On the Logarithmic Summability and Convergence Factors of Fourier Series and Some Associated Series

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DEDICATED TO PROF. B. H. NEUMANN ON HIS 60TH BIRTHDAY

INTRODUCTION

1. Let $\sum a_n$ be an infinite series and $s_n$ be its $n$th partial sum. If the sequence

$$t_n = \frac{1}{\log n} \sum_{k=1}^{n} \frac{s_k}{k}, \quad (n = 1, 2, \ldots),$$

converges to a limit $s$, then the series $\sum a_n$ is said to be summable by the logarithmic mean or summable $(R, \log n, 1)$ to $s$, and we write $\sum a_n = s (R, \log n, 1)$.

Let $f$ be an integrable function with period $2\pi$ and let its Fourier series be

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t).$$

Its conjugate series is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t).$$

The derived Fourier series of $f$ is $\sum nB_n(t)$ and its conjugate series is $\sum nA_n(t)$.

We write

$$\varphi(t) = \varphi_x(t) = f(x + t) + f(x - t) - 2f(x),$$
$$\psi(t) = \psi_x(t) = f(x + t) - f(x - t).$$

This paper consists of five chapters. In Chapter 1, we prove theorems concerning logarithmic summability and convergence factors of Fourier series. Our theorems contain theorems due to Hardy and Zygmund. In
Chapter 2, we generalize theorems due to Mohanty and Nanda concerning the Riesz logarithmic summability of the derived Fourier series and some associated series. In Chapter 3 we generalize another theorem due to Mohanty and Nanda concerning a generalized derivative. In Chapter 4 we prove theorems concerning convergence factors of series associated with Fourier series, which are generalization of theorems due to Mohanty and Ray. In the last chapter we treat convergence of the series $\sum \mid s_n - s \mid/n$, where $s_n$ is the $n$th partial sum of a Fourier series. Each chapter contains introduction and references.

**CHAPTER 1. ON THE LOGARITHMIC SUMMABILITY AND CONVERGENCE FACTORS OF FOURIER SERIES**

1. Introduction and Theorems

1.1. A. Zygmund [1] has proved the following

**THEOREM I.** If

$$\Phi(t) = \int_0^t \varphi(u) \, du = o(t) \quad as \quad t \to 0,$$

then the Fourier series of $f$ is summable $(R, \log n, 1)$ to $f(x)$ at the point $x$.

Further G. H. Hardy [2] proved the

**THEOREM II.** Suppose that

$$\Phi^*(t) = \int_0^t |\varphi(u)| \, du = o\left(t \log \frac{1}{t}\right) \quad as \quad t \to 0.$$  

Then the Fourier series of $f$ is summable $(R, \log n, 1)$ to $f(x)$ at the point $x$, if and only if

$$\int_t^\infty \frac{\varphi(u)}{u} \, du = o\left(\log \frac{1}{t}\right) \quad as \quad t \to 0.$$  

We shall prove a theorem which contains the above two theorems, and incidentally give a simple proof of Theorem II.

**THEOREM 1.** (i) Suppose that

$$\Phi(t) = \int_0^t \varphi(u) \, du = o\left(t \log \frac{1}{t}\right) \quad as \quad t \to 0.$$
Then the Fourier series of \( f \) is summable \((R, \log n, 1)\) to \( f(x) \) at the point \( x \), if and only if

\[
\int_{1/n}^{\pi} \frac{\Phi(t)}{t^2} \left( \frac{\pi}{2} - \sin nt \right) dt = o(\log n) \quad \text{as} \quad n \to \infty. \tag{5}
\]

(ii) The condition (5) cannot be replaced by either

\[
\int_{1/n}^{\pi} \Phi(t) dt = o(\log n) \quad \text{as} \quad n \to \infty \tag{6}
\]

or

\[
\int_{1/n}^{\pi} \frac{\Phi(t)}{t^2} \sin nt dt = o(\log n) \quad \text{as} \quad n \to \infty, \tag{7}
\]

that is, the factor \((\pi/2 - \sin nt)\) in (5) cannot be replaced by either \(\pi/2\) or \(\sin nt\).

This theorem is a generalization of Theorem II, since condition (4) is weaker than (2) and condition (5) is equivalent to (3) under condition (2). Further, Theorem I is a generalization of Theorem I, since condition (1) is stronger than (4) and also than (5).

1.2. G. H. Hardy [3] has proved the following

**Theorem III.** The sequence \((1/\log(n + 1))\) is a sequence of convergence factors of the Fourier series of \( f \) at a point \( x \) satisfying the condition

\[
\Phi^*(t) = o(t) \quad \text{as} \quad t \to 0. \tag{8}
\]

Recently, R. Mohanty [4] generalized this theorem in the following form:

**Theorem IV.** The sequence \((1/\log(n + 1))\) is a sequence of convergence factors of the Fourier series of \( f \) at a point \( x \) satisfying the condition

\[
\int_{t}^{\pi} \frac{\varphi_n(u)}{u} du = o \left( \log \frac{1}{t} \right) \quad \text{as} \quad t \to 0, \tag{9}
\]

It is easy to see that condition (9) is weaker than (8). R. Mohanty proved this theorem by combining Theorem II and a Tauberian theorem. We can prove this theorem directly (without use of Tauberian theorems) and further generalize as follows.

**Theorem 2.** Let \( a > 0 \). Then the sequence \((1/(\log(n + 1))^a)\) is a sequence of convergence factors of the Fourier series of \( f \) at a point \( x \) satisfying the condition

\[
\int_{t}^{\pi} \frac{\varphi_n(u)}{u} du = o \left( \left( \log \frac{1}{t} \right)^a \right) \quad \text{as} \quad t \to 0. \tag{10}
\]
This theorem is further generalized in the following form:

**Theorem 3.** Let \( a > 0 \). Suppose that

\[
\int_t^\infty \varphi(u) \frac{du}{u} = o \left( (\log \frac{1}{t})^a \right) \quad \text{as} \quad t \to 0.
\]

Then the sequence \( (1/((\log(n+1)^a)) \) is a sequence of convergence factors of the Fourier series of \( f \) at the point \( x \), if and only if

\[
\text{the sequence } \{s_n(x; f)/(\log n)^a\} \text{ converges},
\]

where \( s_n(x; f) \) is the \( n \)th partial sum of the Fourier series.

Condition (10) is stronger than both (11) and (12), so that Theorem 2 is contained in this theorem.

As an application of Theorem 3 we can prove

**Corollary.** Condition (10) in Theorem 2 cannot be replaced by (11).

2. Proof of Theorems.

2.1. Proof of Theorem II. We can suppose that \( f(x) = 0 \). By (1) and the Riemann–Lebesgue theorem,

\[
\frac{\pi}{2} \log n \cdot t_n = \sum_{k=1}^{n} \frac{1}{k} \int_0^\pi \varphi(t) \frac{\sin(k + \frac{1}{2})}{2 \sin t/2} dt
\]

\[
- \sum_{k=1}^{n} \int_0^\pi \varphi(t) \frac{\sin kt}{kt} dt + o(\log n)
\]

\[
= \int_0^\pi \varphi(t) \left( \sum_{k=1}^{n} \frac{\sin kt}{k} \right) dt + o(\log n)
\]

\[
= \int_0^\pi \left( \sum_{k=1}^{n} \cos ku \right) du \int_u^\pi \frac{\varphi(t)}{t} dt + o(\log n)
\]

\[
= \int_0^\pi \frac{\sin nu}{u} du \int_u^\pi \varphi(t) \frac{dt}{t} + o(\log n)
\]

\[
= \int_0^\pi \varphi(t) dt \int_0^t \frac{\sin nu}{u} du + o(\log n)
\]

\[
= \int_0^{1/n} \varphi(t) dt \int_0^{nt} \frac{\sin u}{u} du + o(\log n)
\]

\[
= \int_0^{1/n} \varphi(t) dt \left( \frac{\pi}{2} - \int_{nt}^\infty \frac{\sin u}{u} du \right) + o(\log n)
\]

\[
= u_n + \frac{\pi}{2} \int_{1/n}^\pi \varphi(t) \frac{dt}{t} - v_n + o(\log n) \quad \text{as} \quad n \to \infty.
\]
If we can prove that $u_n = o(\log n)$ and $v_n = o(\log n)$ as $n \to \infty$, then by condition (1) the theorem is proved. Now,

$$\left| \frac{u_n}{\log n} \right| \leq \frac{A}{\log n} \int_0^{1/n} |\varphi(t)| \, dt = o(1)$$

and, using integration by parts, we get

$$\left| \frac{v_n}{\log n} \right| \leq \frac{A}{n \log n} \int_{1/n}^n \frac{|\varphi(t)|}{t^2} \, dt \leq \frac{A}{n \log n} \left( \left[ \frac{\Phi^*(t)}{t^2} \right]_{1/n}^n + \int_{1/n}^n \frac{\Phi^*(t)}{t^3} \, dt \right)$$

$$= o(1).$$

Thus we have completed the proof.

2.2. Proof of Theorem 1 (i). By (12),

$$\frac{\pi}{2} \log n \, t_n = u_n + \int_{1/n}^n \frac{\varphi(t)}{t} \, dt \left( \frac{\pi}{2} - \int_{nt}^\infty \frac{\sin u}{u} \, du \right) + o(\log n)$$

$$= u_n + w_n + o(\log n),$$

where

$$u_n = \int_0^{1/n} \frac{\varphi(t)}{t} \, dt \int_0^{nt} \sin u \, du$$

$$= \sum_{j=0}^{\infty} (-1)^j \frac{1}{(2j+1)!} \int_0^{1/n} \frac{\varphi(t)}{t} \, dt \int_0^{nt} u^{2j} \, du$$

$$= \sum_{j=0}^{\infty} (-1)^j \frac{n^{2j+1}}{(2j+1)!} \frac{\varphi(t)}{t^{2j}} \, dt = o(\log n),$$

since

$$\int_0^{1/n} \varphi(t) \, t^{2j} \, dt = [\Phi(t) t^{2j}]_{1/n}^1 - 2j \int_0^{1/n} \Phi(t) t^{2j-1} \, dt$$

$$= o \left( \frac{\log n}{n^{2j+1}} \right) \quad \text{as} \quad n \to \infty.$$

On the other hand

$$w_n = \int_{1/n}^n \frac{\varphi(t)}{t} \left( \frac{\pi}{2} - \int_t^\infty \frac{\sin u}{u} \, du \right) \, dt$$

$$= \left[ \frac{\Phi(t)}{t} \left( \frac{\pi}{2} - \int_t^\infty \frac{\sin u}{u} \, du \right) \right]_{t=1/n}^n$$

$$+ \int_{1/n}^n \frac{\Phi(t)}{t^2} \left( \frac{\pi}{2} - \int_{nt}^\infty \frac{\sin u}{u} \, du \right) \, dt - \int_{1/n}^n \frac{\Phi(t)}{t} \frac{\sin nt}{t} \, dt$$

$$= \int_{1/n}^n \frac{\Phi(t)}{t^2} \left( \frac{\pi}{2} - \sin nt \right) \, dt + o(\log n).$$

Thus we have proved the first part of the theorem.
2.3. Proof of Theorem 1, (ii). We consider the even periodic function
\[ \Phi(t) = \begin{cases} 0 & \text{on } (\pi/2, \pi), \\ c_k t \log \frac{1}{t} \sin n_k t & \text{on } (\pi/n_k, \pi/n_{k-1}) \quad (k = 1, 2, \ldots), \end{cases} \]
where \( n_k = 2^k \), \( c_k = k^{-2} \) \( (2 > r > 1) \). Evidently \( \Phi \) satisfies condition (4).

