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Stability of Runge–Kutta methods for the alternately advanced and retarded differential equations with piecewise continuous arguments☆

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Abstract

This paper deals with the numerical properties of Runge–Kutta methods for the solution of $u'(t) = au(t) + a_0u([t + \frac{1}{2}])$. It is shown that the Runge–Kutta method can preserve the convergence order. The necessary and sufficient conditions under which the analytical stability region is contained in the numerical stability region are obtained. It is interesting that the θ-methods with $0 \leqslant \theta < \frac{1}{2}$ are asymptotically stable. Some numerical experiments are given.

Keywords: Alternately advanced and retarded differential equations; Piecewise continuous arguments; Asymptotical stability

1. Introduction

In this paper we consider the following differential equation with piecewise continuous argument (EPCA):

$$\begin{aligned} &u'(t) = au(t) + bu\left(\left[t + \frac{1}{2}\right]\right), \quad t \geq 0, \\ &u(0) = u_0, \end{aligned} \tag{1.1}$$

where a, b, u_0 are real constants and $[\cdot]$ denotes the greatest integer function. Since the argument deviation for system (1.1), namely $t - [t + \frac{1}{2}]$, is negative in $[n + \frac{1}{2}, n + 1)$ and positive in $[n, n + \frac{1}{2})$, (1.1) is said to be of alternately advanced and retarded type. The general form of this type equation is

$$\begin{aligned} &u'(t) = f(t, u(t), u(\alpha(t))), \quad t \geq 0, \\ &u(0) = u_0, \end{aligned} \tag{1.2}$$

where the arguments $\alpha(t)$ has intervals of constancy and $t - \alpha(t)$ changes the sign in many times.

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These equations are related to impulse and loaded equations and share the properties of certain models of vertically transmitted diseases [6]. The study of EPCA of mixed type is initiated by Aftabizadeh and Wiener [1]. They observe that the change of sign in the argument deviation lead not only to interesting periodic properties but also to complications in the asymptotic and oscillatory behaviour of solutions. Oscillatory, stability and periodic properties of (1.1) have been investigated in [7,10,14,15].

If we consider the quantizer effects of a digital feedback control system, the mathematical models are the form of (1.1) (see [9]). Let

$$\dot{z}(t) = Dz(t) + Ex(t), \qquad v = F^T z, \tag{1.3}$$

and a linear digital controller of the form

$$x_k = Ax_{k-1} + Bv(kT), \tag{1.4}$$

where $D \in \mathbb{R}^{n_1 \times n_1}$, $E \in \mathbb{R}^{n_2 \times n_2}$ and $F \in \mathbb{R}^{n_1}$ while $A \in \mathbb{R}^{n_2 \times n_2}$ and $B \in \mathbb{R}^{n_2}$. Also $k = 0, 1, 2, \ldots$, $T > 0$ denotes the sampling period and

$$x(t) = x_k \quad \text{over the time interval } \left(k - \frac{1}{2}\right) T \leqslant t < \left(k + \frac{1}{2}\right) T.$$

The properties of the analytical solutions have been investigated in the book [16].

Definition 1 (*[16]*). A function $u : [0, \infty) \to \mathbb{R}$ is a solution of (1.1) if the following conditions hold:

(1) $u(t)$ is continuous on $[0, \infty)$;
(2) The derivative $u'(t)$ exists at each point $t \in [0, \infty)$, with the possible exception of the points $t = n + \frac{1}{2}$, $n = 0, 1, 2, \ldots$, where one-sided derivatives exist;
(3) (1.1) is satisfied on $[0, \frac{1}{2})$ and each interval $[n - \frac{1}{2}, n + \frac{1}{2})$ for $n = 1, 2, \ldots$.

