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Schur-convexity and Schur-geometrically concavity of Gini means

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ABSTRACT

The Schur-convexity and the Schur-geometric convexity with variables $(x, y) \in \mathbb{R}^2_{++}$ for fixed (s, t) of Gini means $G(r, s; x, y)$ are discussed. Some new inequalities are obtained.

1. Introduction

Throughout the paper we assume that the set of n-dimensional row vector on the real number field by $\mathbb{R}^n$.

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\},$$

$$\mathbb{R}^n_- = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i < 0, i = 1, \dots, n\}.$$

In particular, $\mathbb{R}^1$, $\mathbb{R}^1_+$ and $\mathbb{R}^1_-$ denoted by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively.

Let $(s, t) \in \mathbb{R}^2$, $(x, y) \in \mathbb{R}^2_+$. The Gini means of (x, y) is defined in [1,2, pp.189] as

$$G(r, s; x, y) = \begin{cases} \left(\dfrac{x^s + y^s}{x^r + y^r}\right)^{1/(s-r)}, & r \neq s, \\ \exp\left(\dfrac{x^s \ln x + y^s \ln y}{x^r + y^r}\right), & r = s. \end{cases}$$

The Gini means are also called the "sum means". Clearly, $G(0, -1; x, y)$ is the harmonic mean, $G(0, 0; x, y)$ is the geometric mean, $G(1, 0; x, y)$ is the arithmetic mean.

Some properties of Gini means are given in next theorem.

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Theorem A *([3, p. 249]).*

(i) $\lim_{s\to r} G(r,s;x,y) = G(r,r;x,y)$;

$$\lim_{s\to\infty} G(r,s;x,y) = \max\{x,y\}; \qquad \lim_{s\to-\infty} G(r,s;x,y) = \min\{x,y\}.$$

(ii) *if* $s_1 \le s_2, r_1 \le r_2$ *then*

$$G(r_1,s_1;x,y) \le G(r_2,s_2;x,y); \tag{1}$$

further if $s_1 \ne s_2$ *or* $r_1 \le r_2$ *then inequality* (1) *is strict unless* $x = y$.

(iii) *if* $s \ge 1 \ge r \ge 0$ *then*

$$G(r,s;x_1+x_2,y_1+y_2) \le G(r,s;x_1,y_1) + G(r,s;x_2,y_2). \tag{2}$$

The Stolarsky means of (x,y) is defined in [4,5] as

$$E(r,s;x,y) = \begin{cases} \left(\dfrac{r}{s}\cdot\dfrac{y^s-x^s}{y^r-x^r}\right)^{1/(s-r)}, & rs(r-s)(x-y)\ne 0,\\ \left(\dfrac{1}{r}\cdot\dfrac{y^r-x^r}{\ln y-\ln x}\right)^{1/r}, & r(x-y)\ne 0;\\ \dfrac{1}{e^{1/r}}\left(\dfrac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, & r(x-y)\ne 0;\\ \sqrt{xy}, \quad x\ne y;\\ x, \quad x=y. \end{cases}$$

The Stolarsky means are sometimes called the "difference means", or the "extended means".

The Schur-convexities of the Stolarsky means $E(r,s;x,y)$ with (r,s) and (x,y) were presented in [6,7] as follows.

Theorem B *([6]). For fixed* $(x,y)\in\mathbb{R}^2_+$ *with* $x\ne y$, $E(r,s;x,y)$ *is Schur-concave on* $\mathbb{R}^2_+$ *and Schur-convex on* $\mathbb{R}^2_-$ *with* (r,s).

Theorem C *([7]). For fixed* $(r,s)\in\mathbb{R}^2$,

(i) *if* $2 < 2r < s$ *or* $2 \le 2s \le r$, *then* $E(r,s;x,y)$ *is Schur-convex on* $\mathbb{R}^2_+$ *with* (x,y),
(ii) *if* $(r,s)\in\{r<s\le 2r, 0<r\le 1\}\cup\{s<r\le 2s, 0<s\le 1\}\cup\{0<s<r\le 1\}\cup\{0<r<s\le 1\}\cup\{s\le 2r<0\}\cup\{r\le 2s<0\}$, *then* $E(r,s;x,y)$ *is Schur-concave on* $\mathbb{R}^2_+$ *with* (x,y).

In a recent paper, József Sándor [8] has proved the following result:

Theorem D. *For fixed* $(x,y)\in\mathbb{R}^2_+$ *with* $x\ne y$, $G(r,s;x,y)$ *is Schur-concave on* $\mathbb{R}^2_+$ *and Schur-convex on* $\mathbb{R}^2_-$ *with* (r,s).