Further, \( \varphi \) is integrable, since
\[ \varphi(t) = \Phi'(t) = c_k \log \frac{1}{t} \sin n_k t - c_k \sin n_k t + c_k n_k t \log \frac{1}{t} \cos n_k t \]
on the interval \( (\pi/n_k, \pi/n_{k-1}) \) and
\[
\int_0^\pi |\varphi(t)| \, dt \leq A \sum_{k=1}^\infty \frac{c_k \log n_{k-1}}{n_k} + A \sum_{k=1}^\infty \frac{c_k n_k \log n_{k-1}}{n_k^2} < \infty.
\]
For any integer \( n \), there is a \( j \) such that \( n_{j-1} \leq n < n_j \) and
\[
\left| \int_{\pi/n}^{\pi} \frac{\Phi(t)}{t^2} \, dt \right| = \left| c_j \int_{\pi/n}^{\pi/n_{j-1}} \log \frac{1}{t} \sin n_j t \, dt \right|
+ \sum_{k=1}^{j-1} c_k \int_{\pi/n_k}^{\pi/n_{k-1}} \log \frac{1}{t} \sin n_k t \, dt \leq A \sum_{k=1}^j \frac{c_k n_k \log n_k}{n_k} = o(\log n) \quad \text{as } n \to \infty.
\]
On the other hand
\[
\int_{\pi/n_i}^{\pi} \frac{\Phi(t)}{t^2} \sin n_i t \, dt = c_i \int_{\pi/n_i}^{\pi/n_{i-1}} \log \frac{1}{t} \sin^2 n_i t \, dt
+ \sum_{k=1}^{i-1} c_k \int_{\pi/n_k}^{\pi/n_{k-1}} \log \frac{1}{t} \sin n_k t \sin n_i t \, dt
\geq Ac_i (\log n_i)^2 - A \sum_{k=1}^{i-1} \frac{c_k n_k \log n_i}{n_i - n_k}
\geq A \log n_i \quad \text{for all } i \geq 1.
\]

Thus we have proved that there is an integrable function \( f \), satisfying condition (4), such that (6) holds but (7) does not.
Next we shall consider the function $\Phi(t) = t \sqrt{\log (2\pi/t)}$ on $(0, \pi)$; then (7) holds but (6) does not.

2.4. Proof of Theorem 3. By Abel’s transformation, we have

$$\sum_{n=M}^{N} \frac{a_n}{(\log n)^a} = -\frac{2}{\pi} \frac{1}{(\log M)^a} \int_{0}^{\pi} D_{M-1}(t) f(t) \, dt$$

$$+ \frac{2}{\pi} \frac{1}{(\log N)^a} \int_{0}^{\pi} D_{N}(t) f(t) \, dt$$

$$+ \frac{2}{\pi} \int_{0}^{\pi} \left( \sum_{n=M}^{N-1} \Delta \left( \frac{1}{(\log n)^a} \right) D_{n}(t) \right) f(t) \, dt$$

$$= -\frac{1}{(\log M)^a} s_{M-1}(0; f) + \frac{1}{(\log N)^a} s_{N}(0; f) + \frac{2}{\pi} R.$$

Now,

$$AR = \sum_{n=M}^{N-1} \frac{1}{n(\log n)^{a+1}} \int_{0}^{\pi} \sin nt \frac{f(t)}{t} \, dt + o(1)$$

$$= \int_{0}^{\pi} \left( \sum_{n=M}^{N-1} \frac{\sin nt}{n(\log n)^{a-1}} \right) \frac{f(t)}{t} \, dt + o(1)$$

$$= \int_{0}^{\pi} \left( \sum_{n=M}^{N-1} \frac{\cos nu}{(\log n)^{a+1}} \right) \, du + o(1)$$

$$= -\frac{1}{(\log M)^{a+1}} \int_{0}^{\pi} D_{M-1}(u) \, du \int_{u}^{\pi} \frac{f(t)}{t} \, dt$$

$$+ \frac{1}{(\log(N-1))^{a+1}} \int_{0}^{\pi} D_{N-1}(u) \, du \int_{u}^{\pi} \frac{f(t)}{t} \, dt$$

$$+ \sum_{n=M}^{N-2} \frac{1}{n(\log n)^{a+2}} \int_{0}^{\pi} D_{n}(u) \, du \int_{u}^{\pi} \frac{f(t)}{t} \, dt + o(1)$$

$$= -S_{M} + S_{N} + T + o(1)$$

and

$$T = \sum_{n=M}^{N-2} \frac{1}{n(\log n)^{a+2}} \int_{0}^{\pi} \sin nu \frac{f(t)}{u} \, du \int_{u}^{\pi} \frac{f(t)}{t} \, dt + o(1)$$

$$= \sum_{n=M}^{N-2} \frac{1}{n(\log n)^{a+2}} \left( \int_{0}^{\pi/n} + \int_{\pi/n}^{\pi} \frac{\sin nu}{u} \, du \int_{u}^{\pi} \frac{f(t)}{t} \, dt + o(1) \right)$$

$$= U + V + o(1)$$
where

\[
V = \sum_{n=1}^{N-2} \frac{1}{n(\log n)^{a+2}} \sum_{k=1}^{n-1} \int_{\pi/(k+1)}^{\pi/k} \sin nu \frac{1}{u} \int_{\pi}^u f(t) \, dt
\]

\[
= \sum_{n=M}^{N-2} \sum_{k=1}^{M-1} + \sum_{n=M}^{N-2} \sum_{k=M}^{n-1} = W + X,
\]

\[
|X| = \left| \sum_{k=M}^{N-3} \int_{\pi/(k+1)}^{\pi/k} \frac{1}{u} \int_{\pi}^u f(t) \, dt \left( \sum_{n=k+1}^{N-2} \frac{\sin nu}{n(\log n)^{a+2}} \right) \right|
\]

\[
\leq A \sum_{k=M}^{N-3} \int_{\pi/(k+1)}^{\pi/k} \frac{1}{u(\log 1/u)^{a+2}} \left| \int_{\pi}^u f(t) \, dt \right| + o(1)
\]

\[
\leq A \int_{\pi/M}^{\pi/N} \frac{1}{u(\log 1/u)^{a+2}} + o(1) = o(1),
\]

and

\[
|W| = \left| \sum_{n=M}^{N-2} \frac{1}{n(\log n)^{a+2}} \int_{\pi/n}^{\pi} \sin nu \frac{1}{u} \int_{\pi}^u f(t) \, dt \right|
\]

\[
= \left| \int_{\pi/M}^{\pi/N} \frac{1}{u} \int_{\pi}^u f(t) \, dt \left( \sum_{n=M}^{N-2} \frac{\sin nu}{n(\log n)^{a+2}} \right) \right|
\]

\[
\leq \frac{A}{M(\log M)^{a+2}} \int_{\pi/M}^{\pi/N} \frac{1}{u^2} \left| \int_{\pi}^u f(t) \, dt \right| = o(1).
\]

Hence we have proved that \( V = o(1) \).

\[
U = \sum_{n=M}^{N-2} \frac{1}{n(\log n)^{a+2}} \int_{a}^{\pi/n} \sin nu \frac{1}{u} \int_{\pi}^u f(t) \, dt
\]

\[
= o \left( \sum_{n=M}^{N-2} \frac{1}{n(\log n)^{a+2}} (\log n)^a \right) = o(1).
\]

Therefore \( T = o(1) \). Finally we get

\[
S_M = \frac{1}{(\log M)^{a+1}} \int_0^\pi \frac{\sin Mu}{u} \int_{\pi}^u f(t) \, dt + o(1)
\]

\[
= \frac{1}{(\log M)^{a+1}} \left( \int_0^{\pi/M} + \int_{\pi/M}^\pi \right) \int_{\pi}^u f(t) \, dt + o(1)
\]

\[
= Y + Z + o(1),
\]
where

\[ Y = o \left( \frac{M}{(\log M)^{a+1}} \int_0^{\pi/M} \left( \frac{\log 1}{u} \right)^a du \right) = o(1) \]

and

\[ Z = \frac{1}{(\log M)^{a+1}} \int_\pi^{\pi/M} \frac{\sin Mu}{u} \left( \int_u^\pi f(t) \frac{dt}{t} \right) du = o \left( \frac{1}{(\log M)^{a+1}} \int_\pi^{\pi/M} \left( \frac{\log(1/u)}{u} \right)^a du \right) = o(1) \]

and similarly \( S_N = o(1) \).

2.5. Proof of Corollary. By Theorem 3, it is sufficient to prove that there is a function satisfying condition (11) but not (12). Consider the function

\[ f(t) = k 2^{(1-a)k} t \sin(M_k + 1/2)t \quad \text{on } (\pi/2^{(1-a)k}, \pi/2^{(1-a)k+1}) \quad (k = 1, 2,...) \]

where \( a \) is a positive number and \( M_k = 2^k (k = 1, 2,...) \). It is easy to verify that condition (11) holds, \( f \) is integrable, and

\[ s_{M_k}(0; f) \geq Ak(\log M_k)^a \quad \text{as } \quad k \rightarrow \infty. \]
We shall prove the

**Theorem 1.** (i) Suppose that

\[ \Psi(t) = \int_0^t \psi_\varepsilon(u) \, du = o \left( t^2 \log \frac{1}{t} \right) \quad \text{as} \quad t \to 0. \]  

Then \( \sum nB_n(x) = O(R, \log n, 1) \) if and only if

\[ \int_{1/n}^\infty \Psi(n) \, u^{-3}(\pi + nu \cos(n + 1) u - 3 \sin nu) \, du = o(\log n) \quad \text{as} \quad n \to \infty. \]  

(ii) In condition (4), the terms \( nu \cos nu - 3 \sin nu \) and \( \pi \) cannot be omitted.

This is an analogue of the Fourier series theorem in Chapter 1. Condition (2) is stronger than (3) and also than (4), so that Theorem II is a particular case of Theorem 1.

1.2. P. C. Rath and R. Mohanty [7] has proved the following

**Theorem III.** If the Cauchy integral

\[ \int_0^\infty \psi(t) \, t^{-2} \, dt = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \psi(t) \, t^{-2} \, dt = \lim_{\varepsilon \to 0} \Psi_1(\varepsilon) \]  

exists and

\[ \lim_{k \to \infty} \lim_{n \to \infty} \sup_{n, \epsilon} \int_{n^2}^n \left[ \frac{\psi(t + \epsilon)}{(t + \epsilon)^2} - \frac{\psi(t)}{t^2} \right] \, dt = 0, \]  

then the series \( \sum s_k'(x)/k \) converges, where \( s_k'(x) \) is the \( k \)th partial sum of the derived Fourier series.

We prove the following generalization:

**Theorem 2.** (i) Suppose that the integral (5) exists. Then the series \( \sum s_k'(x)/k \) converges if and only if

\[ \exists \lim_{n \to \infty} \int_{1/n}^n \Psi_1(u) \, (n \cos(n + 1) u - u^{-1} \sin nu) \, du. \]  

(ii) In the condition (7), the term \( n \cos(n + 1)u \) cannot be omitted.

Another consequence of Theorem 2 is
Corollary. If the integral $\Psi_1(t)$ is of bounded variation in a neighborhood of the origin, then the series $\sum s_k(x)/k$ converges.

1.3. P. C. Rath and R. Mohanty [7] proved the following

Theorem IV. If

$$\int_0^t |\psi(t)| t^{-1} dt = o\left(t \log \frac{1}{t}\right) \quad \text{as} \quad t \to 0, \quad (8)$$

$$\Psi_1(t) = \int_t^\pi \psi(u) u^{-2} du = o\left(\log \frac{1}{t}\right) \quad \text{as} \quad t \to 0 \quad (9)$$

and

$$\int_t^\pi \Psi_1(u) u^{-1} du = o\left(\log \frac{1}{t}\right) \quad \text{as} \quad t \to 0, \quad (10)$$

then the series $\sum s_k(x)/k$ is summable $(R, \log n, 1)$.

Our generalization is as follows.

Theorem 3. (i) Suppose that condition (9) holds. Then the series $\sum s_k(x)/k$ is $(R, \log n, 1)$ summable if and only if

$$\exists \lim_{n \to \infty} \frac{1}{\log n} \int_1^{1/n} \Psi_1(u) u^{-1} \left(\frac{\pi}{2} - \sin nu\right) du. \quad (11)$$

(ii) Condition (9) does not imply (11) and the factor $\left(\pi/2 - \sin nu\right)$ cannot be replaced by either $\pi/2$ or $\sin nu$.

