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TRAVERSABILITY AND CONNECTIVITY OF THE MIDDLE GRAPH OF A GRAPH

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We define a graph $M(G)$ as an intersection graph $\Omega(F)$ on the point set $V(G)$ of any graph G. Let $X(G)$ be the line set of G and $F = V'(G) \cup X(G)$, where $V'(G)$ indicates the family of all one point subsets of the set $V(G)$. Let $M(G) = \Omega(F)$. $M(G)$ is called the middle graph of G. The following theorems result:

Theorem 1. Let G be any graph and G^+ be a graph constructed from G. Then we have $L(G^+) \cong M(G)$, where $L(G^+)$ is the line graph of G^+.

Theorem 2. Let G be a graph. The middle graph $M(G)$ of G is hamiltonian if and only if G contains a closed spanning trail.

Theorem 3. If a graph G is eulerian, then the middle graph $M(G)$ of G is eulerian and hamiltonian.

Theorem 4. If $M(G)$ is eulerian, then G is eulerian and $M(G)$ is hamiltonian.

Theorem 5. Let G be a graph. The middle graph $M(G)$ of G contains a closed spanning trail if and only if G is connected and without points of degree ≤ 1.

Theorem 6. If a graph G is n-line connected, then the middle graph $M(G)$ is n-connected.

1. Introduction

In this paper we shall consider a graph as finite, undirected, with single lines and no loops. All definitions not presented here can be found in [5].

1.1. Let $u_1, u_2, \ldots, u_p; x_1 = \{u_{i_1}, u_{j_1}\}, x_2 = \{u_{i_2}, u_{j_2}\}, \ldots, x_q = \{u_{i_q}, u_{j_q}\}$, denote the points and lines of a (p, q) graph G.

Let

$$V(G) = \{u_1, u_2, \ldots, u_p\},$$
$$X(G) = \{x_1, x_2, \ldots, x_q\},$$
$$V'(G) = \{\{u_1\}, \{u_2\}, \ldots, \{u_p\}\}.$$

Definition 1. We define the *middle graph* of G, denoted $M(G)$, as an intersection graph $\Omega(F)$ on $V(G)$, where

$$F = V'(G) \cup X(G) = \{\{u_1\}, \ldots, \{u_p\}, x_1, x_2, \ldots, x_q\}.$$

1.2. Let $L(G)$, $T(G)$, $S(G)$ denote the line graph, the total graph and the subdivision graph of G, respectively. Then,

$$L(G) \subset M(G) \subset T(G), \quad S(G) \subset M(G);$$
$$|V(M(G))| = |V(G)| + |V(L(G))| = p + q;$$
$$|X(M(G))| = |X(S(G))| + |X(L(G))| = |X(T(G))| - |X(G)|$$
$$= \tfrac{1}{2} \sum_{i=1}^{p} (\rho_G(u_i))^2 + q,$$

where $|A|$ denotes the cardinality of set A and $\rho_G(u_i)$ is the degree of u_i in G.

We define the iterated middle graph $M^k(G)$ as follows: $M^1(G) = M(G)$, $M^2(G) = M(M(G))$, $M^k(G) = M(M^{k-1}(G))$, $k \geq 2$. We abbreviate $L(G)$, $T(G)$, $S(G)$, $M(G)$, $M^k(G)$ as L, T, S, M, M^k, respectively.

The following equalities hold:

(1) $\rho_M(u) = \rho_{M^k}(u) = \rho_G(u)$ for $u \in G$.

(2) $\rho_M(v) = \rho_L(v) + 2 = \rho_G(s) + \rho_G(t)$ for $v \in L(G)$,

where s, t are the two end points of line x in G, corresponding to v.

1.3. Construction of $M(G)$: Construct the subdivision graph $S(G)$ of a given graph G. Construct further the complete graph K_{t_i} ($t_i = \rho_G(u_i) + 1$) for each point u_i of G, containing u_i and $\rho_G(u_i)$ points of $S(G)$ adjacent to u_i ($i = 1, 2, \ldots, p$). Then the result is the middle graph $M(G)$ of G (Fig. 1(i)).