And József Sándor point out that the Schur-convexity problem of $G(r,s;x,y)$ for fixed (s,t) with respect to $(x,y)\in\mathbb{R}^2_+$ are still open.

In this paper, the Schur-convexity and the Schur-geometric convexity with variables $(x,y)\in\mathbb{R}^2_+$ for fixed (s,t) of Gini means $G(r,s;x,y)$ are discussed, and some new inequalities are obtained. We obtain the following results.

Theorem 1. *For fixed* $(r,s)\in\mathbb{R}^2$,

(i) *if* $(r,s)\in\{r\ge 0, s\ge 0, r+s\ge 1\}$, *then* $G(r,s;x,y)$ *is the Schur-convex with* $(x,y)\in\mathbb{R}^2_+$;
(ii) *if* $(r,s)\in\{r\le 0, r+s\le 1\}\cup\{s\le 0, r+s\le 1\}$, *then* $G(r,s;x,y)$ *is the Schur-concave with* $(x,y)\in\mathbb{R}^2_+$.

Theorem 2. *If* $(r,s)\in\mathbb{R}^2_+$, *then* $G(r,s;x,y)$ *is the Schur-geometrically convex with* $(x,y)\in\mathbb{R}^2_+$.

For more information on the Stolarsky means and the Gini means, please refer to [9–15] and the references therein.

2. Definitions and lemmas

We need the following definitions and lemmas.

Definition 1 *([16,17]).* Let $\boldsymbol{x} = (x_1,\ldots,x_n)$ and $\boldsymbol{y} = (y_1,\ldots,y_n)\in\mathbb{R}^n$.

(i) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x}\prec\boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \le \sum_{i=1}^k y_{[i]}$ for $k=1,2,\ldots,n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]}\ge\cdots\ge x_{[n]}$ and $y_{[1]}\ge\cdots\ge y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.
(ii) $\Omega\subset\mathbb{R}^n$ is called a convex set if $(\alpha x_1+\beta y_1,\ldots,\alpha x_n+\beta y_n)\in\Omega$ for any $\boldsymbol{x}$ and $\boldsymbol{y}\in\Omega$, where α and $\beta\in[0,1]$ with $\alpha+\beta=1$.

(iii) let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$.φ is said to be a Schur-concave function on Ω if, and only if, $-\varphi$ is a Schur-convex function.

Definition 2 ([18,19]). Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$.

(i) $\Omega \subset \mathbb{R}^n_+$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for any $\boldsymbol{x}$ and $\boldsymbol{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.

(ii) let $\Omega \subset \mathbb{R}^n_+$, φ: $\Omega \to \mathbb{R}_+$ is said to be a Schur-geometrically convex function on Ω if $(\ln x_1, \ldots, \ln x_n) \prec (\ln y_1, \ldots, \ln y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$.φ is said to be a Schur-geometrically concave function on Ω if, and only if, $-\varphi$ is a Schur-geometrically convex function.

Lemma 1 *([16, p. 58]). Let $\Omega \subset \mathbb{R}^n$ be symmetric with respect to permutations and the convex set, and have a nonempty interior set Ω^0. Let $\varphi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable in Ω^0. Then φ is the Schur-convex(Schur-concave) function if, and only if, it is symmetric on Ω and if*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \ge 0\ (\le 0)$$

holds for any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega^0$.

Lemma 2 *([18, p. 108]). Let $\Omega \subset \mathbb{R}^n_+$ is a symmetric with respect to permutations and the geometrically convex set, and has a nonempty interior set Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0. Then φ is the Schur-geometrically convex (Schur-geometrically concave) function if φ is symmetric on Ω and*

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \ge 0\ (\le 0)$$

holds for any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega^0$.

Lemma 3. *Let $a \le b$, $u(t) = tb + (1-t)a$, $v(t) = ta + (1-t)b$. If $1/2 \le t_2 \le t_1 \le 1$ or $0 \le t_1 \le t_2 \le 1$, then*

$$(u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (a, b). \tag{3}$$

Proof. Case 1. When $1/2 \le t_2 \le t_1 \le 1$, it is easy to see that $u(t_1) \ge v(t_1), u(t_2) \ge v(t_2), u(t_1) \ge u(t_2)$ and $u(t_2) + v(t_2) = u(t_1) + v(t_1) = a + b$, that is (3) holds.