This theorem shows that condition (8) in Theorem IV cannot be omitted and that condition (8) can be replaced by

$$\int_0^t |\Psi_1(u)| du = o(t) \quad \text{as} \quad t \to 0.$$

2. Proof of Theorems

2.1. Proof of Theorem 1, (i). By the definition,

$$\pi s_k(x) = \pi \sum_{j=1}^{k} jB_j(x) = \int_0^\pi \psi(t) \left(\sum_{j=1}^{k} j \sin jt\right) dt$$

$$= -\int_0^\pi \psi(t) \frac{d}{dt} \left(\sum_{j=1}^{k} \cos jt\right) dt = -\int_0^\pi \psi(t) \frac{d}{dt} \left(\frac{\sin(k + \frac{1}{2})t}{2 \sin t/2}\right) dt$$

$$= -\int_0^\pi \psi(t) \left(\frac{(k + \frac{1}{2}) \cos(k + \frac{1}{2})t}{2 \sin t/2} - \frac{\sin(k + \frac{1}{2})t \cos t/2}{(2 \sin t/2)^2}\right) dt$$
and then
\[-\pi \sum_{k=1}^{n} \frac{s_k(x)}{k} = \int_{0}^{\pi} \frac{\psi(t)}{t^2} \left( \sin(n+1)t - \sin t - \sum_{k=1}^{n} \frac{\sin kt}{k} \right) dt\]
\[= \int_{0}^{\pi} \frac{\psi(t)}{t^2} \left( \sin(n+1)t - \sin t - \sum_{k=1}^{n} \frac{\sin kt}{k} \right) dt + O(1)\]
\[= \int_{1/n}^{\pi} dt + \int_{1/n}^{\pi} dt + O(1) = u_n + v_n + O(1) \quad (12)\]
by the condition (3). Using the expansion of the sine function,
\[\sin t = \sum_{j=0}^{\infty} (-1)^j \frac{t^{2j+1}}{(2j+1)!},\]
we get
\[u_n = \sum_{j=1}^{\infty} \frac{(-1)^j}{(2j+1)!} \left( (n+1)^{2j+1} - 1 - \sum_{k=1}^{n} k^{2j} \right) \int_{0}^{1/n} \psi(t) \frac{t^{2j-1}}{u} dt = o(\log n)\]
by condition (3) and
\[v_n = \int_{1/n}^{\pi} \frac{\psi(t)}{t^2} \left( \sin(n+1)t - \sum_{k=1}^{n} \frac{\sin kt}{k} \right) dt + O(1) = w_n - x_n + O(1).\]
We shall estimate \(w_n\) and \(x_n\) under assumption (3).
\[x_n = \int_{1/n}^{\pi} \frac{\psi(t)}{t^2} dt \int_{0}^{t} \left( \sum_{k=1}^{n} \cos ku \right) du\]
\[= \int_{1/n}^{\pi} \frac{\psi(t)}{t^2} dt \int_{0}^{t} \left( \frac{\sin(u/2)}{2 \sin u/2} - \frac{1}{2} \right) du\]
\[= \int_{1/n}^{\pi} \frac{\psi(t)}{t^2} dt \int_{0}^{t} \frac{\sin nu}{u} du + O(1) \quad (13)\]
\[= \int_{1/n}^{\pi} \frac{\psi(t)}{t^2} dt \left( \frac{\pi}{2} - \int_{nt}^{\infty} \frac{\sin u}{u} \right) du + O(1)\]
\[= \int_{1/n}^{\pi} \frac{\psi(t)}{t^3} (\pi - \sin nt) dt + o(\log n) \quad \text{as} \quad n \to \infty\]
and
\[w_n = \int_{1/n}^{\pi} \frac{\psi(t)}{t^2} \sin(n+1)t dt\]
\[= \int_{1/n}^{\pi} \frac{\psi(t)}{t^3} (2 \sin nt - nt \cos(n+1)t) dt + o(\log n) \quad \text{as} \quad n \to \infty.\]
Therefore

\[ v_n = - \int_{1/n}^{\pi/n} \frac{\Psi(t) \left( \pi + nt \cos(n + 1) t - 3 \sin nt \right) dt}{t^3} + o(\log n) \]

and then

\[ \frac{1}{\log n} \sum_{k=1}^{n} \frac{s'_k}{k} = \frac{1}{n \log n} \int_{1/n}^{\pi/n} \frac{\Psi(t) \left( \pi + n \cos(n + 1) t - 3 \sin nt \right) dt}{t^3} + o(1) \]

as \( n \to \infty \), which proves Theorem 1, (i).

2.2. Proof of Theorem 1, (ii). Consider the even periodic function defined by

\[ \Psi(t) = c_k t^2 \log(1/t) \sin n_k t \quad \text{on} \quad (\pi/n_k, \pi/n_{k-1}) \quad (k = 2, 3, \ldots), \]

\( -0 \) on \( (\pi/4, \pi) \)

where \( c_k = 1/k^2, \ n_k = 2^k \). Evidently this function satisfies condition (3).

\( \psi(t) = \Psi''(t) \) is integrable, since

\[ \psi(t) = 2c_k t \log(1/t) \sin n_k t - c_k t \sin n_k t + c_k n_k t^2 \log(1/t) \cos n_k t \]

on \( (\pi/n_k, \pi/n_{k-1}) \quad (k = 2, 3, \ldots) \)

and

\[ \int_{0}^{\pi} |\psi(t)| dt \leq A \sum_{k=1}^{\infty} c_k \int_{\pi/n_k}^{\pi/n_{k-1}} t \log \left( \frac{1}{t} \right) dt + \sum_{k=1}^{\infty} c_k n_k \int_{\pi/n_k}^{\pi/n_{k-1}} t^2 \log \left( \frac{1}{t} \right) dt \]

\[ \leq A \sum_{k=1}^{\infty} \frac{c_k \log n_k - 1}{n_k^2} + A \sum_{k=1}^{\infty} \frac{c_k n_k \log n_k}{n_k^3} < \infty. \]

For any integer \( n \), there is an integer \( j \) such that \( n_{j-1} \leq n < n_j \) and then

\[ \left| \int_{\pi/n}^{\pi} \frac{\Psi(t)}{t^3} dt \right| = \left| c_j \int_{\pi/n_j}^{\pi/n_{j-1}} t^{-1} \log \left( \frac{1}{t} \right) \sin n_j t dt \right| \]

\[ + \sum_{k=1}^{j-1} c_k \int_{\pi/n_k}^{\pi/n_{k-1}} t^{-1} \log \left( \frac{1}{t} \right) \sin n_k t dt \]

\[ \leq \sum_{k=1}^{j} c_k \log n_k = o(\log n) \]
as \( n \to \infty \). On the other hand

\[
\int_{n/n_i}^{\pi} \Psi(t) t^{-3}(3 \sin n_i t - n_i t \cos(n_i + 1) t) \, dt
\]

\[
= \sum_{k=1}^{i} c_k \int_{n/n_k}^{\pi/n_k} t^{-1} \log \left( \frac{1}{t} \right) \, dt
\]

\[
\times (3 \sin n_i t \sin n_k t - n_i t \cos(n_i + 1) t \sin n_k t) \, dt
\]

\[
\geq Ac_i (\log n_i)^2 - Ac_i \log n_i - A \sum_{k=1}^{i} c_k \left( \frac{n_k \log n_k}{n_i - n_k} + \frac{n_i \log n_k}{n_i - n_k + 1} \right)
\]

\[
\geq A \log n_i \quad (i = 2, 3, \ldots).
\]

Thus we have proved that the term \( 3 \sin nt - nt \cos(n + 1)t \) cannot be omitted.

Further, we consider the function

\[
\Psi(t) = c_k t^2 \log(1/t) \quad \text{on} \quad (\pi/n_k, \pi/n_{k-1}) \quad (k = 2, 3, \ldots)
\]

\[
= 0 \quad \text{on} \quad (\pi/4, \pi),
\]

where \( c_k = 1/k \) and \( n_k = 2^k \). Then \( \Psi(t) \) satisfies condition (3) and \( \phi(t) = \Psi'(t) \) is integrable. We can prove that

\[
\left| \int_{1/n}^{\pi} \Psi(t) t^{-3}(3 \sin nt - nt \cos(n + 1) t) \, dt \right| = o(\log n)
\]

as \( n \to \infty \). Therefore, the term \( \pi \) in (4) cannot be omitted.

### 2.3. Proof of Theorem 2, (i).

By formula (12),

\[
- \pi \sum_{k=1}^{n} \frac{s_k'(x)}{k} = \int_0^{\pi} \frac{\psi(t)}{(2 \sin t/2)^2} \left( \sin(n + 1) t - \sin t - \sum_{k=1}^{n} \sin \left( \frac{k t}{k} \right) \right) \, dt
\]

\[
= \int_{0}^{\pi/n} + \int_{\pi/n}^{\pi} = u_n' + v_n'.
\]

By the existence of (3),

\[
\int_0^{\pi/n} \psi(t) \frac{t^{2j+1}}{(2 \sin t/2)^3} \, dt = \int_0^{\pi/n} \psi(t) \frac{t^{2j+3}}{(2 \sin t/2)^3} \, dt
\]

\[
= \int_0^{\pi/n} \left( \Psi_1(t) - \Psi_1(t) \left( \frac{\pi}{n} \right) \right) \left( \frac{d}{dt} \frac{t^{2j+3}}{(2 \sin t/2)^3} \right) \, dt
\]

\[
= o(n^{-2j-1}) \quad (j = 1, 2, \ldots)
\]
where \( \Psi_1(t) = \int_0^\pi \psi(u) u^{-2} \, du \), and then

\[
  u_n = \sum_{j=1}^\infty \frac{(-1)^j}{(2j+1)!} \left( (n+1)^{2j+1} - 1 - \sum_{k=1}^n \frac{k^{2j}}{k^2} \right) \int_0^{\pi/n} \frac{\psi(t) t^{2j+1}}{(2 \sin t/2)^2} \, dt = o(1).
\]

Since \( \lim_{t \to 0} \int_0^\pi \frac{(\psi(u) \sin u)}{(2 \sin u/2)^2} \, du \) exists by assumption (5), we can write

\[
  v_n = \int_{\pi/n}^\pi \frac{\psi(t)}{(2 \sin t/2)^2} \left( \sin(n + 1) t - \sum_{k=1}^n \frac{\sin kt}{k^2} \right) \, dt
  - \int_0^{\pi/n} \frac{\psi(t) \sin t}{(2 \sin t/2)^2} \, dt + o(1)
  = w_n - x_n + A + o(1).
\]

Now, similarly to (13), we get

\[
  x_n = \int_{\pi/n}^\pi \frac{\psi(t)}{(2 \sin t/2)^2} \, dt \int_0^{\pi/n} \frac{\sin(n + 1/2) u}{2 \sin u/2} \, du + A + o(1)
  = \int_{\pi/n}^\pi \frac{\psi(t)}{(2 \sin t/2)^2} \, dt \int_0^{\pi/n} \frac{\sin nu}{u} \, du + A + o(1)
  = \int_{\pi/n}^\pi \frac{\psi(t)}{(2 \sin t/2)^2} \, dt \left( \frac{\pi}{2} - \int_t^\infty \frac{\sin nu}{u} \, du \right) + A + o(1)
  = A - \int_{\pi/n}^\pi \frac{\psi(t)}{(2 \sin t/2)^2} \, dt \int_t^\infty \frac{\sin nu}{u} \, du + o(1)
  = A + \int_{\pi/n}^\pi \Psi_2(t) \frac{\sin nt}{t} \, dt + o(1)
\]

where \( \Psi_2(t) = \int_0^\pi \psi(u)/(2 \sin u/2)^2 \, du \) and

\[
  w_n = \int_{\pi/n}^\pi \frac{\psi(t)}{(2 \sin t/2)^2} \sin(n + 1) t \, dt
  = (n + 1) \int_{\pi/n}^\pi \Psi_2(t) \cos(n + 1) t \, dt + o(1)
  = n \int_{\pi/n}^\pi \Psi_2(t) \cos(n + 1) t \, dt + o(1).
\]