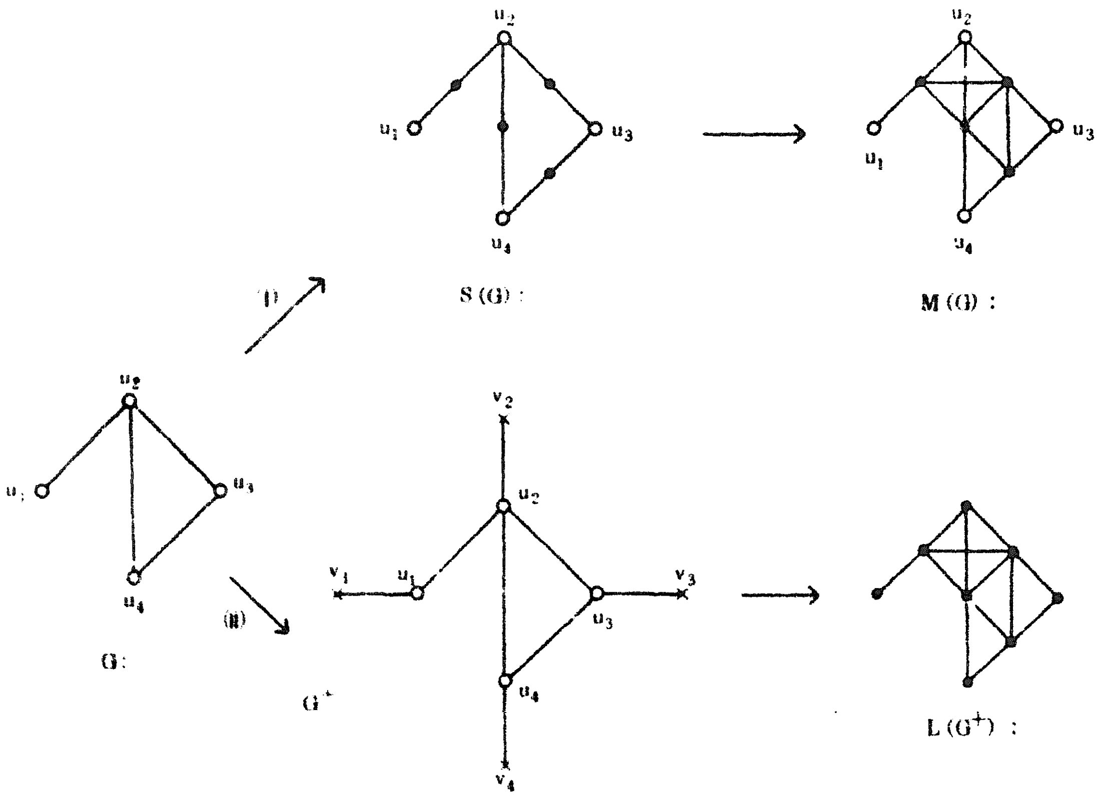

Fig. 1.

1.4. $M(G)$ is partitioned into line-disjoint complete graphs K_{t_i} $(i = 1, 2, \ldots, p)$ in such a way that two distinct K_{t_i}, K_{t_j} $(i \neq j)$ have only the subdivision point of line $\{u_i, u_j\}$ in common if and only if u_i and u_j are adjacent.

Definition 2. Let G be any (p, q) graph and $V(G)$, $X(G)$ be the point set of G and the line set of G respectively (see 1.1). We add to G p points v_i and p lines $\{u_i, v_i\}$ $(i = 1, 2, \ldots, p)$, where v_i are different from the points u_i of G and from each other. Then we obtain a $(2p, p+q)$ graph which consists of the point set $V(G^+) = \{u_1, \ldots, u_p, v_1, \ldots, v_p\}$ and the line set $X(G^+) = \{x_1, \ldots, x_q, \{u_1, v_1\}, \ldots, \{u_p, v_p\}\}$.

Let us denote this graph by G^+ (Fig. 1(ii)).

Theorem 1. *Let G be any graph. Then $L(G^+) \cong M(G)$, where the symbol $\cong$ means isomorphism.*

Proof. By the definitions of line graph and middle graph, we have $L(G^+) = \Omega(X(G^+))$, $M(G) = \Omega(V'(G) \cup X(G))$ (see 1.1).

Let the elements $\{u_i, v_i\}$, x_j of the set $X(G^+)$ correspond to the elements $\{u_i\}$, x_j of the set $V'(G) \cup X(G)$, respectively. Then we have a one to one correspondence between the elements of the two sets. Hence we have a one to one correspondence between the points of the graphs $L(G^+)$ and $M(G)$.

By the assumption that the points v_i are different from the points u_j of G, and that v_i are distinct from each other, we have

$$\{u_i, v_i\} \cap \{u_j, v_j\} = \{u_i\} \cap \{u_j\} = \emptyset \quad (i \neq j)\,,$$

$$\{u_i, v_i\} \cap x_j = \{u_i\} \cap x_j\,, \qquad x_i \cap x_j = x_i \cap x_j\,.$$

This shows that the intersection of the two elements of $X(G^+)$ coincides with that of the corresponding two elements of $V'(G) \cup X(G)$. Hence we see that the adjacency of two points in $L(G^+)$ coincides with that of the corresponding two points in $M(G)$.

Therefore the two graphs $L(G^+)$ and $M(G)$ are isomorphic. Thus the theorem is proved.