Case 2. When $0 \le t_1 \le t_2 \le 1$, then $1/2 \le 1 - t_2 \le 1 - t_1 \le 1$, by the Case 1, it follows

$$(u(1-t_2), v(1-t_2)) \prec (u(1-t_1), v(1-t_1)),$$

i.e. $(u(t_2), v(t_2)) \prec (u(t_1), v(t_1))$. □

Lemma 4 *([20]). Let $l, t, p, q \in \mathbb{R}_+$, $p > q$ and $p + q \le 3(l + t)$. Assume also that $1/3 \le l/t \le 3$ or $q \le l + t$. Then*

$$G(l, t; x, y) \le (p/q)^{1/(p-q)} E(p, q; x, y).$$

Lemma 5. *Let*

$$g(t, z) = \frac{z^t + 1}{t(z^{t-1} - 1)}.$$

Then for fixed $z > 1$,

(i) *$g(t, z)$ is increasing on $(-\infty, 0)$ with t;*
(ii) *$g(t, z)$ is increasing on $(0, \xi_z)$ with t,*
(iii) *$g(t)$ is decreasing on $(\xi_z, 1)$ or $(1, +\infty)$ with t,*

where ξ_z is a zero of the function

$$g_1(t, z) = t(z^t + z^{t-1}) \ln z + (z^t + 1)(z^{t-1} - 1)$$

with $0 < \xi_z < 1/2$.

Proof. Differentiate $g(t,z)$ with respect to t to obtain

$$\frac{\partial g(t,z)}{\partial t} = \frac{tz^t(z^{t-1}-1)\ln z - (z^t+1)(z^{t-1}-1) - tz^{t-1}(z^t+1)\ln z}{t^2(z^{t-1}-1)^2} = -\frac{g_1(t,z)}{t^2(z^{t-1}-1)^2}.$$

For fixed $z > 1$, $g_1(t,z) < 0$ and $\frac{\partial g(t,z)}{\partial t} > 0$ on $(-\infty, 0)$, then $g(t,z)$ increases on $(-\infty, 0)$ with t, and $g_1(t,z) > 0$ and $\frac{\partial g(t,z)}{\partial t} < 0$ on $(1, +\infty)$, then $g(t,z)$ decreases on $(1, +\infty)$ with t.

Differentiate $g_1(t,z)$ with respect to t to obtain

$$\frac{\partial g_1(t,z)}{\partial t} = [2z^{2t-1} + 2z^{t-1} + t(z^t + z^{t-1})\ln z]\ln z.$$

Since $\frac{\partial g_1(t,z)}{\partial t} > 0$ on $(0,1)$, $g_1(t,z)$ increases on $(0,1)$, it following that $g_1(0,z) \le g_1(t,z) \le g_1(1,z)$. Furthermore, $g_1(0,z) = 2(z^{-1}-1) < 0$ and $g_1(1,z) = (z+1)\ln z > 0$, hence there exist $\xi_z \in (0,1)$ such that $g_1(\xi_z, z) = 0$, and $g_1(t,z) \le 0$ and $\frac{\partial g(t,z)}{\partial t} \ge 0$ for $0 < t \le \xi_z$, and $g_1(t,z) > 0$ and $\frac{\partial g(t,z)}{\partial t} < 0$ for $\xi_z < t < 1$. this is, $g(t,z)$ increases on $(0, \xi_z)$ and decreases on $(\xi_z, 1)$.

Differentiate $g_1(t,z)$ with respect to z to obtain

$$\begin{aligned}\frac{\partial g_1(t,z)}{\partial z} &= tz^{t-1}(z^{t-1}-1) + (t-1)z^{t-2}(z^t+1) + t(z^{t-1}+z^{t-2}) + t[tz^{t-1} + (t-1)z^{t-2}]\ln z \\ &= (2t-1)z^{2t-2} + t^2z^{t-1}\ln z + (2t-1)z^{t-2} + (t^2-t)z^{t-2}\ln z.\end{aligned}$$

For $1 > t \ge 1/2$, we have

$$\begin{aligned}\frac{\partial g_1(t,z)}{\partial z} &\ge t^2z^{t-1}\ln z + (2t-1)z^{t-2} + (t^2-t)z^{t-2}\ln z \\ &= (t^2z + t^2 - t)z^{t-2}\ln z \\ &> (2t^2-t)z^{t-2}\ln z = t(2t-1)z^{t-2}\ln z \ge 0.\end{aligned}$$

Hence, for $1 > t \ge 1/2$, $g_1(t,z)$ increases on $(1, +\infty)$ with z, and then

$$g_1(t,z) > \lim_{z\to 1^+} g_1(t,z) = g_1(t,1) = 0.$$

Thus we conclude that $0 < \xi_z < 1/2$. □

Lemma 6. *For fixed (x,y) with $x > y > 0$. If $(r,s) \in \{r > 1, s < 0, r+s \le 1\} \cup \{1 < r \le s\} \cup \{0 < r \le 1-r \le s < 1\} \cup \{1/2 \le r \le s < 1\}$, then*

$$s(x^r + y^r)(x^{s-1} - y^{s-1}) \ge r(x^s + y^s)(x^{r-1} - y^{r-1}), \tag{4}$$

if $(r,s) \in \{s > 1, r < 0, r+s \le 1\} \cup \{r \le s < 0\}$, then (4) is reversed.