Collecting the above estimates, we get

\[
  -\pi \sum_{k=1}^n \frac{s_k(x)}{k} = A + \int_{\pi/n}^\pi \Psi_2(t) (n \cos(n + 1) t - t^{-1} \sin nt) \, dt + o(1)
  = A + \int_{\pi/n}^\pi \Psi_2(t) (n \cos(n + 1) t - t^{-1} \sin nt) \, dt + o(1).
\]

Thus we get Theorem 2, (i).
2.4. Proof of Theorem 2, (ii). We consider the function defined by

\[ \Psi_1(t) = c_k \cos(n_k + 1)t \quad \text{on} \quad (\pi/n_k, \pi/n_{k-1}) \quad (k = 2, 3, \ldots) \]

\[ = 0 \quad \text{on} \quad (\pi/4, \pi) \]

where \( c_k = 1/k \) \( 2^k \) and \( n_k = 2^{2^k} \). Then \( \psi(t) = -t^2 \Psi_1'(t) \) is integrable, \( \Psi_1(0) \) exists and

\[ \limsup_{n \to \infty} n \int_{\pi/n}^\pi \Psi_1(t) \cos(n + 1)t \, dt = \infty, \]

\[ \exists \lim_{n \to \infty} \int_{\pi/n}^\pi \Psi_1(t) t^{-1} \sin nt \, dt. \]

2.5. Proof of Theorem 3, (i). Let \( S_k(x) \) be the \( k \)th partial sum of the series \( \sum_{k=1}^\infty s_k(x)/k \); then, by (12),

\[ -\pi S_n(x) = \int_0^\pi \frac{\psi(t)}{(2 \sin t/2)^2} \left( \sin(n + 1)t - \sin t - \sum_{k=1}^n \frac{\sin kt}{k} \right) \, dt \]

and

\[ -\pi \sum_{n=1}^m S_n(x) = \int_0^\pi \frac{\psi(t)}{(2 \sin t/2)^2} \left( \sum_{n=1}^m \frac{\sin(n + 1)t}{n} - \sin t \sum_{n=1}^m \frac{1}{n} \right. \]

\[ - \left. \sum_{n=1}^m \frac{1}{n} \sum_{k=1}^n \frac{\sin kt}{k} \right) \, dt \]

\[ = \int_0^{1/m} \, dt + \int_{1/m}^\pi \, dt = u_m + v_m. \]

Using the expansion of the sinc function and assumption (9), we get

\[ u_m = \frac{\sum_{j=1}^{\infty} (-1)^j}{(2j + 1)!} \sum_{n=1}^{m} \frac{1}{n} \left( (n + 1)^{2j+1} - 1 - \sum_{k=1}^n k^{2j} \right) \int_0^{1/m} \frac{\psi(t) t^{2j+1}}{(2 \sin t/2)^2} \, dt \]

\[ = o(\log n) \quad \text{as} \quad m \to \infty. \]

Since \( \int_0^\pi \psi(t)/(2 \sin t/2) \, dt \) exists,

\[ v_m = \int_{1/m}^\pi \frac{\psi(t)}{(2 \sin t/2)^2} \left( \sum_{n=1}^m \frac{\sin(n + 1)t}{n} \right) \, dt \]

\[ - \int_{1/m}^\pi \frac{\psi(t)}{(2 \sin t/2)^2} \left( \sum_{n=1}^m \frac{1}{n} \sum_{k=1}^n \frac{\sin kt}{k} \right) \, dt + (A + o(1)) \log m \]

\[ = w_m - x_m + (A + o(1)) \log m \quad \text{as} \quad m \to \infty, \]
where, by (13),

\[ w_m = \int_1^t \frac{\psi(t)}{t} \left( \sum_{n=1}^{m+1} \frac{\sin nt}{n} - \sin t \right) dt + (A + o(1)) \log m \]

\[ = \int_1^t \frac{\psi(t)}{t^2} \left( \sum_{n=1}^{m+1} \frac{\sin nt}{n} \right) dt + (A + o(1)) \log m \]

and then

\[ \sum_{n=1}^{m+1} \frac{\sin nt}{n} = \sum_{n=1}^{m} \frac{1}{n} \sum_{k=1}^{n} \cos kt \]

\[ = \sum_{n=1}^{m} \frac{1}{n} \int_0^t \left( \frac{\sin(n + \frac{1}{2}) \sin nt}{2 \sin \frac{t}{2}} - \frac{1}{2} \right) du \]

\[ = \int_0^t \left( \frac{1}{2 \sin \frac{t}{2}} \sum_{n=1}^{m} \frac{\sin \left( n + \frac{1}{2} \right) u}{n} \right) du - \frac{t}{2} \sum_{n=1}^{m} \frac{1}{n} \]

\[ = \int_0^t \left( \frac{1}{2 \tan \frac{u}{2}} \sum_{n=1}^{m} \frac{\sin nu}{n} - \frac{1}{2} \sum_{n=1}^{m} \frac{1 - \cos nu}{n} \right) du \]

\[ = \int_0^t \frac{du}{2 \tan \frac{u}{2}} \left( \int_0^u \frac{m^2}{2 \tan \frac{v}{2}} dv + \frac{m^2}{2} \right) \]

and then

\[ x_m = \int_1^t \frac{\psi(t)}{t} \left( \sum_{n=1}^{m+1} \frac{\sin nt}{n} - \sin t \right) dt + (A + o(1)) \log m \]

\[ = \int_1^t \frac{\psi(t)}{t^2} \left( \sum_{n=1}^{m+1} \frac{\sin nt}{n} \right) dt + (A + o(1)) \log m \]

\[ = \int_1^t \frac{\psi(t)}{t^2} \left( \sum_{n=1}^{m+1} \frac{\sin nt}{n} \right) dt + (A + o(1)) \log m \]
where
\[ y_m = \int_{1/m}^{\pi} \frac{\psi(t)}{t^2} dt + \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2 (2j+1)!} = o(\log m) \]

and
\[ z_m = \frac{\pi}{2} \int_{1/m}^{\pi} \frac{\Psi_1(t)}{t} dt + o(\log m). \]

Collecting the above estimates, we get
\[ \frac{1}{\log m} \sum_{n=1}^{m} S_n(x) - \frac{1}{\pi \log m} \int_{1/m}^{\pi} \frac{\Psi_1(t)}{t} \left( \frac{\pi}{2} - \sin mt \right) dt = A + o(1). \]

2.6. Proof of Theorem 3, (ii). We shall consider the function
\[ \Psi_1(t) = c_k \log(1/t) \sin n_k t \quad \text{on} \quad (\pi/n_k, \pi/n_{k-1}) \quad (k = 2, 3, \ldots) \]
\[ = 0 \quad \text{on} \quad (\pi/4, \pi) \]

where \( c_k = k^{-k}, n_k = 2^k \). This function satisfies condition (9), \( \psi(t) = -t^2\Psi_1'(t) \) is integrable and \( \int_{\pi/n_k}^{\pi} \Psi_1(t) t^{-1} dt = o(\log n) \) as \( n \to \infty \), but
\[ \int_{\pi/n_k}^{\pi} \Psi(t) \frac{\sin n_k t}{t} dt \geq Ac_k (\log n_k)^2 \quad (k = 2, 3, \ldots). \]

On the other hand, we consider the function
\[ \psi(t) = \frac{t}{\sqrt{\log(2\pi/t)}} \quad \text{on} \quad (0, \pi) \]

which is integrable on \( (0, \pi) \); here
\[ \Psi_1(t) = \int_{\pi/n_k}^{\pi} \psi(u) u^{-2} du \approx \sqrt{\log(2\pi/t)} \quad \text{as} \quad t \to 0, \]
\[ \int_{\pi/n_k}^{\pi} \Psi_1(t) t^{-1} dt \approx (\log n)^{3/2}, \]
\[ \int_{\pi/n_k}^{\pi} \Psi_1(t) t^{-1} \sin nt dt \approx \sqrt{\log n} \quad \text{as} \quad n \to \infty. \]

Thus Theorem 3, (ii) is proved.
CHAPTER 3. ON THE GENERALIZED DERIVATIVE AND LOGARITHMIC MEANS OF THE CONJUGATE DERIVED FOURIER SERIES

1. Introduction and Theorems

R. Mohanty and M. Nanda [8] have proved the following

**Theorem 1.** If

\[
\int_{t}^{\pi} \left| \frac{\varphi(u)}{2 \sin \frac{u}{2}} - d \right| \frac{du}{u} = o \left( \log \frac{1}{t} \right) \quad \text{as} \quad t \to 0, \tag{1}
\]

then

\[
\lim_{n \to \infty} (i_{n}^{\prime}(x) - i_{n}(x)) = \left( \frac{d}{\pi} \right) \log 2, \tag{2}
\]

where \(i_{n}(x)\) denotes the \(n\)th logarithmic mean of the conjugate series of the derived Fourier series of \(f\).

This theorem is easily deduced from the case \(d = 0\) (see [8]), so we can suppose always \(d = 0\) without loss of generality. Our generalization of this theorem with \(d = 0\) is as follows.

**Theorem 1.** (i) Suppose that

\[
\Phi(t) = \int_{t}^{\pi} \varphi(u) u^{-2} du = o \left( \log \frac{1}{t} \right) \quad \text{as} \quad t \to 0, \tag{3}
\]

then relation (2) with \(d = 0\) holds if and only if

\[
\int_{n}^{2n} \frac{dv}{\log v} \int_{1/v}^{\pi} \frac{\varphi(u)}{u} \sin uv du + \int_{n}^{2n} \frac{dv}{v \log v} \int_{1/v}^{\pi} \frac{\varphi(u)}{u^2} \cos uv du = o(1)
\]

as \(n \to \infty\). \tag{4}

(ii) There is a function \(f\) satisfying condition (3) but not (4) and for this function condition (1) with \(d = 0\) in Theorem 1 cannot be replaced by (3). Further the two integrals on the left side of (4) cannot be replaced by either one alone.

**Theorem 2.** (i) If (3) holds,

\[
\int_{1/v}^{\pi} \varphi(t) t^{-1} \sin vt dt = o \left( \frac{\log v}{v} \right) \quad \text{as} \quad v \to \infty, \tag{5}
\]
and
\[
\int_{1/e}^{\pi} \varphi(t) t^{-2} \cos vt \, dt = o(\log v) \quad \text{as} \quad v \to \infty, \quad (6)
\]
then relation (2) with \( d = 0 \) holds.

(ii) Three conditions (3), (5) and (6) are independent, that is, there are functions satisfying any two of (3), (5) and (6), but not the remaining one.

Theorem I is a corollary of Theorem 2, (i) and further Theorem 2, (i) is a corollary of Theorem 1, (i). The examples used in the proof of Theorem 2, (ii), can be used for the proof of Theorem 1, (ii), so we omit the proof of Theorem 1, (ii) and also the proof of Theorem 2, (i).