2. The traversability of $M(G)$

Proposition 8 in [6] states: $L(G)$ is hamiltonian if and only if there is a tour in G which includes at least one end-point of each line of G (a tour is the same as the closed trail).

If G contains a closed spanning trail, say T, then G^+ contains a tour T, that is a closed trail as is described in [6, Proposition 8]. Conversely let us suppose that G^+ contains a tour T, which includes at least one end-point of each line of G^+. Then it is evident that T does not pass points v_i $(i = 1, ..., p)$ (see Definition 2).

Assume that T does not pass a point u_i (see Definition 2). Then T cannot pass any one of the endpoints of line $\{u_i, v_i\}$, contradicting the hypothesis. Hence T passes all the points u_i $(i = 1, ..., p)$ of G, i.e., G contains a closed spanning trail T. The above considerations and Theorem 1 lead to:

Theorem 2. *Let G be a graph. The middle graph $M(G)$ of G is hamiltonian if and only if G contains a closed spanning trail.*

Theorem 3. *If a graph G is eulerian, then the middle graph $M(G)$ of G is eulerian and hamiltonian.*

Proof. It is evident that $M(G)$ is eulerian from the equalities (1) and (2). $M(G)$ is hamiltonian by Theorem 2.

Corollary 3.1. *If G is eulerian, then $M^k(G)$, $k \geqslant 1$, is eulerian and hamiltonian.*

Theorem 4. *If $M(G)$ is eulerian, then G is eulerian and $M(G)$ is hamiltonian.*

Proof. G is eulerian by (1), and hence $M(G)$ is hamiltonian by Theorem 3.

The proof of the following Theorem 5 is due to the referee of this paper.

Theorem 5. *Let G be a graph. The middle graph $M(G)$ of G contains a closed spanning trail if and only if G is connected and without points of degree $\leqslant 1$.*

Proof. If G is disconnected, then clearly $M(G)$ is also disconnected. Accordingly, $M(G)$ cannot contain a spanning trail. If G has a point of degree $\leqslant 1$, then $M(G)$ has also a point of degree $\leqslant 1$ from the equality (1). Accordingly, $M(G)$ cannot contain a closed trail. Therefore the necessity is established. Let us turn to the proof of sufficiency.

Let G be connected and without points of degree less than 2 and let T be a spanning tree of G. By the very definitions, $M(T)$ is a subgraph of $M(G)$. Let us construct a closed almost-spanning trail of $M(T)$, where we define an almost-spanning trail as a trail which passes through all points of $M(T)$ except the pendant ones (which are also pendant in T, as obvious).

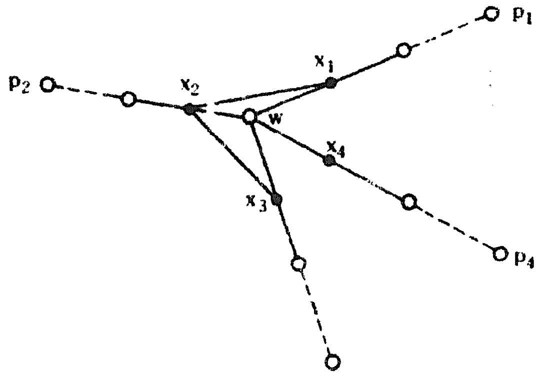

Fig. 2.

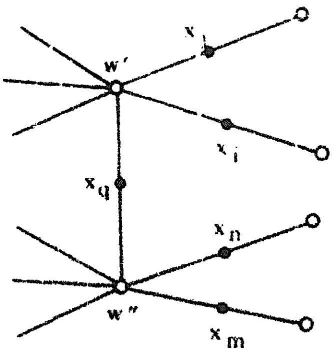

Fig. 3.

The construction is as follows: First take a path P in T linking two pendant vertices, p_1 and p_2. Starting from a point in $M(P)$ adjacent to say p_1 (which represents the pendant edge of P incident to p_1) follow the lines in $L(P)$ and return by those in $S(P)$. We obtain a trail as required corresponding to P. If $T \neq P$, then points of degree higher than 2 exist in P. Let w be such a point and p_3 ($\neq p_1, p_2$) a pendant vertex in a branch of T rooted at w. A path $P'(w, p_3)$ exists. The non-pendant points in $M(P')$ will be included in an extended trail obtained from the preceding one by taking, in an obvious notation, $x_2, x_3, L(P'), S(P'), x_3, w, x_1$ instead of x_2, w, x_1 (see Fig. 2, where not all lines are pictured).

If the degree of w is higher than 3 (as in the case shown in Fig. 2), then consider another pendant vertex, say p_4, and the path $P''(w, p_4)$ and insert now $x_3, x_4, L(P''), S(P''), x_4, w, x_1$ instead of x_3, w, x_1, and so on.