Theorem 2 (*[16]*). *If $b \neq \frac{a}{e^{\frac{a}{2}} - 1}$, then (1.1) has on $[0, \infty)$ a unique solution*

$$u(t) = m(T(t))\lambda^{[t+\frac{1}{2}]} u_0, \tag{1.5}$$

where

$$m(t) = e^{at} + (e^{at} - 1)a^{-1}b, \qquad T(t) = t - \left[t + \frac{1}{2}\right], \qquad \lambda = \frac{m(\frac{1}{2})}{m(-\frac{1}{2})}.$$

The solution $u(t)$ is asymptotically stable ($u(t) \to 0$ as $t \to \infty$) for any given u_0, if and only if $|\lambda| < 1$, i.e.,

$$\begin{aligned} &-\frac{a(e^a + 1)}{(e^{\frac{a}{2}} - 1)^2} < b < -a, \quad \text{for } a > 0, \\ &b < -a \quad \text{or} \quad b > -\frac{a(e^a + 1)}{(e^{\frac{a}{2}} - 1)^2}, \quad \text{for } a < 0, \\ &b < 0, \quad \text{for } a = 0. \end{aligned} \tag{1.6}$$

The convergence and the stability of numerical solutions for the linear EPCA of the retarded type and the advanced type have been investigated in [8,11,12], but the authors are not aware of any published results on the numerical solutions of (1.1).

In this paper we investigate the numerical properties of Runge–Kutta methods for the solution of (1.1) and show the numerical solution is of order p for the pth-order Runge–Kutta method. We also give the necessary and sufficient conditions under which the analytical stability region is contained in the numerical stability region. It is interesting that the analytical stability region is contained in the numerical stability region for the even stage Radau IA and IIA methods and the θ-methods with $0 \leqslant \theta < \frac{1}{2}$, which is different from the results for the EPCA of the retarded type or the advanced type [8,11,12]. Some numerical experiments are given.

2. Runge–Kutta methods

In this section we consider the adaptation of the Runge–Kutta methods (A, B, C) given by the Butcher tabular $\begin{array}{c|c} C & A \\ \hline & B^{\mathrm{T}} \end{array}$ where the matrix $A = (a_{ij})_{\mu\times\mu}$ and the vectors $B = (B_1, B_2, \ldots, B_\mu)^{\mathrm{T}}$ and $C = (C_1, C_2, \ldots, C_\mu)^{\mathrm{T}}$. Let $h = \frac{1}{2m}$ be a given stepsize with integer $m \geqslant 1$ and the gridpoints t_n be defined by $t_n = nh$ $(n = 0, 1, 2, \ldots)$. For the Runge–Kutta methods we always assume that $B_1 + B_2 + \cdots + B_\nu = 1$ and $0 \leqslant C_1 \leqslant C_2 \leqslant \cdots \leqslant C_\nu \leqslant 1$. The adaptation of the Runge–Kutta methods to (1.2) leads to a numerical process of the following type, generating approximations $u_1, u_2, u_3, \ldots$ to the exact solution $u(t)$ of (1.2) at the gridpoints t_n $(n = 1, 2, 3, \ldots)$

$$u_{n+1} = u_n + h\sum_{i=1}^{\nu} B_i\, f(t_n + C_i h, y_i^{(n)}, z_i^{(n)}), \tag{2.1}$$

where $y_1^{(n)}, y_2^{(n)}, \ldots, y_\nu^{(n)}$ satisfy

$$y_i^{(n)} = u_n + h\sum_{j=1}^{\nu} a_{ij}\, f(t_n + C_j h, y_j^{(n)}, z_j^{(n)}), \tag{2.2}$$

and the argument $z_i^{(n)}$ denotes the given approximation to $u(\alpha(t_n + C_i h))$, $i = 1, 2, \ldots, \nu$, $n = 0, 1, 2, \ldots$.