2. Proof of Theorems

2.1. Proof of Theorem 1, (i). We shall assume (3) only. By the definition,
\[
kA_k(x) = \frac{1}{\pi} \int_0^\pi \varphi(t) k \cos kt \, dt = \frac{1}{\pi} \int_0^\pi \varphi(t) \left( \frac{d}{dt} \sin kt \right) \, dt
\]
and then the \( k \)th partial sum of the conjugate series of the derived Fourier series of \( f \) is
\[
\tilde{s}_k'(x) = -\frac{1}{\pi} \int_0^\pi \varphi(t) \frac{d}{dt} \left( \sum_{j=1}^k \sin jt \right) \, dt.
\]
Hence we get
\[
- \log n \tilde{s}_n'(x) = -\pi \sum_{k=1}^n \left( \frac{s_k'(x)}{k} \right)
\]
\[
= \sum_{k=1}^n \frac{1}{k} \int_0^\pi \varphi(t) \frac{d}{dt} \left( \sum_{j=1}^k \sin jt \right) \, dt
\]
\[
= \sum_{k=1}^n \frac{1}{k} \int_0^{1/n} + \sum_{k=1}^n \frac{1}{k} \int_0^\pi = u_n + v_n.
\]
By assumption (3),
\[
\int_0^{1/n} \varphi(t) t^{2i} \, dt = o \left( \frac{\log n}{n^{2i+2}} \right) \quad \text{as} \quad n \to \infty \quad \text{for} \quad i = 0, 1, 2, \ldots
\]
and then
\[
\left| u_n \right| \leq A \sum_{i=0}^{\infty} \frac{n^{2i+2}}{(2i)!} \int_0^{1/n} \varphi(t) t^{2i} \, dt = o(\log n) \quad \text{as} \quad n \to \infty.
\]
On the other hand,

\[
\varphi_n = \sum_{k=1}^{n} \frac{1}{k} \int_{1/n}^{\pi} \varphi(t) \frac{d}{dt} \left( \frac{1}{2 \tan t/2} - \frac{\cos(k + \frac{1}{2}) t}{2 \sin t/2} \right) dt
\]

\[
= \sum_{k=1}^{n} \frac{1}{k} \int_{1/n}^{\pi} \varphi(t) \left( - \frac{1}{(2 \sin t/2)^2} + \frac{(k + \frac{1}{2}) \sin(k + \frac{1}{2}) t}{2 \sin t/2} + \frac{\cos(k + \frac{1}{2}) t \cos t/2}{(2 \sin t/2)^3} \right) dt
\]

\[
= \int_{1/n}^{\pi} \varphi(t) \left( \sum_{k=1}^{n} \frac{1}{k} \sin \left( k + \frac{1}{2} \right) t \right) dt - \int_{1/n}^{\pi} \varphi(t) \left( \sum_{k=1}^{n} \frac{1 - \cos kt}{k} \right) dt
\]

\[
= \int_{1/n}^{\pi} \varphi(t) \frac{1}{2 \sin t/2} (\cos t - \cos(n + 1) t) dt - \int_{1/n}^{\pi} \varphi(t) \frac{1}{2 \sin t/2} \left( \frac{1}{2 \tan u/2} - \frac{\cos(n + \frac{1}{2}) u}{2 \sin u/2} \right) du
\]

\[
= \varphi_n - \varphi^*_n.
\]

Similarly we write

\[
- \pi \log(2n) \varphi_{2n}(x) = \sum_{k=1}^{2n} \frac{1}{k} \int_{0}^{1/n} + \sum_{k=1}^{2n} \frac{1}{k} \int_{1/n}^{\pi} = u_n^* + \varphi^*_n
\]

and

\[
\varphi_{n*} = \int_{1/n}^{\pi} \varphi(t) \frac{1}{2 \sin t/2} (\cos t - \cos(2n + 1) t) dt - \int_{1/n}^{\pi} \varphi(t) \frac{1}{2 \sin t/2} \left( \frac{1}{2 \tan u/2} - \frac{\cos(2n + \frac{1}{2}) u}{2 \sin u/2} \right) du
\]

\[
= \varphi_{n*} - \varphi_{n*}.
\]

and further we get \( u_{n*} = o(\log n) \) as \( n \to \infty \), in the same way as for \( u_n \).

Now

\[
\frac{\varphi_n}{\log n} - \frac{\varphi_{n*}}{\log(2n)} = \int_{1/n}^{\pi} \varphi(t) \frac{1}{2 \sin t/2} \left( \frac{\cos(2n + 1) t}{\log(2n)} - \frac{\cos(n + 1) t}{\log n} \right) dt + o(1)
\]

\[
= \int_{1/n}^{\pi} \varphi(t) \frac{1}{2 \sin t/2} dt \int_{n}^{2n} \frac{d}{dv} \left( \frac{\cos(v + 1) t}{\log v} \right) dv + o(1)
\]

\[
= - \int_{1/n}^{\pi} \varphi(t) \frac{1}{2 \sin t/2} dt \int_{n}^{2n} \left( \frac{(v + 1) t}{\log v} + \frac{\cos(v + 1) t}{v(\log v)^2} \right) dv + o(1)
\]

\[
= - y_n - y_{n*} + o(1)
\]
where
\[ y_n^* = \int_{1/n}^{\pi} \frac{q(t)}{(2 \sin t/2)^2} dt \int_n^{2n} \frac{\cos(v + 1) t}{v(\log v)^{\delta}} dv \]
\[ = \int_{1/n}^{\pi} \frac{dv}{v(\log v)^{\delta}} \int_{1/n}^{\pi} \frac{q(t)}{(2 \sin t/2)^2} \cos(v + 1) t dt \]
\[ = \int_{1/n}^{\pi} \frac{dv}{v(\log v)^{\delta}} \left( \Phi_1 \left( \frac{1}{n} \right) \frac{v + 1}{n} - \int_{1/n}^{\pi} (v + 1) \Phi_1(t) \sin(v + 1) t dt \right) \]
\[ = \int_{1/n}^{\pi} \Phi_1(t) dt \int_n^{2n} \frac{\sin(v + 1) t}{(1 + 1/v)(\log v)^{\delta}} dv + o(1) \]
\[ = O \left( \frac{1}{(\log n)^{\delta}} \int_{1/n}^{\pi} \Phi_1(t) t^{-1} dt \right) + o(1) = o(1), \]

since \( \Phi_1(t) = \int_t^{\pi} q(u)(2 \sin u/2)^{-2} du = o(\log 1/t) \) as \( t \to 0 \). Further we have
\[ \frac{x_n - x_n^*}{\log n - \log(2n)} \]
\[ = \int_{1/n}^{\pi} \frac{q(t)}{(2 \sin t/2)^2} dt \int_{1/n}^{t} \left( \frac{\cos(2n \frac{1}{2}) u}{\log(2n)} - \frac{\cos(n \frac{1}{2}) u}{\log n} \right) \frac{du}{2 \sin u/2} + o(1) \]
\[ = \int_{1/n}^{\pi} \frac{q(t)}{(2 \sin t/2)^2} dt \int_{1/n}^{t} \frac{1}{u} \left( \frac{\cos(2n \frac{1}{2}) u}{\log(2n)} - \frac{\cos(n \frac{1}{2}) u}{\log n} \right) \frac{du}{u} + o(1) \]
\[ - \int_{1/n}^{\pi} \frac{q(t)}{(2 \sin t/2)^2} dt \int_{1/n}^{t} \frac{du}{u} \int_n^{2n} \frac{d}{dv} \left( \frac{\cos(v + 1) u}{\log v} \right) dv + o(1) \]
\[ = - \int_{1/n}^{\pi} \frac{q(t)}{(2 \sin t/2)^2} \int_{1/n}^{t} \frac{du}{u} \]
\[ \times \int_n^{2n} \left( \frac{u \sin(v + 1) u}{\log v} + \frac{\cos(v + 1) u}{v(\log v)^{\delta}} \right) dv + o(1) \]
\[ = \int_{1/n}^{\pi} \frac{dv}{v(\log v)^{\delta}} \int_{1/n}^{\pi} \frac{q(t)}{(2 \sin t/2)^2} \cos \left( v + \frac{1}{2} \right) dt \]
\[ - \int_{1/n}^{\pi} \frac{dv}{v(\log v)^{\delta}} \int_{1/n}^{\pi} \frac{q(t)}{(2 \sin t/2)^2} dt \int_{1/n}^{t} \frac{\cos(v + 1) u}{u} du + o(1) \]
\[ = z_n - z_n^* + o(1) \]

where
\[ z_n^* = \int_{1/n}^{\pi} \frac{dv}{v(\log v)^{\delta}} \int_{1/n}^{\pi} \frac{q(t)}{(2 \sin t/2)^2} dt \frac{1}{u} \int_{1/n}^{t} \frac{q(t)}{(2 \sin t/2)^2} dt \cos \left( v + \frac{1}{2} \right) u \]
\[ = \int_{1/n}^{\pi} \frac{du}{u} \int_{1/n}^{\pi} \frac{q(t)}{(2 \sin t/2)^2} dt \int_{1/n}^{2n} \frac{\cos(v + 1) u}{v(\log v)^{\delta}} dv \]
\[ = O \left( \int_{1/n}^{\pi} \frac{\Phi_1(u)}{u} \frac{1}{u m(\log n)^{\delta}} du \right) \]
\[ = o \left( \frac{1}{m(\log n)^{\delta}} \int_{1/n}^{1} \frac{\log 1/u}{u^2} du \right) = o(1). \]
Collecting the above estimates we get

\[- \pi(t_{2n}(x) - t_n'(x)) = \int_{1/n}^{2n} \frac{dv}{(v + \frac{1}{2}) \log v} \int_{1/n}^{\pi} \frac{\varphi(t)}{(2 \sin t/2)^2} \cos \left( v + \frac{1}{2} \right) t \, dt + o(1)\]

\[= \int_{1/n}^{2n} \frac{dv}{(v + \frac{1}{2}) \log v} \int_{1/n}^{\pi} \frac{\varphi(t)}{t^2} \cos \left( v + \frac{1}{2} \right) t \, dt + o(1)\]

\[= \int_{1/n}^{2n} \frac{dv}{v \log v} \int_{1/v}^{\pi} \frac{\varphi(t)}{t^2} \cos vt \, dt + o(1)\]

which was proved by using assumption (3) alone. Therefore the condition (4) is necessary and sufficient for \( t_{2n}(x) - t_n'(x) = o(1) \) as \( n \to \infty \). Thus we have proved Theorem 1, (i).

2.2. Proof of Theorem 2, (ii). We shall consider the function \( \varphi \) defined on the interval \((0, \pi)\) by

\[\varphi(t) = c_k t \sin nk t \quad \text{on} \quad (1/n_k, 1/n_{k-1}) \quad (k = 2, 3, \ldots)\]

\[= 0 \quad \text{on} \quad (1/4, \pi) \quad (7)\]

where \( n_k = 2^k \) and \( c_k = 2^k/2^{k-1} \) \((= n_{k-1} \log n_k/n_k)\). Then

\[\int_{1/n_k}^{\pi} \frac{\varphi(t)}{t} \sin nk t \, dt = c_k \int_{1/n_k}^{1/n_{k-1}} \sin^2 nk t \, dt + \sum_{j=1}^{k-1} c_j \int_{1/n_k}^{1/n_{k-1}} \sin nk t \sin n_j t \, dt\]

\[\geq A \frac{c_k}{n_{k-1}} \geq A \frac{\log n_k}{n_k} \quad (k = 2, 3, \ldots)\]

Hence the function (7) does not satisfy condition (5).

Now, for any \( n \), there is an \( i \) such that \( n_{i-1} \leq v < n_i \), and

\[\int_{1/v}^{\pi} \frac{\varphi(t)}{t^2} \cos vt \, dt \]

\[= c_{i-1} \int_{1/v}^{1/n_{i-1}} \frac{\sin nk t \cos vt}{t} \, dt + \sum_{k=1}^{i-1} c_k \int_{1/n_k}^{1/n_{i-1}} \frac{\sin n_k t \cos vt}{t} \, dt\]

\[= O(c_{i-1} + c_{i-2}) + O \left( \frac{1}{v} \sum_{k=1}^{i-2} c_k n_k \right) = o(\log v) \quad \text{as} \quad v \to \infty,\]
so that condition (6) is satisfied by (7). Similarly, condition (3) is also satisfied by (7). Thus we have proved that there is a function which satisfies (3) and (6), but not (5).

Next, we shall take the function

\[ q(t) = c_k t^\alpha \cos nk t \quad \text{on} \quad (1/n_k, 1/n_{k-1}) \quad (k = 2, 3, \ldots) \]

\[ = 0 \quad \text{on} \quad (1/4, \pi) \] (8)

where \( n_k = 2^{2^k} \) and \( c_k = 2^k 2^{k-1} (= n_{k-1} \log n_k) \). This function satisfies conditions (3) and (5), but not (6).

Finally, the function

\[ \varphi(t) = t \quad \text{on} \quad (0, \pi) \]

satisfies (5) and (6), but not (3).

Condition (4) is not satisfied by either (7) or (8). Thus we have shown that each integral on the left side of (4) cannot be dropped.

