t}(z,t) \), we have
\[
\frac{\partial F_\Phi(z,t)}{\partial t} = \left( \frac{\partial f(z_1,t)}{\partial t}, e^{(1-\beta_2-\gamma_2)t} s_2(z_1,t) z_2, \ldots, e^{(1-\beta_n-\gamma_n)t} s_n(z_1,t) z_n \right),
\]
where
From (2.13) and (2.14), we obtain
\[\{\langle \partial_t (z) \rangle \} = (z_1 f'(z_1, t) p(z_1, t), b_2, \ldots, b_n)\]
for a.e. \(t \geq 0\) and \(z \in B\), where
\[b_j = z_j e^{(1-\beta_j-\gamma_j) t} \left( \frac{f(z_1, t)}{z_1} \right)^{\beta_j} \left( f'(z_1, t) \right)^{\gamma_j} \left[ 1 - \beta_j - \gamma_j + \beta_j z_1 f'(z_1, t) p(z_1, t) + \gamma_j \frac{z_1 f''(z_1, t)}{f'(z_1, t)} p(z_1, t) \right], \quad j = 2, \ldots, n.\]

Straightforward calculation yields
\[DF\Phi(z, t) = \begin{pmatrix}
  f'(z_1, t) & 0 & \cdots & 0 \\
  v_2 & \left( \frac{f(z_1, t)}{z_1} \right)^{\beta_2} e^{\gamma_2 t} \left( f'(z_1, t) \right)^{\gamma_2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  v_n & 0 & \cdots & \left( \frac{f(z_1, t)}{z_1} \right)^{\beta_n} e^{\gamma_n t} \left( f'(z_1, t) \right)^{\gamma_n}
\end{pmatrix},\]
and
\[\left( DF\Phi(z, t) \right)^{-1} = \begin{pmatrix}
  \frac{1}{f'(z_1, t)} & 0 & \cdots & 0 \\
  w_2 & \left( \frac{f(z_1, t)}{z_1} \right)^{\beta_2} e^{\gamma_2 t} \left( f'(z_1, t) \right)^{\gamma_2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  w_n & 0 & \cdots & \left( \frac{f(z_1, t)}{z_1} \right)^{\beta_n} e^{\gamma_n t} \left( f'(z_1, t) \right)^{\gamma_n}
\end{pmatrix},\]
where
\[v_j = z_j e^{(1-\beta_j-\gamma_j) t} \left( \frac{f(z_1, t)}{z_1} \right)^{\beta_j} \left( f'(z_1, t) \right)^{\gamma_j} \left[ \beta_j \left( f'(z_1, t) \right)^{\gamma_j} - \frac{1}{z_1} + \gamma_j \frac{f''(z_1, t)}{f'(z_1, t)} \right], \quad j = 2, \ldots, n.\]

From (2.13) and (2.14), we obtain
\[\frac{\partial F\Phi}{\partial t}(z, t) = DF\Phi(z, t) h(z, t)\]
for a.e. \(t \geq 0\) and \(z \in B\), where
\[h(z, t) = (z_1 p(z_1, t), z_2 [1 - \beta_2 + \gamma_2 p(z_1, t) + \gamma_2 z_1 p'(z_1, t)], \ldots, z_n [1 - \beta_n - \gamma_n + (\beta_n + \gamma_n) p(z_1, t) + \gamma_n z_1 p'(z_1, t)]].\]

Clearly, \(h(\cdot, t) \in H(B)\), \(h(0, t) = 0\), \(Dh(0, t) = I\).

Next, we prove that \(h(z, t) \in M\) for \(t \geq 0\).

Since \(p_j > 1\) for \(j = 1, 2, \ldots, n\), then \(\rho(z)\) is differentiable on \(\Omega_{p_1, p_2, \ldots, p_n} \setminus \{0\}\). From Lemma 2.9, we have \(T(\cdot) = \{2(\cdot, \frac{\partial \rho}{\partial z})\}\) for \(z \in \Omega_{p_1, p_2, \ldots, p_n} \setminus \{0\}\). In order to prove \(h(z, t) \in M\) for \(t \geq 0\), we only prove that \(\text{Re}\{\langle h(z, t), \frac{\partial \rho}{\partial z}(z) \rangle\} \geq 0\) for a.e. \(t \geq 0\) and \(z \in B \setminus \{0\}\).
From (2.2), (2.15), Lemmas 2.3, 2.4 and 2.6, we get

$$\text{Re} \left\{ h(z,t), \frac{\partial \rho}{\partial z} (z) \right\} = \frac{\partial \rho(z)}{\partial z_1} z_1 \text{Re} p(z_1,t) + \sum_{j=2}^{n} (1 - \beta_j - \gamma_j) \frac{\partial \rho(z)}{\partial z_j} z_j$$