We are interested in the application of (2.1) and (2.2) to (1.1). The application of the process (2.1) and (2.2), in the case of (1.1), yields

$$\begin{aligned} u_{n+1} &= u_n + h\sum_{i=1}^{\nu} B_i\, (a y_i^{(n)} + b z_i^{(n)}), \\ y_i^{(n)} &= u_n + h\sum_{j=1}^{\nu} a_{ij}\, (a y_j^{(n)} + b z_j^{(n)}), \end{aligned} \tag{2.3}$$

where $z_i^{(n)}$ is the approximation to $u([t_n + C_i h + \frac{1}{2}])$ $(n = 0, 1, \ldots)$. If we denote $\mathcal{L}(k) = \{0, 1, \ldots, m-1\}$ for $k = 0$ and $\mathcal{L}(k) = \{-m, -m+1, \ldots, m-1\}$ for $k \geqslant 1$, $n = 2km + l$, then $z_i^{(2km+l)}$ can be defined as u_{2km} according to Definition 1 $(i = 1, 2, \ldots, \nu, l \in \mathcal{L}(k))$. Let $Y^{(n)} = (y_1^{(n)}, y_2^{(n)}, \ldots, y_\nu^{(n)})^{\mathrm{T}}$, then (2.3) reduces to

$$\begin{aligned} u_{2km+l+1} &= u_{2km+l} + haB^{\mathrm{T}}Y^{(2km+l)} + hbu_{2km}, \\ Y^{(2km+l)} &= u_{2km+l}e + haAY^{(2km+l)} + hbAeu_{2km}, \end{aligned} \tag{2.4}$$

where $e = (1, 1, \ldots, 1)^{\mathrm{T}}$.

3. Convergence

Now we assume $a \neq 0$. From (2.4) we can see that if $I - xA$ is invertible, then

$$u_{2km+l+1} = R(x)u_{2km+l} + \frac{b}{a}(R(x) - 1)u_{2km}, \tag{3.1}$$

where $x = ha$, $R(x) = 1 + xB^{\mathrm{T}}(I - xA)^{-1}e$ is the stability function of the method and $l \in \mathcal{L}(k)$.

Assume that $R(x) \neq 0$ and $b \neq \frac{a}{R^m(x)-1}$. Then it follows from (3.1) that

$$u_{2km+l} = \tilde{m}(l)u_{2km}, \tag{3.2}$$

and

$$u_{2(k+1)m} = \tilde{\lambda}u_{2km}, \tag{3.3}$$

where $\tilde{\lambda} = \frac{\tilde{m}(m)}{\tilde{m}(-m)}$ and

$$\tilde{m}(l) = \left(R^l(x) + \frac{b}{a}(R^l(x) - 1)\right), \quad -m \leqslant l \leqslant m.$$

The process (2.4) can be represented as

$$\begin{aligned} &u_l = \tilde{m}(l)u_0, \quad l \in \mathcal{L}(0), \\ &u_{2km} = \tilde{\lambda} u_{2(k-1)m}, \quad k \geqslant 1, \\ &u_{2km+l} = \tilde{m}(l)u_{2km}, \quad l \in \mathcal{L}(k). \end{aligned} \tag{3.4}$$

Lemma 3. *Assume that the Runge–Kutta method is of order p. Then there exist constants K and h_0 such that for $h = \frac{1}{2m} < h_0$*

$$\max_{-m \leqslant l \leqslant m} \left| \tilde{m}(l) - m\left(\frac{l}{2m}\right) \right| \leqslant K h^p. \tag{3.5}$$

What is more, if $u_{2km} = u(k)$, then

$$|u_{2(k+1)m} - u(k+1)| = O(h^p). \tag{3.6}$$

Proof. Since the method is of order p, there exist constants K_1 and $\delta_1 > 0$ such that for $|x| < \delta_1$

$$|R(x) - e^x| \leqslant K_1 |x|^{p+1}.$$

Therefore there exist constant K and $\delta > 0$ such that for $|x| < \delta$

$$\max_{-m \leqslant l \leqslant m} |R^l(x) - e^{lx}| \leqslant K|x|^p.$$

It is easily seen that

$$\tilde{\lambda} = \frac{\tilde{m}(m)}{\tilde{m}(-m)} = \lambda + O(h^p),$$

which implies from (1.5) and (3.4) that (3.6) is true. □

Theorem 4. *Assume that $a \neq 0$, the Runge–Kutta method is of order p and u_n, $n = 0, 1, \ldots, 2m$, is the numerical solution on the interval* [0, 1]. *Then there exist constants $c_1 > 0$ and $h_1 > 0$ such that for all $h < h_1$ and $t_n \in [0, 1]$*