**Chapter 4. On the Convergence Factors of Fourier Series and of Some Associated Series**

1. Introduction and Theorems

1.1. R. Mohanty and B. K. Ray [9] have proved the following

**Theorem 1.** If (i)

\[ \int \varphi(u) u^{-1} du = o \left( \log \frac{1}{t} \right) \quad \text{as} \quad t \to 0 \]

and (ii) the function \( \varphi(t)/\log(2\pi/t) \) is of bounded variation on an interval \((0, c) \) \( (0 < c < \pi) \), then the series \( \sum A_n(x)/\log n \) is convergent.

In Chapter 1, we have proved that the condition (ii) in the above theorem cannot be omitted. The condition (i) in the above theorem also cannot be omitted, since the function \( \varphi(u) = \log(2\pi/u) \) on the interval \((0, \pi) \) satisfies the condition (ii) but not (i), and its Fourier series is

\[ a_0 - \sum_{n=1}^{\infty} \frac{\cos nx}{n} \int_0^{\pi} \frac{\sin t}{t} \, dt. \]

We prove the

**Theorem 1.** We suppose that (i) the function \( g(t) = \varphi(t)/\log(2\pi/t) \) is of bounded variation on an interval \((0, c) \) \( (0 < c < \pi) \) and (ii) \( g(t) \to 0 \) as \( t \to 0 \).
Then the series \( \sum A_n(x)/\log n \) is convergent if and only if the integral (in the Cauchy sense)

\[
\int_{+\infty}^{\infty} \frac{g(t)}{t \log(2\pi/t)} \, dt \text{ converges.} \tag{1}
\]

Theorem I is a particular case of our theorem. Further Theorem 1 shows that the condition (i) in Theorem I can be replaced by

\[
\int_{t}^{\infty} \varphi(u) u^{-1} \, du = o \left( \frac{(\log 1/t)^2}{(\log \log 1/t)^2} \right) \quad \text{as} \quad t \to 0.
\]

1.2. R. Mohanty and B. K. Ray [9] proved the following theorems.

**THEOREM II.** If (i)

\[
\int_{t}^{\infty} \varphi(u) u^{-2} \, du = o \left( \frac{1}{t} \right) \quad \text{as} \quad t \to 0
\]

and (ii) the function \( \varphi(t)/t \log(2\pi/t) \) is of bounded variation on the interval \((0, \pi)\), then the series \( \sum nB_n(x)/\log n \) is convergent.

**THEOREM III.** If (i)

\[
\int_{t}^{\infty} |\varphi(u)| u^{-2} \, du = o \left( \frac{1}{t} \right) \quad \text{as} \quad t \to 0
\]

and (ii) the function \( \varphi(t)/t \log(2\pi/t) \) is of bounded variation on the interval \((0, \pi)\), then

\[
\sum_{k=1}^{N} kA_k(x) = o(\log N) \quad \text{as} \quad N \to \infty. \tag{2}
\]

Our generalization is as follows:

**THEOREM 2.** Suppose that (i) the function \( h(t) = \psi(t)/t \log(2\pi/t) \) is of bounded variation on \((0, \pi)\) and that (ii) \( h(t) \to 0 \) as \( t \downarrow 0 \). Then the series \( \sum nB_n(x)/\log n \) is convergent if and only if the integral

\[
\int_{+\infty}^{\infty} \frac{h(t)}{t \log(2\pi/t)} \, dt \text{ converges.} \tag{3}
\]

This theorem shows that condition (i) in Theorem II can be replaced by

\[
\int_{t}^{\infty} \psi(u) u^{-2} \, du = o \left( \frac{(\log 1/t)^2}{(\log \log 1/t)^2} \right) \quad \text{as} \quad t \to 0.
\]
THEOREM 3. Suppose that (i) the function \( k(t) = \frac{v(t)}{t \log(2\pi/t)} \) is of bounded variation on \((0, \pi)\) and that (ii) \( k(t) \to 0 \) as \( t \to 0 \). Then relation (2) holds if and only if

\[
\int_t^\pi \varphi(u) u^{-2} \, du = o \left( \log \frac{1}{t} \right) \quad \text{as} \quad t \to 0.
\]

Condition (ii) in Theorem 3 is a consequence of conditions (i) and (ii) in Theorem III, so that our theorem is a generalization of Theorem III.

1.3. We can generalize Theorem 1 as follows:

THEOREM 1'. Let \( 0 < a < 1 \). Suppose that (i) the function \( g_a(t) = \frac{\varphi(t)}{(\log 2\pi/t)^a} \) is of bounded variation on an interval \((0, c) \) \((0 < c < \pi)\) and (ii) \( g_a(t) \to 0 \) as \( t \to 0 \). Then the series \( \sum_{n=1}^\infty A_n(x)/(\log n)^a \) is convergent if and only if the integral

\[
\int_0^\pi \frac{g_a(t)}{t \log(2\pi/t)} \, dt \text{ converges.}
\]

Similarly we can generalize Theorems 2 and 3. We can prove these theorems by methods similar to the case \( a = 1 \), so that we omit the proofs.

1.4. We shall prove Theorem 1 and sketch the proof of Theorems 2 and 3. For the proof of these theorems we use the following lemma.

**Lemma.** If \( p(u) \) is a non-increasing continuous function such that \( p(n) = p_n \) for all \( n \), then

\[
\left| \sum_{n=1}^N p_n \cos nt - \int_1^N p(u) \cos ut \, du \right| \leq A p_1
\]

uniformly for all \( t, 0 < t < \pi \).

This is essentially a lemma of van der Corput and is stated by R. Salem in this form (see [10]).

2. Proof of Theorems

2.1. Proof of Theorem 1. By the definition

\[
\pi \sum_{n=M}^N A_n(x) \frac{1}{\log n} = \int_0^{\pi} \left( \sum_{n=M}^N \frac{\cos nt}{\log n} \right) \varphi(t) \, dt
\]

\[
= \int_0^{\pi/N} + \int_{\pi/N}^{\pi/M} + \int_{\pi/M}^{\pi} = U + V + W.
\]
By integration by parts

\[ U = g \left( \frac{\pi}{N} \right) \int_{0}^{\pi/N} \left( \sum_{n=M}^{N} \frac{\cos nt}{\log n} \right) \log \frac{2\pi}{t} dt \]

\[ - \int_{0}^{\pi/N} dg(t) \int_{0}^{\pi/N} \left( \sum_{n=M}^{N} \frac{\cos nu}{\log n} \right) \log \frac{2\pi}{u} du = U_{1} - U_{2} \]

where

\[ |U_{1}| = \left| g \left( \frac{\pi}{N} \right) \sum_{n=M}^{N} \frac{1}{\log n} \int_{0}^{\pi/N} \cos nt \log \frac{2\pi}{t} dt \right| \]

\[ \leq A \left| g \left( \frac{\pi}{N} \right) \frac{\log N}{N} \sum_{n=M}^{N} \frac{1}{\log n} \right| \leq A \left| g \left( \frac{\pi}{N} \right) \right| = o(1) \quad \text{as} \quad N \to \infty \]

and similarly

\[ |U_{2}| \leq A \int_{0}^{\pi/N} \left| dg(t) \right| = o(1) \quad \text{as} \quad N \to \infty. \]

Therefore \( U = o(1) \) as \( N \to \infty. \)

\[ W = \int_{\pi/M}^{\pi} \left( \sum_{n=M}^{N} \frac{\cos nt}{\log n} \right) \log \frac{2\pi}{t} g(t) dt \]

\[ = g \left( \frac{\pi}{M} \right) \int_{\pi/M}^{\pi} \left( \sum_{n=M}^{N} \frac{\cos nt}{\log n} \right) \log \frac{2\pi}{t} dt \]

\[ + \int_{\pi/M}^{\pi} dg(t) \int_{t}^{\pi} \left( \sum_{n=M}^{N} \frac{\cos nu}{\log n} \right) \log \frac{2\pi}{u} du = W_{1} + W_{2}. \]

Since

\[ \int_{\pi/M}^{\pi} \cos nt \log \frac{2\pi}{t} dt = - \frac{\sin(n\pi/M)}{n} \log 2M + \frac{1}{n} \int_{\pi/M}^{\pi} \frac{\sin nt}{t} dt \]

we have

\[ W_{1} = - g \left( \frac{\pi}{M} \right) \log 2M \sum_{n=M}^{N} \frac{\sin(n\pi/M)}{n \log n} \]

\[ + g \left( \frac{\pi}{M} \right) \sum_{n=M}^{N} \frac{1}{n \log n} \int_{\pi/M}^{\pi} \frac{\sin nt}{t} dt = o(1) \quad \text{as} \quad M \to \infty. \]

By a similar estimate

\[ W_{2} = O(1) \int_{\pi/M}^{\pi} \left| dg(t) \right| + o(1) \quad \text{as} \quad M \to \infty. \]
for any $\epsilon > 0$, and, for any $\delta > 0$, there is an $\epsilon$ such that

$$
\int_{n/M}^{e} |dg(t)| < \delta.
$$

Thus $W_2 = o(1)$ as $M \to \infty$, and then $W = o(1)$ as $M \to \infty$. We shall now estimate $V$.

$$
V = \int_{1/N}^{\pi/M} \left( \sum_{n=M}^{N} \frac{\cos nt}{\log n} \right) \log \frac{2}{t} g(t) dt
$$

$$
= \pi \int_{1/N}^{\pi/M} \left( \sum_{n=M}^{N} \frac{\cos n\pi t}{\log n} \right) \log \frac{2}{t} g(\pi t) dt
= \pi \int_{1/N}^{1/M} \left( \sum_{n=M}^{N} \frac{\cos n\pi t}{\log n} \right) dt + \pi \int_{1/N}^{1/M} \left( \sum_{n=1/t+1}^{N} \frac{\cos n\pi t}{\log n} \right) dt
$$

$$
= \pi V_1 + \pi V_2.
$$

Then

$$
V_1 = \int_{1/N}^{\pi/M} \log \frac{2}{t} g(t) \int_{1}^{\pi/M} \left( \frac{1}{u} \sum_{n=M}^{N} \frac{\cos n\pi u}{\log n} \right) \log \frac{2}{u} du
$$

$$
+ g \left( \frac{\pi}{N} \right) \int_{1/N}^{\pi/M} \left( \frac{1}{u} \sum_{n=M}^{N} \frac{\cos n\pi u}{\log n} \right) \log \frac{2}{u} du = X_1 + X_2.
$$

By the Lemma, we have

$$
\int_{1}^{\pi/M} \frac{1}{u} \sum_{n=M}^{N} \frac{\cos n\pi u}{\log n} \log \frac{2}{u} du = \int_{1}^{\pi/M} \frac{\cos n\pi u}{\log n} \log \frac{2}{u} du + O \left( \frac{1}{\log M} \right)
$$

$$
= \int_{1}^{\pi/M} \frac{\cos n\pi u}{\log n} \log \frac{2}{u} du + O \left( \frac{1}{\log M} \right)
$$

and then

$$
\int_{1}^{\pi/M} \sum_{n=M}^{N} \frac{\cos n\pi u}{\log n} \log \frac{2}{u} du
$$

$$
= \int_{1}^{\pi/M} \log \frac{2}{u} du \int_{1}^{\pi/M} \frac{\cos(\pi u/w)}{w^2 \log (1/w)} dw + o(1)
$$

$$
= \int_{1}^{\pi/M} \frac{dw}{w^2 \log (1/w)} \int_{1}^{w} \cos \frac{\pi u}{w} \log \frac{2}{u} du + o(1)
$$

$$
= \frac{1}{\pi} \int_{1}^{\pi/M} \frac{dw}{w^2 \log (1/w)} \left( -w \log \frac{2}{t} \sin \frac{\pi t}{w} + w \int_{t}^{w} \frac{\sin(\pi u/w)}{u} du \right) + o(1)
$$

$$
= -\frac{1}{\pi} \log \frac{2}{t} \int_{1}^{\pi/M} \frac{dw}{w \log (1/w)} + \frac{1}{\pi} \int_{1}^{\pi/M} \frac{du}{u} \int_{1}^{\pi/M} \frac{\sin(\pi u/w)}{w \log (1/w)} dw + o(1)
$$

$$
= A_1 \int_{1}^{\pi/M} \frac{dw}{w \log (1/w)} + O(1),
$$
where

\[ A_1 = \sum_{k=0}^{\infty} \frac{(-1)^k n^{2k}}{(2k+1)! (2k+1)} > \frac{1}{2}, \]

using the power series expansion of the sine function. Thus we get

\[ X_1 = A_1 \int_{1/N}^{1/M} d g(\pi t) \int_{1/N}^{1/M} \frac{du}{u \log u} + o(1) \]
\[ = A_1 \int_{1/N}^{1/M} g(\pi u) - g(\pi/N) \frac{du}{u \log 1/u} + o(1). \]