Proof. Let $g(t) = \frac{z^t+1}{t(z^{t-1}-1)}$ with $z = x/y > 1$. Notice that $y > 0$, it is easy to see that (4) equivalent to $g(r) \ge g(s)$. For $r > 1$, we first prove that $g(r) \ge g(1-r)$, i.e.

$$\frac{y(z^r+1)}{r(z^{r-1}-1)} \ge \frac{y(z^{1-r}+1)}{(1-r)(z^{-r}-1)} = \frac{y(z^r+z)}{(r-1)(z^r-1)}.$$

It is sufficient prove that

$$h(z) := (r-1)(z^r-1)(z^r+1) - r(z^{r-1}-1)(z^r+z) \ge 0.$$

Directly calculating yields

$$\begin{aligned}&h(z) = (r-1)z^{2r} - rx^{2r-1} + rx - r + 1, \\ &h'(z) = 2r(r-1)z^{2r-1} - r(2r-1)z^{2r-2} + r, \\ &h''(z) = 2r(r-1)(2r-1)z^{2r-3}(z-1).\end{aligned}$$

By $r > 1$, and $z > 1$, it follows $h''(z) > 0$. Therefore, $h'(z) > h'(1) = 0$, moreover, $h(z) > h(1) = 0$, i.e. $g(r) \ge g(1-r)$.

If $r > 1, s < 0, r+s \le 1$, then $s \le 1-r < 0$, from (i) of Lemma 5, we have $g(r) \ge g(s)$, i.e. (4) holds.

If $s > 1, r < 0, r+s \le 1$, replacing r by s and replacing s by r in the above case, it follows that $g(r) \le g(s)$, i.e. (4) is reversed.

If $0 < r \le 1/2 \le 1-r \le s < 1$, then $h''(z) > 0$, it follows $h'(z) > h'(1) = 0$, moreover, $h(z) > h(1) = 0$, i.e. $g(r) \ge g(1-r)$, from (iii) of Lemma 5, we have $g(r) \ge g(1-r) \ge g(s)$, i.e. (4) holds.

If $1/2 \le r \le s < 1$ or $1 < r \le s$, from (iii) of Lemma 5, we have $g(r) \ge g(s)$ i.e. (4) holds.

If $r \le s < 0$, from (i) of Lemma 5, we have $g(r) \le g(s)$ i.e. (4) is reversed. □

3. Proofs of main results

Proof of Theorem 1

Proof. Let $\varphi(x,y)=\frac{x^s+y^s}{x^r+y^r}$. When $r\neq s$, for fixed $(x,y)\in\mathbb{R}^2$, we have

$$\frac{\partial\varphi}{\partial x}=\frac{sx^{s-1}(x^r+y^r)-rx^{r-1}(x^s+y^s)}{(x^r+y^r)^2},$$
$$\frac{\partial\varphi}{\partial y}=\frac{sy^{s-1}(x^r+y^r)-ry^{r-1}(x^s+y^s)}{(x^r+y^r)^2}.$$
$$\begin{aligned}\frac{\partial\varphi}{\partial x}-\frac{\partial\varphi}{\partial y}&=\frac{s(x^r+y^r)(x^{s-1}-y^{s-1})-r(x^s+y^s)(x^{r-1}-y^{r-1})}{(x^r+y^r)^2}\\&=\frac{s(x^{r-1}-y^{r-1})}{(x^r+y^r)}\left[\frac{s-1}{r-1}\cdot\frac{(r-1)(x^{s-1}-y^{s-1})}{(s-1)(x^{r-1}-y^{r-1})}-\frac{r}{s}\cdot\frac{x^s+y^s}{x^r+y^r}\right]\\&=\frac{s(x^{r-1}-y^{r-1})}{(x^r+y^r)}\left[\frac{s-1}{r-1}\cdot E^{s-r}(r-1,s-1;x,y)-\frac{r}{s}\cdot G^{s-r}(r,s;x,y)\right],\end{aligned}$$

and then

$$\begin{aligned}\Delta&:=(x-y)\left(\frac{\partial G}{\partial x}-\frac{\partial G}{\partial y}\right)=\frac{x-y}{s-r}\left(\frac{\partial\varphi}{\partial x}-\frac{\partial\varphi}{\partial y}\right)\varphi^{\frac{1}{s-r}-1}(x,y)\\&=\frac{s(x-y)(x^{r-1}-y^{r-1})}{(s-r)(x^r+y^r)}\left[\frac{s-1}{r-1}\cdot E^{s-r}(r-1,s-1;x,y)-\frac{r}{s}\cdot G^{s-r}(r,s;x,y)\right]\varphi^{\frac{1}{s-r}-1}(x,y).\end{aligned}$$