$$|u_n - u(t_n)| \leqslant c_1 h^p. \tag{3.7}$$

Proof. From (1.5) and (3.4) and Lemma 3, we can see that if $t_n \in [0, \frac{1}{2})$ then

$$|u_n - u(t_n)| \leqslant K h^p, \quad \text{for } h < \frac{\delta}{|a|},$$

where K and δ are given by Lemma 3. Specially, we have $|u_{2m} - u(1)| \leqslant K h^p$. Therefore there exist constant K' and $\delta' > 0$ such that for $t_n \in [\frac{1}{2}, 1]$

$$|u_n - u(t_n)| = \left| \tilde{m}(l)u_{2m} - m\left(\frac{l}{2m}\right)u(1) \right| \leqslant K' h^p, \quad \text{for } h < \frac{\delta'}{|a|}.$$

Hence (3.8) holds for $c_1 = \max\{K, K'\}$ and $h_1 = \min\{\frac{\delta}{|a|}, \frac{\delta'}{|a|}\}$.

If $a = 0$, then from (1.1) and (2.4) we have $u_n = u(t_n)$ for all $t_n \geqslant 0$. □

Theorem 5. *Assume that the Runge–Kutta method is of order p. Then the convergence of the numerical solution u_n approximating the analytic solution $u(t)$ is of order p, namely, for any integer $k \geqslant 0$, there exist constants $c_k > 0$ and $h_k > 0$ such that for all $h < h_k$*

$$\max_{0\leqslant t_n\leqslant k} |u_n - u(t_n)| \leqslant c_k h^p. \tag{3.8}$$

4. Numerical stability

Definition 6. Process (2.1) for (1.1) is called asymptotically stable at (a, b) if and only if there exists a constant number M such that for all $h = \frac{1}{2m}$, $m > M$ and any given u_0, relation (3.4) defines u_n that satisfies $u_n \to 0$ for $n \to \infty$.

Definition 7. The set of all points (a, b) at which the process (2.1) for (1.1) is asymptotically stable is called the asymptotical stability region denoted by S.

For any given Runge–Kutta method we assume that $\delta_1 < 0 < \delta_2$ are such that

$$\begin{aligned} &1 < R(x) < \infty \quad \text{for } 0 < x < \delta_2,\\ &0 < R(x) < 1 \quad \text{for } \delta_1 < x < 0, \end{aligned} \tag{4.1}$$

which implies

$$0 < \frac{R(x) - 1}{x} < \infty \quad \text{for } \delta_1 < x < \delta_2. \tag{4.2}$$

Lemma 8. *For any given Runge–Kutta method, if (4.1) holds, then there exists a constant number $N > 0$ independent of k and l such that*

$$|u_{2km+l}| \leqslant N|u_{2km}| \quad \textit{for all } k \geqslant 0 \textit{ and } l \in \mathcal{L}(k). \tag{4.3}$$

Proof. From (4.1), we can obtain that there is an $N > 0$ such that

$$|\tilde{m}(l)| \leqslant N \quad \text{for all } k \geqslant 0 \text{ and } l \in \mathcal{L}(k),$$

which implies from (3.2) that the result is true. □

Corollary 9. *$u_n \to 0$ as $n \to \infty$ if and only if $u_{2km} \to 0$ as $k \to \infty$.*

It is well known by (3.4) that $u_{2km} \to 0$ as $k \to \infty$ if and only if

$$|\tilde{\lambda}| < 1, \tag{4.4}$$

which is in view of (4.1) equivalent to

$$\begin{aligned} &-\frac{a(R^{2m}(x) + 1)}{(R^m(x) - 1)^2} < b < -a, \qquad \text{for } a > 0,\\ &b < -a \quad \text{or} \quad b > -\frac{a(R^{2m}(x) + 1)}{(R^m(x) - 1)^2}, \qquad \text{for } a < 0,\\ &b < 0, \quad \text{for } a = 0. \end{aligned} \tag{4.5}$$