Similarly

\[ X_2 = A_1 g \left( \frac{\pi}{N} \right) \int_{1/N}^{1/M} \frac{du}{u \log 1/u} + o(1). \]

Therefore

\[ V_1 = A_1 \int_{1/N}^{1/M} \frac{g(\pi u)}{u \log 1/u} du + o(1). \]

Concerning \( V_2 \),

\[ V_2 = - \int_{1/N}^{1/M} d g(\pi t) \int_{1/N}^{t} \left( \sum_{n=1}^{N} \frac{\cos n\pi u}{\log n} \right) \log \frac{2}{u} du \]
\[ + g \left( \frac{\pi}{M} \right) \int_{1/N}^{1/M} \left( \sum_{n=1}^{N} \frac{\cos n\pi u}{\log n} \right) \log \frac{2}{u} du + o(1) \]
\[ = - Y_1 + Y_2 + o(1). \]

Using the formula

\[ \sum_{n=1}^{N} \frac{\cos n\pi u}{\log n} = \int_{1/N}^{u} \frac{\cos(u\pi/w)}{w^2 \log 1/w} dw + O \left( \frac{1}{\log 1/u} \right), \]

we get

\[ \int_{1/N}^{t} \left( \sum_{n=1}^{N} \frac{\cos n\pi u}{\log n} \right) \log \frac{2}{u} du \]
\[ = \int_{1/N}^{t} \log \frac{2}{u} du \int_{1/N}^{u} \frac{\cos(u\pi/w)}{w^2 \log 1/w} dw + o(1) \]
\[ = \int_{1/N}^{t} \frac{dw}{w^2 \log 1/w} \int_{w}^{t} \cos \frac{u\pi}{w} \log \frac{2}{u} du + o(1) \]
\[ = \frac{1}{\pi} \int_{1/N}^{t} \frac{dw}{w^2 \log 1/w} \left( w \log \frac{2}{t} \sin \frac{t\pi}{w} + w \int_{w}^{t} \sin \frac{u\pi/w}{u} du \right) + o(1) \]
\[ - \frac{1}{\pi} \int_{1/N}^{t} \frac{dw}{w \log 1/w} \int_{w}^{t} \sin \frac{u\pi/w}{u} du + O(1) \]
and then

\[ Y_1 = \frac{1}{\pi} \int_{1/N}^{1/M} d\eta(t) \int_{1/w}^{t} \frac{dw}{w \log 1/w} \int_{w}^{t} \frac{\sin(\eta/w)}{u} du + o(1) \]

\[ = \frac{1}{\pi} \int_{1/N}^{1/M} \frac{dw}{w \log 1/w} \int_{w}^{1/M} \frac{\sin(\eta/w)}{u} du + o(1) \]

\[ = \frac{1}{\pi} \int_{1/N}^{1/M} \frac{dw}{w \log 1/w} \int_{w}^{1/M} \frac{\sin(\eta/w)}{u} du \left( g\left( \frac{\eta}{M} \right) - g(\eta u) \right) du + o(1). \]

Similarly

\[ Y_2 = \frac{1}{\pi} g\left( \frac{\pi}{M} \right) \int_{1/N}^{1/M} \frac{dw}{w \log 1/w} \int_{w}^{1/M} \frac{\sin(\eta/w)}{u} du + o(1) \]

and then

\[ V_2 = \frac{1}{\pi} \int_{1/N}^{1/M} \frac{dw}{w \log 1/w} \int_{w}^{1/M} \frac{\sin(\eta/w)}{u} du g(\eta u) du + o(1) \]

\[ = \frac{1}{\pi} \int_{1/N}^{1/M} \frac{g(\eta u)}{u} du \int_{w}^{u} \frac{\sin(\eta/w)}{w} dw + o(1) \]

\[ = \frac{1}{\pi} \int_{1/N}^{1/M} \frac{g(\eta u)}{u} du \left( \frac{1}{\log 1/u} \int_{1/N}^{u} \frac{\sin(\eta/w)}{w} dw + O\left( \frac{1}{(\log 1/u)^2} \right) \right) + o(1) \]

\[ = \frac{1}{\pi} \int_{1/N}^{1/M} \frac{g(\eta u)}{u \log 1/u} du \int_{1/N}^{u} \frac{\sin(\eta/w)}{w} dw + o(1) \]

\[ = \frac{1}{\pi} \int_{1/N}^{1/M} \frac{g(\eta u)}{u \log 1/u} du \int_{1/N}^{\pi \sqrt{n}} \frac{\sin v}{v} dv + o(1). \]

Collecting the above estimates, we get

\[ V = \int_{1/N}^{1/M} \frac{g(\eta u)}{u \log 1/u} du \left( \pi A_1 + \int_{\pi}^{\infty} \sin v \frac{dv}{v} \right) + o(1). \]

Putting \( A_2 = \pi A_1 + \int_{\pi}^{\infty} (\sin v)/v \ dv (> 0) \), we have proved that

\[ \sum_{n=M}^{N} \frac{A_n(x)}{\log n} = A_2 \int_{1/N}^{1/M} \frac{g(\eta u)}{u \log 1/u} du + o(1) \quad \text{as} \quad M, N \to \infty. \]

This is a stronger form than Theorem 1.

2.2. Proof of Theorem 2. By the definition in the Introduction,

\[ B_n(x) = b_n \cos nx - a_n \sin nx = \frac{1}{\pi} \int_{0}^{\pi} \psi(t) \sin nt \ dt \]
and
\[ \pi \sum_{n=M}^{N} \frac{nB_n(x)}{\log n} = \int_{0}^{\pi} \frac{\psi(t)}{\log n} \left( \sum_{n=M}^{N} n \sin nt \right) dt \]
\[ = \int_{\pi/2N}^{\pi/2M} + \int_{\pi/2N}^{\pi/2M} + \int_{\pi/2M}^{\pi} = U + V + W. \]

We can easily see that \( U = o(1) \). We get, by integration by parts,

\[
W = \int_{\pi/2M}^{\pi} dh(t) \int_{t}^{\pi} \left( \sum_{n=M}^{N} \frac{n \sin nu}{\log n} \right) u \log \frac{2\pi}{u} du 

+ h \left( \frac{\pi}{2M} \right) \int_{\pi/2M}^{\pi} \left( \sum_{n=M}^{N} \frac{n \sin nu}{\log n} \right) u \log \frac{2\pi}{u} du = W_1 + W_2.
\]

Since
\[
\int_{t}^{\pi} \left( \sum_{n=M}^{N} \frac{n \sin nu}{\log n} \right) u \log \frac{2\pi}{u} du 

= -\pi \log 2 \sum_{n=M}^{N} \frac{1}{\log n} + \sum_{n=M}^{N} \cos nt \log \frac{2\pi}{t}

+ \sum_{n=M}^{N} \frac{\sin nt}{n \log n} \int_{t}^{\pi} \left( \sum_{n=M}^{N} \frac{\cos nu}{\log n} \right) \log \frac{2\pi}{u} du,
\]
we get \( W_1 = o(1) \) using the estimate of the last integral in section 2.1, and similarly \( W_2 = o(1) \). Finally,

\[
V = A \int_{1/N}^{1/M} \left( \sum_{n=M}^{N} \frac{n \sin(nt/2)}{\log n} \right) t \log \frac{2}{t} h \left( \frac{t \pi}{2} \right) dt
\]
\[ = A \int_{1/N}^{1/M} \left( \sum_{n=M}^{N} \frac{1}{n} + \sum_{n=1}^{N} \frac{1}{n+1} \right) dt = A(V_1 + V_2). \]

Estimating \( V_1 \) and \( V_2 \) by a method similar to that of section 2.1, we get

\[
\pi \sum_{n=M}^{N} \frac{nB_n(x)}{\log n} = \frac{A_3}{\pi} \int_{1/N}^{1/M} h(v \pi/2) \frac{dv}{v \log 1/v} + o(1) \quad \text{as} \quad M, N \to \infty.
\]

This completes the proof of Theorem 2.

2.3. Proof of Theorem 3. By the definition

\[
\pi \sum_{n=1}^{N} nA_n(x) = \int_{0}^{\pi} \psi(t) \left( \sum_{n=1}^{N} n \cos nt \right) dt = \int_{0}^{\pi/n} + \int_{\pi/n}^{\pi} = U + V,
\]
where we can easily see that $U = o(\log n)$. Using the formula
\[
\sum_{n=1}^{N} n \cos nt = \frac{N \sin(N + \frac{1}{2}) t}{2 \sin t/2} - \frac{1 - \cos Nt}{4 \sin^2 t/2}
\]
in the estimate of $V$, we get
\[
\pi \sum_{n=1}^{N} nA_n(x) = -2 \int_{\pi/N}^{\pi} \varphi(u) u^{-2} du + o(\log N) \quad \text{as} \quad N \to \infty.
\]
This proves the theorem.

CHAPTER 5. ON THE ABSOLUTE CONVERGENCE OF SERIES ASSOCIATED WITH FOURIER SERIES

1. Introduction

1.1. Let $f$ be an integrable function with period $2\pi$ and let its Fourier series be
\[
f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x).
\]
By $S_n(x)$ we denote the $n$th partial sum of the series (1). We write
\[
\Phi(t) = \int_{0}^{t} \varphi(u) du \quad \text{and} \quad \varphi_1(t) = \frac{\Phi(t)}{2} \sin \frac{t}{2} \quad \text{for} \quad t > 0.
\]
R. Mohanty and S. Mohapatra [11] (cf. S. M. Mazar [12]) proved the following

THEOREM 1. If (i) the function $\varphi_1(t) \log 2\pi / t$ is of bounded variation on the interval $(0, \pi)$, (ii) the function $\varphi_1(t) / t$ is absolutely integrable on a neighborhood of the origin and (iii) the sequence $(n^a A_n(x))$ is of bounded variation for an $a > 0$, then the series
\[
\sum_{n=1}^{\infty} \frac{(S_n(x) - f(x))}{n}
\]
is absolutely convergent.

We shall prove the following generalization.

THEOREM 1. Suppose that (i) the function $g(t) = \varphi_1(t) \log 2\pi / t$ is of bounded variation on the interval $(0, \pi)$, (ii) $g(+0) = 0$ and (iii) the series
\[ \sum_{n=1}^{\infty} A_n(x)/n^b < \infty \text{ for } a, b, 0 < b < 1. \text{ Then the series } (2) \text{ is absolutely convergent if and only if } \frac{\varphi(t)}{t} \text{ is absolutely integrable.} \]

Condition (iii) of Theorem 1 is satisfied when the sequence \((n^a A_n(x))\) is bounded for an \(a\) such that \(0 < a < 1\), and condition (ii) of Theorem 1 is a consequence of (i) and (ii) of Theorem 1.