In Lemma 4, taking $l=r$, $t=s$, $p=r-1$, $q=s-1$, we have

$$\begin{cases}l>0,t>0,p>0,q>0\\p>q\\p+q\le 3(l+t)\\1/3\le l/t\le 3\end{cases}\Leftrightarrow\begin{cases}r>1,s>1\\r>s\\r+s\ge -1\\s/3\le r\le 3s\end{cases}\Leftrightarrow 3s\ge r>s>1$$

and

$$\begin{cases}l>0,t>0,p>0,q>0\\p>q\\p+q\le 3(l+t)\\q\le l+t\end{cases}\Leftrightarrow\begin{cases}r>1,s>1\\r>s\\r+s\ge -1\\r\ge -1\end{cases}\Leftrightarrow r>s>1.$$

Hence, when $r>s>1$, we have

$$G(r,s;x,y)\le\left(\frac{r-1}{s-1}\right)^{\frac{1}{r-s}}E(r-1,s-1;x,y),$$

i.e.

$$G^{s-r}(r,s;x,y)\ge\frac{s-1}{r-1}\cdot E^{s-r}(r-1,s-1;x,y). \tag{5}$$

When $r>s>1$, we have $s-r<0$ and $(x-y)(x^{r-1}-y^{r-1})\ge 0$. Combining with (3), it follows that $\Delta\ge 0$. By Lemma 1, $G(r,s;x,y)$ is the Schur-convex with $(x,y)\in\mathbb{R}^2_{++}$.

Now we consider other cases. Notice that

$$(x-y)\left(\frac{\partial\varphi}{\partial x}-\frac{\partial\varphi}{\partial y}\right)=\frac{s(x^r+y^r)(x-y)(x^{s-1}-y^{s-1})-r(x^s+y^s)(x-y)(x^{r-1}-y^{r-1})}{(x^r+y^r)^2},$$

when $r\ge 1$, $0\le s\le 1$, since t^{r-1} and t^{s-1} is increasing and decreasing in $\mathbb{R}_+$ respectively, it follows that $(x-y)(x^{s-1}-y^{s-1})\ge 0$ and $(x-y)(x^{r-1}-y^{r-1})\le 0$, moreover, $(x-y)\left(\frac{\partial\varphi}{\partial x}-\frac{\partial\varphi}{\partial y}\right)\le 0$ and

$$\Delta=\frac{x-y}{s-r}\left(\frac{\partial\varphi}{\partial x}-\frac{\partial\varphi}{\partial y}\right)\varphi^{\frac{1}{s-r}-1}(x,y)\ge 0.$$

That is, when $r\ge 1$, $0\le s\le 1$, $G(r,s;x,y)$ is the Schur-convex with $(x,y)\in\mathbb{R}^2_+$.

When $r < 0, 0 < s \le 1$, since t^{r-1} and t^{s-1} are decreasing in $\mathbb{R}_{++}$, it follows that $(x-y)(x^{s-1}-y^{s-1}) \le 0$ and $(x-y)(x^{r-1}-y^{r-1}) \le 0$, moreover, $(x-y)\left(\frac{\partial\varphi}{\partial x}-\frac{\partial\varphi}{\partial y}\right) \le 0$ and $\Delta \le 0$, that is, when $r<0, 0<s\le 1$, $G(r,s;x,y)$ is the Schur-concave with $(x,y)\in\mathbb{R}^2_+$.

Without loss of generality, we may assume $x>y>0$. Notice that

$$\Delta = \frac{x-y}{s-r}\cdot\frac{s(x^r+y^r)(x^{s-1}-y^{s-1})-r(x^s+y^s)(x^{r-1}-y^{r-1})}{(x^r+y^r)^2}\varphi^{\frac{1}{s-r}-1}(x,y).$$

When $r>1, s<0, r+s\le 1$, from Lemma 6, it following that $\Delta \le 0$, i.e. $G(r,s;x,y)$ is the Schur-concave with $(x,y)\in\mathbb{R}^2_+$.

Similarly, we can prove that when $r \le s < 0$, $G(r,s;x,y)$ is the Schur-concave with $(x,y) \in \mathbb{R}^2_+$, and when $0<r\le 1-r\le s$ or $1/2 \le r \le s < 1$, $G(r,s;x,y)$ is the Schur-convex with $(x,y)\in\mathbb{R}^2_+$.