$$+ \text{Re} p(z_1,t) \sum_{j=2}^{n} \beta_j \frac{\partial \rho(z)}{\partial z_j} z_j + \text{Re} \left[ p(z_1,t) + z_1 p'(z_1,t) \right] \sum_{j=2}^{n} \gamma_j \frac{\partial \rho(z)}{\partial z_j} z_j$$

$$\geq \frac{\partial \rho(z)}{\partial z_1} z_1 \text{Re} p(z_1,t) + \frac{1 - 2|z_1| - |z_1|^2}{1 - |z_1|^2} \text{Re} p(z_1,t) \sum_{j=2}^{n} \gamma_j \frac{\partial \rho(z)}{\partial z_j} z_j$$

$$= \text{Re} p(z_1,t) \left[ \frac{\partial \rho(z)}{\partial z_1} z_1 + \frac{1 - 2|z_1| - |z_1|^2}{1 - |z_1|^2} \sum_{j=2}^{n} \gamma_j \frac{\partial \rho(z)}{\partial z_j} z_j \right] \geq 0$$

for a.e. $t \geq 0$ and $z \in B \setminus \{0\}$. Since $e^{-t} f(\cdot, t)$ is locally uniformly bounded on $U$ for $t \geq 0$, then $\{e^{-t} f(\cdot, t)\}_{t \geq 0}$ is a normal family. Hence there exists a nonnegative sequence $\{t_m\}$, increasing to $+\infty$, such that

$$\lim_{m \to +\infty} e^{-t_m} f(z_1, t_m) = g(z_1)$$

locally uniformly on $U$. It follows that $\lim_{m \to +\infty} e^{-t_m} f'(z_1, t_m) = g'(z_1)$ locally uniformly on $U$. Hence we get

$$\lim_{m \to +\infty} e^{-t_m} F_{\Phi}(z, t_m) = \Phi_{n, \beta_2, \gamma_2, \ldots, \beta_n, \gamma_n}(g)(z)$$

locally uniformly on $B$. By Lemma 2.2, we obtain that $F_{\Phi}(z, t)$ is a Loewner chain on $B$ and $F_{\Phi}(z) = F_{\Phi}(z, 0)$. This is $F_{\Phi} \in S^I(B)$. This completes the proof. □

**Remark 2.1.** Setting $p_1 = 2$ in Theorem 2.1, we obtain Theorem 1 in [14]. From Theorem 2.1, we give the answer to the problem in [14]. Setting $p_1 = p_2 = \cdots = p_n = 2$ in Theorem 2.1, we may obtain Theorem 2.7 in [5], Theorem 2.1 in [6] and Theorem 2.1 in [10].

From Lemmas 2.2, 2.7 and Theorem 2.1, we may obtain the following corollaries.

**Corollary 2.1.** Suppose that $n \geq 2$, $p_1 > 1$, $p_j > 1$, $\beta_j + \gamma_j \leq 1$, $\beta_j \in [0, 1]$, $\gamma_j \in [0, 1/q_j]$ in (2.3), and $\Omega_{p_1, p_2, \ldots, p_n}$ is defined by (1.1). Then we have $\Phi_{a, \beta_2, \gamma_2, \ldots, \beta_n, \gamma_n}(S^* (U)) \subset S^* (\Omega_{p_1, p_2, \ldots, p_n})$.

**Corollary 2.2.** Suppose that $n \geq 2$, $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $p_1 > 1$, $p_j > 1$, $\beta_j + \gamma_j \leq 1$, $\beta_j \in [0, 1]$, $\gamma_j \in [0, 1/q_j]$ in (2.3), and $\Omega_{p_1, p_2, \ldots, p_n}$ is defined by (1.1). Then we have $\Phi_{a, \beta_2, \gamma_2, \ldots, \beta_n, \gamma_n}(S^* (U)) \subset S^* (\Omega_{p_1, p_2, \ldots, p_n})$.

**Corollary 2.3.** Suppose that $f \in S(U)$, $n \geq 2$, $p_1 > 1$, $p_j > 1$, $\beta_j + \gamma_j \leq 1$, $\beta_j \in [0, 1]$, $\gamma_j \in [0, 1/q_j]$ in (2.3), and $\Omega_{p_1, p_2, \ldots, p_n}$ is defined by (1.1). Let $\rho(z)$ be the Minkowski functional of $\Omega_{p_1, p_2, \ldots, p_n}$. Then we have

$$\frac{\rho(z)}{1 + \rho(z)^2} \leq \rho(\Phi_{a, \beta_2, \gamma_2, \ldots, \beta_n, \gamma_n}(f)(z)) \leq \frac{\rho(z)}{(1 - \rho(z)^2)^2}, \quad z \in \Omega_{p_1, p_2, \ldots, p_n}.$$

**Remark 2.2.** Setting $p_1 = 2$ in Corollary 2.2, we get Theorem 3.1 in [20] and Theorem 2.1 in [21]. However, their methods of proof were different from Corollary 2.2. Let $\gamma_j = 0$ ($j = 1, 2, \ldots, n$) in Theorem 2.1, from the proof of Theorem 2.1, we have the following results.