1.2. For the proof of Theorem 1, we use the following lemma.

**Lemma.**

\[
\int_0^\pi \frac{\sin nu}{2 \sin u/2 \cdot \log 2\pi/u} \, du = \frac{n}{2} \frac{1}{\log n} + O \left( \frac{1}{(\log n)^2} \right) \quad \text{as} \quad n \to \infty.
\]

**Proof.** We have

\[
\int_0^\pi \frac{\sin nu}{2 \sin u/2 \cdot \log 2\pi/u} \, du = \int_0^\pi \frac{\sin nu}{u \cdot \log 2\pi/u} \, du + O \left( \frac{1}{n} \right)
\]

\[= \int_0^{n\pi/2} \frac{\sin u}{u \log 2\pi n/u} \, du + O \left( \frac{1}{n} \right).\]

Now

\[
\int_0^{n\pi/2} \frac{\sin u}{u \log 2\pi n/u} \, du - \frac{1}{\log n} \int_0^{n\pi/2} \frac{\sin u}{u} \, du
\]

\[= \int_0^{n\pi/2} \sin u \left( \frac{1}{\log 2\pi n/u} - \frac{1}{\log n} \right) \, du = \frac{1}{\log n} \int_0^{n\pi/2} \sin u \log 2\pi/u \, du
\]

\[= \frac{1}{\log n} \left( \int_0^{2\pi/n} + \int_{2\pi/n}^{2\pi} + \int_{2\pi}^{6\pi} + \int_{6\pi}^{n\pi/2} \right) = \frac{1}{\log n} (p_n + q_n + r_n + s_n)
\]

where

\[|p_n| \leq \left| \int_0^{2\pi/n} \frac{\sin u}{u} \left( \log n/\log 2\pi/u \right) + 1 \, du \right| \leq \frac{A}{n},\]

\[|q_n| \leq \int_{2\pi/n}^{2\pi} \frac{\left| \sin u \right| \log 2\pi/u}{\log n} \, du \leq \frac{A}{\log n}
\]

and

\[|r_n| \leq \int_{2\pi}^{6\pi} \frac{\log u/2\pi}{u \log u} \, du \leq \frac{A}{\log n}.
\]

In order to estimate \(s_n\), consider the function

\[h(u) = \frac{1}{u} \frac{\log u/2\pi}{\log 2\pi n/u} \quad \text{on the interval} \quad \left( 6\pi, \frac{n\pi}{2} \right).
\]
for large $n$. By logarithmic differentiation

$$\frac{h'(u)}{h(u)} = -\frac{1}{u} \left( 1 - \frac{\log n}{\log u/2\pi \cdot \log 2\pi n/u} \right)$$

where, putting $x = \log u/2\pi$,

$$1 - \frac{\log n}{\log u/2\pi \cdot \log 2\pi n/u} = -\frac{x^2 - x \log n + \log n}{\log u/2\pi \cdot \log 2\pi n/u};$$

the quadratic function has two real roots $a = 1 + O(1/\log n)$ and $b = \log n - 1 + O(1/\log n)$. When $u$ lies between $6\pi$ and $n\pi/2$, $x$ lies inside the interval $(a, b)$ and then the numerator of the right side is positive. Thus the function $h$ decreases in $(6\pi, n\pi/2)$ and we get

$$|s_n| \leq \left| \int_{6\pi}^{n\pi/2} \frac{\sin u \log 2\pi u}{u \log 2\pi n/u} \, du \right| \leq \frac{A}{\log n}.$$

Therefore

$$\int_0^{n\pi/2} \frac{\sin nu}{2 \sin u/2 \cdot \log 2\pi/u} \, du = \frac{1}{\log n} \int_0^{n\pi/2} \frac{\sin u}{u} \, du + O \left( \frac{1}{(\log n)^2} \right)$$

$$= \frac{\pi}{2 \log n} + O \left( \frac{1}{(\log n)^2} \right) \quad \text{as} \quad n \to \infty,$$

which has to be proved.

2. Proof of Theorem 1.

Without loss of generality, we can suppose $\Phi(\pi) = g(\pi) = 0$. By the definition,

$$\frac{\pi(s_n(x) - f(x))}{n} = \frac{1}{n} \int_0^\pi \varphi(t) \frac{\sin(n + \frac{1}{2})}{2 \sin t/2} \, dt$$

$$= -\int_0^\pi \Phi(t) \frac{\cos(n + \frac{1}{2})}{2 \sin t/2} \, dt + \frac{1}{n} \int_0^\pi \Phi(t) \frac{\sin nt}{(2 \sin t/2)^2} \, dt$$

$$= -\int_0^\pi \varphi_1(t) \cos \left( n + \frac{1}{2} \right) \, dt + \frac{1}{n} \int_0^\pi \varphi_1(t) \frac{\sin nt}{2 \sin t/2} \, dt$$

$$= -u_n + v_n.$$  

Integrating by parts, we write

$$u_n = \int_0^\pi g(t) \frac{\cos(n + \frac{1}{2})}{\log 2\pi/t} \, dt = \int_0^\pi dg(t) \int_t^\pi \frac{\cos(n + \frac{1}{2})}{\log 2\pi/u} \, du$$

$$= -\frac{1}{n + \frac{1}{2}} \int_0^\pi \sin(n + \frac{1}{2}) \, dg(t) - \frac{1}{n + \frac{1}{2}} \int_0^\pi dg(t) \int_t^\pi \frac{\sin(n + \frac{1}{2})}{u(\log 2\pi/u)^2} \, du$$

$$= -w_n - x_n.$$  

(4)
and

\[ \sum_{n=1}^{\infty} |w_n| \leq \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{1/n^c}^{\infty} \frac{\sin(n + \frac{1}{2}) t}{\log 2\pi/t} \, dg(t) \right| \]

\[
\leq \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{1/n^c}^{1/n^c} + \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{1/k c}^{\infty} \right| = W_1 + W_2
\]

where \( c \) is a positive constant < 1, which will be determined later. Now,

\[
W_1 \leq \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=n^c}^{\infty} \int_{1/(k+1)}^{1/k} \frac{|dg(t)|}{\log 2\pi/t} \leq \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} \frac{|dg(t)| k^{1/n}}{\log 2\pi/t} \sum_{n=1}^{\infty} \frac{1}{n}
\]

\[
\leq A \int_{0}^{\infty} |dg(t)| < \infty.
\]

Since \( dg(t) = d((1/t) \log 2\pi/t) \Phi(t) + (1/t) \log 2\pi/t \varphi(t) \, dt \), we have

\[
W_2 \leq \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{1/n^c}^{\infty} \frac{\sin(n + \frac{1}{2}) t}{t} \varphi(t) \, dt \right|
\]

\[
+ \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{1/n^c}^{\infty} \frac{1 + \log 2\pi/t}{t^2 \log 2\pi/t} \sin \left( n + \frac{1}{2} \right) t \Phi(t) \, dt \right|
\]

\[ = X_1 + X_2. \]

Substituting \( \varphi(t) \sim \sum_{k=1}^{\infty} A_k(x) \cos kt \),

\[
\int_{1/n^c}^{\infty} \frac{\sin(n + \frac{1}{2}) t}{t} \varphi(t) \, dt
\]

\[ = \sum_{k=1}^{\infty} A_k(x) \int_{1/n^c}^{\infty} \frac{\cos kt \sin(n + \frac{1}{2}) t}{t} \, dt
\]

\[ = \frac{1}{2} \sum_{k=1}^{\infty} A_k(x) \int_{1/n^c}^{\infty} (\sin(k + n + \frac{1}{2}) t + \sin(n - k + \frac{1}{2}) t) t^{-1} \, dt
\]

\[ = O \left( \sum_{k=1}^{\infty} \left| A_k(x) \right| n^c \right)
\]

and then

\[ |X_1| \leq A \sum_{n=1}^{\infty} \frac{1}{n^{1-c}} \sum_{k=1}^{\infty} \left| \frac{A_k(x)}{n - k + \frac{1}{2}} \right| \leq A \sum_{k=1}^{\infty} \frac{|A_k(x)| \log k}{k^{1-c}} \]
which is finite for a $c$ less than $1 - b$. Using the Fourier expansion

$$\Phi(t) \sim \sum_{k=1}^{\infty} A_k(x) k^{-1} \sin kt,$$

we get

$$\int_{1/n^e}^{\pi} \frac{1 + \log 2\pi/t}{t^2 \log 2\pi/t} \sin \left(n + \frac{1}{2}\right) t \Phi(t) \, dt$$

$$= \sum_{k=1}^{\infty} \frac{A_k(x)}{k} \int_{1/n^e}^{\pi} \frac{1 + \log 2\pi/t}{t^2 \log 2\pi/t} \sin \left(n + \frac{1}{2}\right) t \sin kt \, dt$$

$$= O \left( \sum_{k=1}^{\infty} \frac{|A_k(x)| n^{2c}}{k |n - k + \frac{1}{2}|} \right)$$

and then

$$X_2 \leq A \sum_{n=1}^{\infty} \frac{1}{n^{1-2c}} \sum_{k=1}^{\infty} \frac{|A_k(x)|}{k |n - k + \frac{1}{2}|} \leq A \sum_{k=1}^{\infty} \frac{A_k(x) \log k}{k^{2-2c}} < \infty.$$ 

Thus we have proved that $W_2$ is finite and hence the left side of (5) is finite. We shall now consider $x_n$ in (4). We write

$$\sum_{n=1}^{\infty} |x_n| \leq \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{0}^{1/n^e} \sin(n + \frac{1}{2}) \frac{u}{u(\log 2\pi/u)^2} \, du \right| + A \sum_{n=1}^{\infty} \frac{|g(1/n^e)|}{n(\log n)^2}$$

$$\leq A \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{\infty} \frac{1}{n^{1/(k+1)}} \int_{0}^{1/k} \frac{|dg(t)|}{\log 2\pi/t} + A \leq A \int_{0}^{\pi} |dg(t)| + A$$

and

$$Y_2 \leq A \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{0}^{1/n^e} \frac{|dg(t)|}{t(\log 2\pi/t)^2} \leq A \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n^e} \int_{0}^{1/k} \frac{|dg(t)|}{t(\log 2\pi/t)^2}$$

$$\leq A \sum_{k=1}^{\infty} \int_{0}^{1/k} \frac{|dg(t)|}{t(\log 2\pi/t)^2} \sum_{n=1}^{n^e} \frac{1}{n^2} \leq A \int_{0}^{\pi} |dg(t)|.$$

Hence the left side of (6) is finite, and then $\sum |u_n| < \infty$. Finally we shall estimate $v_n$ in (3).

$$v_n = \frac{1}{n} \int_{0}^{\pi} \varphi(t) \frac{\sin nt}{2 \sin t/2} \, dt = \frac{1}{n} \int_{0}^{\pi} g(t) \frac{\sin nt}{2 \sin t/2 \cdot \log 2\pi/t} \, dt$$

$$= \frac{1}{n} \int_{0}^{\pi} dg(t) \int_{t/2}^{\pi} \frac{\sin nt}{2 \sin u/2 \cdot \log 2\pi/u} \, du$$

$$= \frac{1}{n} \int_{0}^{1/\log n} + \frac{1}{n} \int_{1/\log n}^{\pi} = y_n + z_n.$$
where
\[
\sum_{n=2}^{\infty} | z_n | \leq \sum_{n=2}^{\infty} n^{-2} \int_{1/\log n}^{\infty} \frac{| \log(t) |}{t \log 2\pi/t} \leq \sum_{n=2}^{\infty} n^{-2} \int_{1/(k+1)}^{1/k} \frac{| \log(t) |}{t \log 2\pi/t} \leq A \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} \frac{| \log(t) |}{t \log 2\pi/t} \leq A \int_{0}^{\infty} | \log(t) | \, dt.
\]

We write
\[
y_n' = \frac{1}{n} \int_{0}^{1/\log n} \sin nu \, du - \frac{1}{2} \sin u/2 \cdot \log 2\pi/\log 2\pi/du - \int_{0}^{t} \frac{\sin nu}{2 \sin u/2 \cdot \log 2\pi/du}.
\]

where
\[
\sum_{n=2}^{\infty} | y_n'' | \leq \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} \frac{| \log(t) |}{t \log 2\pi/t} \leq A \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} \frac{| \log(t) |}{t \log 2\pi/t} \cdot \frac{k}{\log k} < \infty.
\]

Since the inner integral of \( y_n' \) is \( \pi/(2 \log n) + O(1/(\log n)^2) \), the above estimates give
\[
\sum_{n=1}^{\infty} \frac{| s_n(x) - f(x) |}{n} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n \log n} \left| g \left( \frac{1}{n \log n} \right) \right| + O(1) = \frac{1}{2} \int_{0}^{\pi} \frac{\varphi(t)}{t} \, dt + O(1),
\]
which is the required result.

REFERENCES