When $r=s\ge 1$, let

$$\psi(x,y)=\frac{x^s\ln x+y^s\ln y}{x^r+y^r}=\frac{x^s\ln x+y^s\ln y}{x^s+y^s}.$$

Then

$$\frac{\partial\psi}{\partial x}=\frac{x^{s-1}h(x,y)}{(x^s+y^s)^2},\qquad \frac{\partial\psi}{\partial y}=\frac{y^{s-1}k(x,y)}{(x^s+y^s)^2},$$

where

$$h(x,y)=(s\ln x+1)(x^s+y^s)-s(x^s\ln x+y^s\ln y),$$
$$k(x,y)=(s\ln y+1)(x^s+y^s)-s(x^s\ln x+y^s\ln y).$$

By computing,

$$\begin{aligned}x^{s-1}h(x,y)-y^{s-1}k(x,y) &= (x^s+y^s)\left[x^{s-1}(s\ln x+1)-y^{s-1}(s\ln y+1)\right]-s(x^s\ln x+y^s\ln y)(x^{s-1}-y^{s-1})\\ &= s^{s-1}y^{s-1}(x+y)(\ln x-\ln y)+(x^{s-1}-y^{s-1})(x^s+y^s),\end{aligned}$$

and then,

$$\begin{aligned}(x-y)\left(\frac{\partial G}{\partial x}-\frac{\partial G}{\partial y}\right) &= (x-y)\left(\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial y}\right)\mathrm{e}^{\psi(x,y)}\\ &= \frac{sx^{s-1}y^{s-1}(x+y)(x-y)(\ln x-\ln y)+(x-y)(x^{s-1}-y^{s-1})(x^s+y^s)}{(x^s+y^s)^2}\mathrm{e}^{\psi(x,y)}.\end{aligned}$$

Since $\ln t$ and t^{s-1} are increasing in $\mathbb{R}_+$ with t for $s\ge 1$, therefore, $(x-y)(\ln x-\ln y)\ge 0$ and $(x-y)(x^{s-1}-y^{s-1})\ge 0$, moreover, $(x-y)\left(\frac{\partial G}{\partial x}-\frac{\partial G}{\partial y}\right)\ge 0$. That is, when $r=s\ge 1$, $G(r,s;x,y)$ is the Schur-convex with $(x,y)\in\mathbb{R}^2_+$.

In conclusion, if $(r,s)\in\{r>s>1\}\cup\{r=s\ge 1\}\cup\{r\ge 1, 0\le s\le 1\}\cup\{0<r\le 1-r\le s\}\cup\{1/2\le r\le s<1\}$, then $G(r,s;x,y)$ is the Schur-convex with $(x,y)\in\mathbb{R}^2_+$, and if $(r,s)\in\{r<0, 0<s\le 1\}\cup\{r>1, s<0, r+s\le 1\}\cup\{r\le s<0\}$, then $G(r,s;x,y)$ is the Schur-concave with $(x,y)\in\mathbb{R}^2_+$.

Since $G(r,s;x,y)$ is symmetric with (r,s), if $(r,s)\in\{s>r>1\}\cup\{s\ge 1, 0\le r\le 1\}\cup\{0<s\le 1-s\le r\}\cup\{1/2\le s\le r<1\}$, then $G(r,s;x,y)$ is also the Schur-convex with $(x,y)\in\mathbb{R}^2_+$, and if $(r,s)\in\{s<0, 0<r\le 1\}\cup\{s>1, r<0, r+s\le 1\}\cup\{s\le r<0\}$, then $G(r,s;x,y)$ is also the Schur-concave with $(x,y)\in\mathbb{R}^2_+$.

The proof is complete. □

Remark 1. The Schur-convexity of the function $G(r,s;x,y)$ on the set $\{s<0, r+s>1\}$ or $\{r<0, r+s>1\}$ or $\{r>0, s>0, r+s<1\}$ with (x,y) is uncertain.

Example 1. Let $(r,s)=(2.5,-1.2)$. It is clear that $(2.5,-1.2)\in\{s<0, r+s>1\}$. For $(3,3)\prec(5,1)$, directly calculating yields

$$G(2.5,-1.2;3,3)=3.000000000>G(2.5,-1.2;5,1)=2.873884533.$$

But, for $(1.25,1.25)\prec(1.5,1)$, directly calculating yields

$$G(2.5,-1.2;1.25,1.25)=1.25.0000000<G(2.5,-1.2;1.5,1)=1.256253447.$$

Example 2. Let $(r,s)=(-0.2,1.5)$. It is clear that $(-0.2,1.5)\in\{r<0, r+s>1\}$. For $(8,8)\prec(15,1)$, directly calculating yields