In this section we will discuss the stability of the Runge–Kutta methods. We introduce the set H consisting of all points $(a, b) \in \mathbb{R}^2$ satisfying (1.6). In the following we will investigate which conditions lead to $H \subseteq S$. For convenience, we divide the region H into three parts:

$$\begin{aligned} H_0 &= \{(a, b) \in H : a = 0\},\\ H_1 &= \{(a, b) \in H\backslash H_0 : a < 0\},\\ H_2 &= \{(a, b) \in H\backslash H_0 : a > 0\}, \end{aligned}$$

and in the similar way we denote

$$S_0 = \{(a,b) \in S : a = 0\},$$
$$S_1 = \{(a,b) \in S \backslash S_0 : a < 0\},$$
$$S_2 = \{(a,b) \in S \backslash S_0 : a > 0\}.$$

It is easy to see that $H = H_0 \cup H_1 \cup H_2$, $S = S_0 \cup S_1 \cup S_2$ and

$$H_i \cap H_j = \emptyset, \qquad S_i \cap S_j = \emptyset, \qquad H_i \cap S_j = \emptyset, \quad i \neq j, i, j = 0, 1, 2.$$

Therefore we can conclude that $H \subseteq S$ is equivalent to $H_i \subseteq S_i$, $i = 0, 1, 2$.

4.1. The padé approximation to the exponential function

In this subsection, we will investigate the stability of Runge–Kutta method with the stability function which is given by the (r, s)-Padé approximation to e^x.

The following lemmas will be useful to determine the stability conditions:

Lemma 10 (*[2–4,13]*). *The (r, s)-Padé approximation to e^z is given by*

$$R(z) = \frac{P_r(z)}{Q_s(z)} \tag{4.6}$$

where

$$P_r(z) = 1 + \frac{r}{r+s} z + \frac{r(r-1)}{(r+s)(r+s-1)} \frac{z^2}{2!} + \cdots + \frac{r!\,s!}{(r+s)!} \frac{z^r}{r!},$$
$$Q_s(z) = 1 - \frac{s}{r+s} z + \frac{s(s-1)}{(r+s)(r+s-1)} \frac{z^2}{2!} + \cdots + (-1)^s \frac{s!\,r!}{(r+s)!} \frac{z^s}{s!},$$

with error

$$e^z - R(z) = (-1)^s \frac{r!\,s!}{(r+s)!\,(r+s+1)!} z^{r+s+1} + O(z^{r+s+2}). \tag{4.7}$$

It is the unique rational approximation to e^z of order $r+s$, such that the degrees of numerator and denominator are r and s, respectively.

Following [2,4,5,13], we define the order star

$$D = \{z \in \mathbb{C} : |R(z)| > |e^z|\}.$$

Lemma 11 (*[2,4,5,13]*). *If the Runge–Kutta method is of order p, then for $z \to 0$, D behaves like a star with $p+1$ sectors of equal width $\frac{\pi}{p+1}$, separated by $p+1$ similar white sectors of the complement of D, each of the same width.*

Lemma 12 (*[2,4,5,13]*). *If $R(z)$ is the (r, s)-Padé approximation to e^z, then*

(1) *there are s star sectors in the right-half plane, each containing a pole of $R(z)$;*
(2) *there are r white sectors in the left-half plane, each containing a zero of $R(z)$;*
(3) *all sectors are symmetrical with respect to the real axis.*

Corollary 13. *Suppose $R(z)$ is the (r, s)-Padé approximation to e^z. Then $R(x) \leqslant e^x$ for all $x > 0$ if and only if s is even, and there exists a $\delta > 0$ such that $0 < R(x) \leqslant e^x$ for $-\delta < x < 0$ if and only if r is odd.*

Lemma 14. *Let $f(x) = \frac{x^2+1}{(x-1)^2}$. Then $f(x)$ is increasing on $[0, 1)$ and decreasing on $(1, \infty)$.*