Repeat this construction as long as points of degree higher than 2 are to be considered. A closed almost-spanning trail of $M(T)$ will be obtained.

Now let us add to T, successively, the edges in $E(G) - E(T)$. We successively extend the trail so that the points representing these edges in

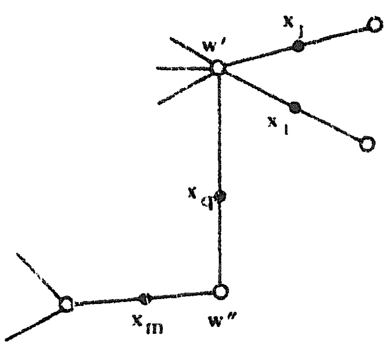

Fig. 4.

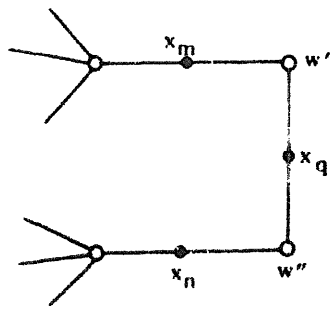

Fig. 5.

$M(G)$ are included:

(a) If the added edge x_q links two non-pendant vertices w', w'' (Fig. 3) in the graph we have at this stage, then, in the trail ... x_i, w', x_j, ..., x_m, w'', x_n, ..., we insert x_m, x_q, w'', x_n instead of x_m, w'', x_n.

(b) If x_q links a pendant vertex w'' to a non-pendant vertex w' (Fig. 4), then, in the trail ... x_m, ..., x_i, w', x_j, ..., we set x_i, x_q, x_m, w'', x_q, w', x_j instead of x_i, w', x_j.

(c) If x_q links two pendant vertices w', w'' (Fig. 5), then, in the trail ... x_m, ..., x_n, ..., we set x_n, x_q, x_m, w', x_q, w'', x_n instead of x_n.

These additions of lines do not use already used lines. Moreover, all points will be introduced in the trail.

This completes the proof of the theorem.

Corollary 5.1. *The graph $M^k(G)$, $k \geqslant 2$, is hamiltonian if and only if G is connected and without points of degree $\leqslant 1$.*

3. The connectivity of $M(G)$

The connectivity and the line-connectivity of $M(G)$ are simpler than those of $T(G)$, $L(G)$ (see [1,3,4,7,8]).

Theorem 6. *If a graph G is n-line connected, then the middle graph $M(G)$ of G is n-connected*

Proof. The theorem is evident for $n = 0$ or $G = K_2$. Therefore we assume that $n \geqslant 1$ and $G \neq K_2$ in the following. It suffices to show that there exist at least n disjoint paths between any two distinct points u, v in $M(G)$ (see [5, Theorem 5.10]).

(i) When $u \in G$, $v \in G$, there exist at least n line-disjoint paths between u and v in G. If we subdivide all lines of all these paths, then they may be considered as n line-disjoint paths between u, v in $M(G)$. Suppose that some of them have a point w in common. Further let u_1, v_1 ($\in L(G)$) be two points on one of these paths and adjacent to w. Then the new path obtained by replacing subpath $u_1 w v_1$ with line (u_1, v_1) is a path between u, v in $M(G)$ which does not pass w. Let us perform similar constructions on all other paths passing w. Repeating this process for all paths which have points in common, we obtain n disjoint paths between u, v in $M(G)$.

(ii) Let $u \in G$, $v \in L(G)$ and let $v', v'' \in G$ be the end-points of the edge v in G. There are at least n line-disjoint paths in G between u and v' and consequently, as shown in case (i), n point-disjoint paths in $M(G)$. Consider the n lines in G incident to v', say $(w_1, v'), \ldots, (w_n, v')$, each one belonging to a distinct path. Let $u_1, \ldots, u_n$ be the points in $L(G)$ corresponding to these lines (eventually $u_j = v$). We obtain n point-disjoint paths between u and v, by taking now, for each i, the subpath $w_i u_i v$ instead of the subpath $w_i u_i v'$. If $u_j = v$, then take (v'', v) instead of the subpath $v'' u_j v'$.

(iii) When $u \in L(G)$, $v \in L(G)$, there are two distinct points u, v in $L(G)$, since $n \geqslant 1$, $G \neq K_2$. Therefore there exist n disjoint paths between u, v in $M(G)$ (see [3, Theorem 1]).

Thus the proof of Theorem 6 is completed.

Corollary 6.1. *If G is n-line connected, then $M^k(G)$ is n-connected for all $k \geqslant 1$.*

Corollary 6.2. *If G is n-connected, then $M^k(G)$ is n-connected for all $k \geqslant 1$.*

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