$$G(-0.2,1.5;8,8)=8.000000000<G(-0.2,1.5;15,1)=8.412747770.$$

But, for $(25.5, 25.5) \prec (50, 1)$, directly calculating yields

$$G(-0.2, 1.5; 25.5, 25.5) = 25.5.0000000 > G(-0.2, 1.5; 50, 1) = 25.32833093.$$

Example 3. Let $(r, s) = (0.6, 0.2)$. It is clear that $(0.6, 0.2) \in \{r > 0, s > 0, r + s < 1\}$. For $(10.5, 10.5) \prec (20.9, 0.1)$, directly calculating yields

$$G(0.6, 0.2; 10.5, 10.5) = 10.5.0000000 < G(0.6, 0.2; 20.9, 0.1) = 11.03249418.$$

But, for $(10.5, 10.5) \prec (18, 3)$, directly calculating yields

$$G(0.6, 0.2; 10.5, 10.5) = 10.50000000 > G(0.6, 0.2; 18, 3) = 9.970045812.$$

Proof of Theorem 2

Proof. Let

$$\varphi(x, y) = \frac{x^s + y^s}{x^r + y^r}.$$

When $r \neq s$, for fixed $(x, y) \in \mathbb{R}^2$, we have

$$x\frac{\partial\varphi}{\partial x} = \frac{sx^s(x^r + y^r) - rx^r(x^s + y^s)}{(x^r + y^r)^2},$$

$$y\frac{\partial\varphi}{\partial y} = \frac{sy^s(x^r + y^r) - ry^r(x^s + y^s)}{(x^r + y^r)^2}.$$

$$\begin{aligned} x\frac{\partial\varphi}{\partial x} - y\frac{\partial\varphi}{\partial y} &= \frac{s(x^r + y^r)(x^s - y^s) - r(x^s + y^s)(x^r - y^r)}{(x^r + y^r)^2} \\ &= \frac{s(x^r - y^r)}{x^r + y^r}\left[\frac{s}{r} \cdot \frac{r(x^s - y^s)}{s(x^r - y^r)} - \frac{r}{s} \cdot \frac{x^s + y^s}{x^r + y^r}\right] \\ &= \frac{s(x^r - y^r)}{x^r + y^r}\left[\frac{s}{r} \cdot E^{s-r}(r, s; x, y) - \frac{r}{s} \cdot G^{s-r}(r, s; x, y)\right], \end{aligned}$$

and then

$$\begin{aligned} (\ln x - \ln y)\left(x\frac{\partial G}{\partial x} - y\frac{\partial G}{\partial y}\right) &= \frac{\ln x - \ln y}{s - r}\left(x\frac{\partial\varphi}{\partial x} - y\frac{\partial\varphi}{\partial y}\right)\varphi^{\frac{1}{s-r}-1}(x, y) \\ &= \frac{s(\ln x - \ln y)(x^r - y^r)}{(s - r)(x^r + y^r)}\left[\frac{s}{r} \cdot E^{s-r}(r, s; x, y) - \frac{r}{s} \cdot G^{s-r}(r, s; x, y)\right]\varphi^{\frac{1}{s-r}-1}(x, y). \end{aligned}$$

In Lemma 4, taking $l = p = r$, $t = q = s$, we have

$$\begin{cases} l > 0, t > 0, p > 0, q > 0 \\ p > q \\ p + q \leq 3(l + t) \\ 1/3 \leq l/t \leq 3 \end{cases} \Leftrightarrow \begin{cases} r > 0, s > 0 \\ r > s \\ r + s \geq -1 \\ s/3 \leq r \leq 3s \end{cases} \Leftrightarrow 3s \geq r > s > 0$$

and

$$\begin{cases} l > 0, t > 0, p > 0, q > 0 \\ p > q \\ p + q \leq 3(l + t) \\ q \leq l + t \end{cases} \Leftrightarrow \begin{cases} r > 0, s > 0 \\ r > s \\ r + s \geq -1 \\ r \geq 0 \end{cases} \Leftrightarrow r > s > 0.$$

Hence, when $r > s > 0$, we have

$$G(r, s; x, y) \leq \left(\frac{r}{s}\right)^{\frac{1}{r-s}} E(r, s; x, y),$$

i.e.

$$G^{s-r}(r, s; x, y) \geq \frac{s}{r} \cdot E^{s-r}(r, s; x, y). \tag{6}$$

When $r > s > 0$, we have $s - r < 0$, and since $\ln t$ and t^r are increasing in $\mathbb{R}_+$ with t, therefore $(\ln x - \ln y)(x^r - y^r) \geq 0$. Combining with (6), it follows that $(\ln x - \ln y)\left(x\frac{\partial G}{\partial x} - y\frac{\partial G}{\partial y}\right) \geq 0$. By Lemma 2, $G(r, s; x, y)$ is the Schur-geometrically convex with (x, y) in $\mathbb{R}^2_+$. Since $G(r, s; x, y)$ is symmetric with (r, s), when $s > r > 0$, $G(r, s; x, y)$ is also the Schur-geometrically convex with $(x, y) \in \mathbb{R}^2_+$.