Theorem 15. *Suppose that the stability function $R(x)$ of the Runge–Kutta method is given by the (r, s)-Padé approximation to e^x. Then $H_1 \subseteq S_1$ if and only if r is odd and $H_2 \subseteq S_2$ if and only if s is even.*

Table 1
The higher order Runge–Kutta methods

	Gauss–Legendre	Radau IA, IIA	Lobatto IIIA, IIIB	Lobatto IIIC
(r, s)	(ν, ν)	$(\nu - 1, \nu)$	$(\nu - 1, \nu - 1)$	$(\nu - 2, \nu)$
$H_1 \subseteq S_1$	ν is odd	ν is even	ν is even	ν is odd
$H_2 \subseteq S_2$	ν is even	ν is even	ν is odd	ν is even

Proof. In view of (1.6), (4.5) and (4.1), we know that $H_1 \subseteq S_1$ if and only if

$$-\frac{a(R^{2m}(x)+1)}{(R^m(x)-1)^2} \leqslant -\frac{a(e^a+1)}{(e^{\frac{a}{2}}-1)^2} \quad \text{for } m > M, \tag{4.8}$$

which from Lemma 14 is equivalent to

$$0 < R(x) \leqslant e^x.$$

Similarly $H_2 \subseteq S_2$ if and only if

$$-\frac{a(R^{2m}(x)+1)}{(R^m(x)-1)^2} \leqslant -\frac{a(e^a+1)}{(e^{\frac{a}{2}}-1)^2} \quad \text{for } m > M, \tag{4.9}$$

which from Lemma 14 is equivalent to

$$1 < R(x) \leqslant e^x.$$

As a consequence of Corollary 13, the proof is complete. □

Theorem 16. *For all Runge–Kutta methods, we have $H_0 = S_0$.*

Remark 17. If we define

$$M = \begin{cases} \dfrac{a}{2\delta_2}, & a > 0, \\ \dfrac{a}{2\delta_1}, & a < 0, \end{cases}$$

then from the above discussion, we have $I - xA$ is invertible for $m > M$. And in view of (4.4), it is easy to see $1 - \frac{b}{a}(R(x)^m - 1) \neq 0$ in H_1 and H_2. Therefore, the process (3.4) is well defined for $m > M$.

Remark 18. Assume $R(x)$ is given by the (r, s)-Padé approximation to e^x, then we have

(1) if r is odd, then $\delta_1 > -\infty$ is the zero of $R(z)$;
(2) if s is even and $s \leqslant r$, then $\delta_2 = +\infty$, in particular $\delta_2 = +\infty$ for the ν-stage explicit Runge–Kutta method of order p with $p = \nu$;
(3) if s is even and $s > r$, then $\delta_2 < +\infty$ is the smallest positive zero of the function $R(x) - 1$.

Remark 19. For the A-stable higher order Runge–Kutta methods, it is easy to see from Theorem 15 (see Table 1):

(1) For the ν-stage Radau IA and IIA methods, $H \subseteq S$ if and only if ν is even;
(2) For the ν-stage Lobatto IIIA and IIIB methods, $H_1 \subseteq S_1$ if and only if ν is even and $H_2 \subseteq S_2$ if and only if ν is odd;
(3) For the ν-stage Gauss–Legendre and Lobatto IIIC methods, $H_1 \subseteq S_1$ if and only if ν is odd and $H_2 \subseteq S_2$ if and only if ν is even.