**Theorem 2.2.** Suppose that $n \geq 2$, $\beta_j \in [0, 1]$, $j = 2, 3, \ldots, n$, $\Omega$ is a bounded convex Reinhardt domain and its Minkowski functional $\rho(z)$ of $\Omega$ is a $C^1$ function on $\Omega \setminus \{0\}$. Let $f$ be univalent in the unit disk $U$. Then the mapping
\[ \Psi_{n, \beta_1, \ldots, \beta_n}(f)(z) = \left( f(z_1), \left( \frac{f(z_1)}{z_1} \right)^{\beta_2} z_2, \ldots, \left( \frac{f(z_1)}{z_1} \right)^{\beta_n} z_n \right) \tag{2.16} \]

can be embedded in a Loewner chain on the unit ball \( B \) of \( \mathbb{C}^n \) for the norm \( \| \cdot \| = \rho(\cdot), \) where \( z = (z_1, z_2, \ldots, z_n) \in \Omega, \) and the branch of the power functions are chosen such that \( f(z_1) \) is defined by

**Corollary 2.4.** Suppose that \( \rho(z) \) is the Minkowski functional of \( \Omega_{p_1, p_2, \ldots, p_n} \) and \( n \geq 2, \) \( p_j > 1, \) \( \beta_j \in [0, 1], \) \( j = 2, 3, \ldots, n, \) where \( \Omega_{p_1, p_2, \ldots, p_n} \) is defined by (1.1). Let \( f \) be univalent in the unit disk \( U. \) Then the mapping \( \Psi_{n, \beta_1, \ldots, \beta_n}(f) \) can be embedded in a Loewner chain on the unit ball \( B \) of \( \mathbb{C}^n \) for the norm \( \| \cdot \| = \rho(\cdot), \) where \( \Psi_{n, \beta_1, \ldots, \beta_n}(f) \) is defined in (2.16).

**Remark 2.3.** From Corollary 2.4, we may get Theorem 2 in [14]. Setting \( p_1 = p_2 = \cdots = p_n = 2 \) and \( \beta_1 = \beta_2 = \cdots = \beta_n \) in Corollary 2.4, we obtain Theorem 3.2 in [8]. Lemmas 2.5 and 2.6 are very useful, we may use them to obtain the following theorem, its proof will appear in the sequential paper.

**Theorem 2.3.** Suppose that \( n \geq 2, \) \(-\frac{\pi}{2} < \alpha < \frac{\pi}{2}, \) \( 0 \leq \lambda \leq 1, \) \( p_j > 1, \) \( j = 2, 3, \ldots, n, \) where \( a = a(p_1) \) is defined by (2.3), and \( \Omega_{p_1, p_2, \ldots, p_n} \) is defined by (1.1). The operator \( \Phi_{n, \beta_1, \gamma_1, \ldots, \beta_n, \gamma_n}(f) \) is defined by

\[ \Phi_{n, \beta_1, \gamma_1, \ldots, \beta_n, \gamma_n}(f)(z) = \left( f(z_1), \left( \frac{f(z_1)}{z_1} \right)^{\beta_2} z_2, \ldots, \left( \frac{f(z_1)}{z_1} \right)^{\beta_n} z_n \right) \tag{2.17} \]

for \( z = (z_1, z_2, \ldots, z_n) \in \Omega_{p_1, p_2, \ldots, p_n}. \) Then

1. \( \Phi_{n, \beta_1, \gamma_1, \ldots, \beta_n, \gamma_n}(S_\lambda(U)) \subset S_\lambda' (\Omega_{p_1, p_2, \ldots, p_n}), \)
2. \( \Phi_{n, \beta_1, \gamma_1, \ldots, \beta_n, \gamma_n}(S_\lambda(U)) \subset S_\lambda' (\Omega_{p_1, p_2, \ldots, p_n}), \) and
3. if \( p_j > 1 \) for \( j = 1, 2, \ldots, n, \) then \( \Phi_{n, \beta_1, \gamma_1, \ldots, \beta_n, \gamma_n}(f) \in S_\lambda' (\Omega_{p_1, p_2, \ldots, p_n}) \) if and only if \( f \in S_\lambda(U), \) and \( \Phi_{n, \beta_1, \gamma_1, \ldots, \beta_n, \gamma_n}(f) \in \tilde{S}_\lambda (\Omega_{p_1, p_2, \ldots, p_n}) \) if and only if \( f \in \tilde{S}_\lambda(U), \) where \( S_\lambda' (\Omega_{p_1, p_2, \ldots, p_n}) \) is the class of all normalized starlike mappings of order \( \lambda \) on \( \Omega_{p_1, p_2, \ldots, p_n} \).

**References**