Now we consider other cases.

Without loss of generality, we may assume $x > y > 0$. Notice that

$$\Lambda = \frac{\ln x - \ln y}{s - r} \cdot \frac{s(x^r + y^r)(x^s - y^s) - r(x^s + y^s)(x^r - y^r)}{(x^r + y^r)^2} \varphi^{\frac{1}{s-r} - 1}(x, y),$$

when $r = s > 0$, we have

$$\frac{\partial \psi}{\partial x} = \frac{x^{s-1} h(x, y)}{(x^s + y^s)^2}, \qquad \frac{\partial \psi}{\partial y} = \frac{x^{s-1} k(x, y)}{(x^s + y^s)^2},$$

where $h(x, y)$, $k(x, y)$ and $\psi(x, y)$ are the same as in Theorem 2.

By computing,

$$x^s h(x, y) - y^s k(x, y) = s^s y^s (x + y)(\ln x - \ln y) + (x^s - y^s)(x^s + y^s),$$

and then,

$$\begin{aligned}(\ln x - \ln y)\left(x\frac{\partial G}{\partial x} - y\frac{\partial G}{\partial y}\right) &= (\ln x - \ln y)\left(x\frac{\partial \psi}{\partial x} - y\frac{\partial \psi}{\partial y}\right) \mathrm{e}^{\psi(x,y)} \\ &= \frac{sx^s y^s (x + y)(\ln x - \ln y)^2 + (\ln x - \ln y)(x^s - y^s)(x^s + y^s)}{(x^s + y^s)^2} \mathrm{e}^{\psi(x,y)}.\end{aligned}$$

Since when $s > 0$, $\ln t$ and t^s are increasing in $\mathbb{R}_+$, $(\ln x - \ln y)(x^s - y^s) \geq 0$, moreover, $(\ln x - \ln y)\left(x\frac{\partial G}{\partial x} - y\frac{\partial G}{\partial y}\right) \geq 0$. That is, when $r = s > 0$, $G(r, s; x, y)$ is the Schur-geometrically convex with $(x, y) \in \mathbb{R}^2_+$.

In conclusion, if $(r, s) \in \{r > s > 0\} \cup \{s > r > 0\} \cup \{r = s > 0\} = \mathbb{R}^2_+$, $G(r, s; x, y)$ is the Schur-geometrically convex with $(x, y) \in \mathbb{R}^2_+$.

The proof is complete. □

4. Applications

Theorem 3. *Let $(x, y) \in \mathbb{R}^2_{++}$, $u(t) = ty + (1-t)x$, $v(t) = tx + (1-t)y$. Assume also that $\frac{1}{2} \leq t_2 \leq t_1 \leq 1$ or $0 \leq t_1 \leq t_2 \leq 1$. If $(r, s) \in \{r \geq 0, s \geq 0, r + s \geq 1\} \subseteq \mathbb{R}^2$, then for fixed $(r, s) \in \mathbb{R}^2$, we have*

$$\begin{aligned}G\left(r, s; \frac{x+y}{2}, \frac{x+y}{2}\right) &\leq G(r, s; u(t_2), v(t_2)) \\ &\leq G(r, s; u(t_1), v(t_1)) \leq G(r, s; x, y) \leq G(r, s; x + y, 0).\end{aligned} \tag{7}$$

If $(r, s) \in \{r \leq 0, r + s \leq 1\} \cup \{s \leq 0, r + s \leq 1\} \subseteq \mathbb{R}^2$, then inequalities in (7) are all reversed.

Proof. From Lemma 3, we have

$$\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (r, s)$$

and it is clear that $(x, y) \prec (x + y - \varepsilon, \varepsilon)$, where ε is enough small positive number.

If $(r, s) \in \{r \geq 1, s > 0\} \cup \{0 < r < 1, s \geq 1\}$, by Theorem 1, and let $\varepsilon \to 0$, it follows that (7) holds. If $(r, s) \in \{r < 0, 0 < s < 1\} \cup \{0 < r < 1, s < 0\}$, then inequalities in (7) are all reversed.

The proof is complete. □

Theorem 4. *Let $(x, y) \in \mathbb{R}^2_{++}$. For fixed $(r, s) \in \mathbb{R}^2_+$, we have*

$$G\left(r, s; \sqrt{xy}, \sqrt{xy}\right) \leq G(r, s; x, y). \tag{8}$$

Proof. Since $(\ln \sqrt{xy}, \ln \sqrt{xy}) \prec (\ln x, \ln y)$, by Theorem 2, it follows that (8) holds.

The proof is complete. □

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