4.2. The θ-methods

Lemma 20. *Assume that $\phi(x) : \mathbb{R} \to \mathbb{R}$ is defined by $\phi(x) = \frac{1}{x} - \frac{1}{e^x - 1}$. Then $\phi(x)$ is a monotone decreasing function of x with $\phi(+\infty) = 0$, $\phi(0) = \frac{1}{2}$ and $\phi(-\infty) = 1$.*

Theorem 21. *For the one-leg θ-methods, $H_1 \subseteq S_1$ if and only if*

$$0 \leqslant \theta \leqslant \frac{1}{2}, \quad \textit{for } \frac{1}{\theta - 1} < x < 0,$$

and $H_2 \subseteq S_2$ if and only if

$$0 \leqslant \theta < \frac{1}{2}, \quad \textit{for } 0 < x \leqslant \phi^{-1}(\theta).$$

Proof. For the one-leg θ-methods, we have

$$R(x) = \frac{1 + (1 - \theta)x}{1 - \theta x}, \qquad e^x - R(x) = \frac{(e^x - 1)x}{1 - \theta x}(\phi(x) - \theta).$$

Therefore $\delta_1 = \frac{1}{\theta - 1}$ and $\delta_2 = \infty$. Similarly to the proof of Theorem 15, we can obtain that $H_1 \subseteq S_1$ if and only if $\phi(x) \geqslant \theta$, which from Lemma 20 is equivalent to $\theta \leqslant \phi(0)$, i.e. $0 \leqslant \theta \leqslant \frac{1}{2}$.

From the proof of Theorem 15, we can obtain that $H_2 \subseteq S_2$ if and only if $\phi(x) \geqslant \theta$ and $x\theta < 1$. From Lemma 20 and $\phi(x) < \frac{1}{x}$, we can obtain that $H_2 \subseteq S_2$ if and only if

$$0 \leqslant \theta < \frac{1}{2}, \quad \text{for } 0 < x \leqslant \phi^{-1}(\theta).$$

The proof is complete. $\square$

Remark 22. Applying the one-leg θ-method and the linear θ-method to the Eq. (1.1), we obtain the same recurrence relation. Hence the stability function of the two θ-methods are the same. Therefore Theorem 21 is also valid for the linear θ-methods.

Remark 23. For the alternately advanced and retarded type, the stability condition of the θ-methods for $a < 0$ given by Theorem 21 is different from the stability conditions for the linear EPCA of the retarded type or the advanced type, which is $\frac{1}{2} < \theta \leqslant 1$ for $a < 0$ and $a + a_0 < 0$ (see [8,11,12]).

5. Numerical experiments

In this section we give some examples to illustrate the conclusions in the paper.

Consider the following problems:

$$u'(t) = u(t) - 2u\left(\left[t + \frac{1}{2}\right]\right), \qquad u(0) = 1. \tag{5.1}$$

We shall use several methods listed in Table 2 with the stepsize $h = \frac{1}{2m}$ to get the numerical solution at $t = 10$, where the true solutions are $u(10) \approx 1.036440451448093\text{E–}6$. In Table 2, we have listed the absolute errors (AE) and relative errors (RE) at $t = 10$ and the *Ratio* of the errors of the case $m = 50$ over that of $m = 100$. We can see from Table 2 that the methods preserve their order of convergence.

In Fig. 1 we draw the numerical solutions of these methods with $m = 20$. It is easy to see that the numerical solutions are asymptotically stable for these methods.

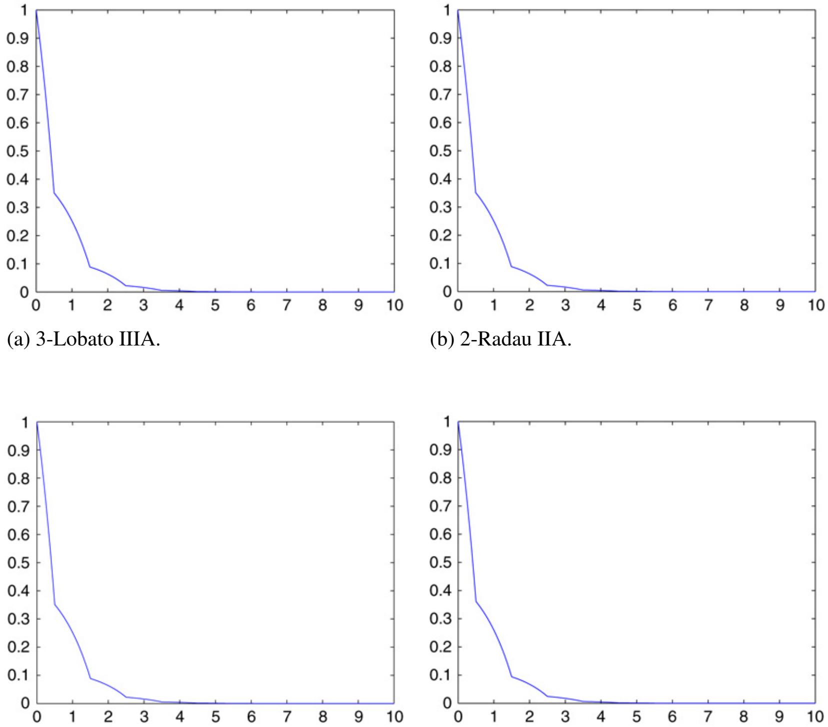
Next we consider the application of the one-leg θ-method with $m = 10$ to the following equation:

$$u'(t) = -u(t) + 9u\left(\left[t + \frac{1}{2}\right]\right), \qquad u(0) = 1. \tag{5.2}$$

In Fig. 2(a) $\theta = 0$ and in Fig. 2(b) $\theta = 1$. We can see from Fig. 2 that $y_n \to 0$ as $n \to \infty$ for $\theta = 0$ and $y_n \to \infty$ as $n \to \infty$ for $\theta = 1$, which show that the stability condition of the one-leg θ-method for $a < 0$ is different from the results in the papers [8,11,12].

Table 2
Problem (5.1)

	3-Lobato IIIA		2-Radau IIA		$\theta = 0.5$	
	AE	RE	AE	RE	AE	RE
$m = 2$	1.4474E−10	1.3965E−4	6.2100E−9	5.9917E−3	1.3155E−7	1.2693E−1
$m = 3$	2.8531E−11	2.7528E−5	1.7917E−9	1.7287E−3	6.0149E−8	5.8034E−2
$m = 5$	3.6936E−12	3.5638E−6	3.7946E−10	3.6612E−4	2.1969E−8	2.1197E−2
$m = 10$	2.3075E−13	2.2264E−7	4.6772E−11	4.5127E−5	5.5259E−9	5.3316E−3
$m = 20$	1.4420E−14	1.3913E−8	5.8067E−12	5.6026E−6	1.3836E−9	1.3349E−3
$m = 50$	3.6897E−16	3.5600E−10	3.7013E−13	3.5712E−7	2.2147E−10	2.1368E−4
$m = 100$	2.2864E−17	2.2060E−11	4.6205E−14	4.4580E−8	5.5370E−11	5.3423E−5
Ratio	16.138	16.138	8.0108	8.0108	3.9998	3.9998



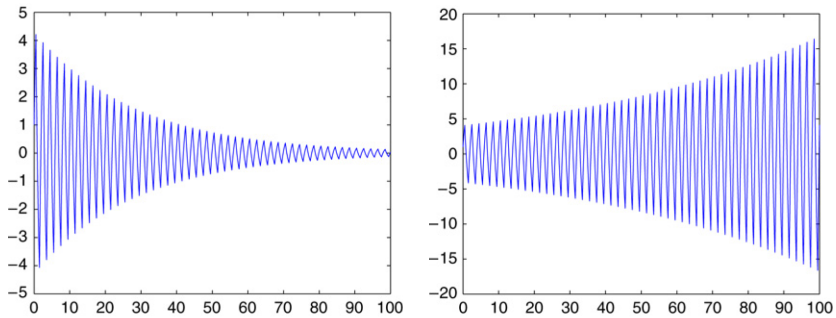
(a) 3-Lobato IIIA.

(b) 2-Radau IIA.

(c) The one-leg θ-method with $\theta = 0.5$.

(d) The one-leg θ-method with $\theta = 0$.

Fig. 1. The numerical solution for (5.1).



(a) $\theta = 0$.

(b) $\theta = 1$.

Fig. 2. The numerical solution of one-leg θ-method for (5.2).

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