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Applications of a Subordination Theorem

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The object of the present paper is to give applications of a subordination theorem due to Hallenbeck and Ruscheweyh [*Proc. Amer. Math. Soc.* **52** (1975), Theorem 1]. Our results have some interesting corollaries and examples as special cases.

1. INTRODUCTION

Let $A(n)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in N = \{1, 2, 3, \ldots\}) \tag{1.1}$$

which are analytic in the open unit disk $U = \{z: |z| < 1\}$. For analytic functions $g(z)$ and $h(z)$ with $g(0) = h(0)$, $g(z)$ is said to be subordinate to $h(z)$ if there exists an analytic function $w(z)$ so that $w(0) = 0$, $|w(z)| < 1$ $(z \in U)$ and $g(z) = h(w(z))$. We denote this subordination by

$$g(z) \propto h(z). \tag{1.2}$$

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If $h(z)$ is univalent in U, then the subordination (1.2) is equivalent to $g(0) = h(0)$ and $g(U) \subset h(U)$ (cf. Duren [1]).

With this subordination, Hallenbeck and Ruscheweyh [2] have shown

LEMMA 1. *Let $F(z)$ be convex univalent in U, $F(0) = 1$. Let $f(z)$ be analytic in U, $f(0) = 1$, $f'(0) = \cdots = f^{(n-1)}(0) = 0$, and let $f(z) \propto F(z)$ in U. Then for all $\gamma \neq 0$, $\mathrm{Re}(\gamma) \geqq 0$,*

$$\gamma z^{-\gamma} \int_0^z t^{\gamma-1} f(t)\, dt \propto \gamma z^{-\gamma/n} \int_0^{z^{1/n}} t^{\gamma-1} F(t^n)\, dt. \tag{1.3}$$

In this paper, we give some interesting results, applying the above lemma by Hallenbeck and Ruscheweyh [2].

2. APPLICATIONS OF A SUBORDINATION THEOREM

An application of Lemma 1 leads us to

THEOREM 1. *Let $p(z)$ be analytic in U with $p(0) = 1$, $p'(0) = \cdots = p^{(n-1)}(0) = 0$. If*

$$\mathrm{Re}\{p(z) + \alpha z p'(z)\} > \beta \qquad (z \in U), \tag{2.1}$$

then

$$\mathrm{Re}\{p(z)\} > \beta + (1-\beta)\left\{2\int_0^1 \frac{1}{1+\rho^{n\mathrm{Re}(\alpha)}}\, d\rho - 1\right\} \qquad (z \in U), \tag{2.2}$$

where $\alpha \neq 0$, $\mathrm{Re}(\alpha) \geqq 0$, and $\beta < 1$.

Proof. Letting

$$F(z) = \beta + (1-\beta)\frac{1-z}{1+z} \tag{2.3}$$

and

$$f(z) = p(z) + \alpha z p'(z) \tag{2.4}$$

in Lemma 1, we see that $F(z)$ is convex univalent in U, $F(0) = 1$, and that $f(z)$ is analytic in U, $f(0) = 1$, $f'(0) = \cdots = f^{(n-1)}(0) = 0$. Note that

$$p(z) = \frac{1}{\alpha} z^{-1/\alpha} \int_0^z t^{1/\alpha - 1} f(t)\, dt. \tag{2.5}$$

Therefore, using Lemma 1, we have that if $f(z) \propto F(z)$, then

$$p(z) \propto \frac{1}{\alpha} z^{-1/\alpha n} \int_0^{z^{1/n}} t^{1/\alpha - 1} F(t^n)\, dt. \tag{2.6}$$

Since

$$\begin{aligned}
&\frac{1}{\alpha} z^{-1/\alpha n} \int_0^{z^{1/n}} t^{1/\alpha-1} F(t^n)\, dt \\
&= \frac{1}{\alpha} z^{-1/\alpha n} \int_0^{z^{1/n}} t^{1/\alpha-1} \left(\beta + (1-\beta)\frac{1-t^n}{1+t^n}\right) dt \\
&= \frac{2\beta-1}{\alpha} z^{-1/\alpha n} \int_0^{z^{1/n}} t^{1/\alpha-1}\, dt + \frac{2(1-\beta)}{\alpha} z^{-1/\alpha n} \int_0^{z^{1/n}} \frac{t^{1/\alpha-1}}{1+t^n}\, dt \qquad (2.7) \\
&= 2\beta - 1 + \frac{2(1-\beta)}{\alpha} \int_0^1 \frac{u^{1/\alpha-1}}{1+zu^n}\, du \\
&= 2\beta - 1 + 2(1-\beta) \int_0^1 \frac{1}{1+z\rho^{\alpha n}}\, d\rho,
\end{aligned}$$

we conclude that

$$p(z) \propto q(z) = 2\beta - 1 + 2(1-\beta) \int_0^1 \frac{1}{1+z\rho^{\alpha n}}\, d\rho. \tag{2.8}$$

This implies that if $p(z)$ satisfies the inequality (2.1), then

$$\operatorname{Re}\{p(U)\} > \operatorname{Re}\{q(U)\}. \tag{2.9}$$

Now, it is easy to see that

$$\begin{aligned}
\operatorname{Re}\{q(z)\} &= \operatorname{Re}\left\{2\beta - 1 + 2(1-\beta)\int_0^1 \frac{1}{1+z\rho^{\alpha n}}\, d\rho\right\} \\
&= \beta + (1-\beta)\left\{2\int_0^1 \operatorname{Re}\left(\frac{1}{1+z\rho^{\alpha n}}\right) d\rho - 1\right\} \\
&\geqq \beta + (1-\beta)\left\{2\int_0^1 \frac{1}{1+|\rho^{\alpha n}|}\, d\rho - 1\right\} \qquad (2.10) \\
&= \beta + (1-\beta)\left\{2\int_0^1 \frac{1}{1+\rho^{n\operatorname{Re}(\alpha)}}\, d\rho - 1\right\}.
\end{aligned}$$

This completes the assertion of Theorem 1.

Letting $\text{Re}(\alpha) = 1/n$ in Theorem 1, we have

COROLLARY 1. *Let $p(z)$ be analytic in U with $p(0) = 1$, $p'(0) = \cdots = p^{(n-1)}(0) = 0$. If*

$$\text{Re}\{p(z) + \alpha z p'(z)\} > \beta \qquad (z \in U), \tag{2.11}$$

then

$$\text{Re}\{p(z)\} > 2\beta - 1 + 2(1 - \beta)\log 2 \qquad (z \in U), \tag{2.12}$$

where $\text{Re}(\alpha) = 1/n$ *and* $\beta < 1$.

Making $p(z) = f'(z)$ for $f(z) \in A(n)$ in Theorem 1, we have

EXAMPLE 1. If $f(z) \in A(n)$ satisfies

$$\text{Re}\{f'(z) + \alpha z f''(z)\} > \beta \qquad (z \in U). \tag{2.13}$$

then

$$\text{Re}\{f'(z)\} > \beta + (1 - \beta)\left\{2\int_0^1 \frac{1}{1 + \rho^{n\text{Re}(\alpha)}}\, d\rho - 1\right\} \qquad (z \in U), \tag{2.14}$$

where $\alpha \neq 0$, $\text{Re}(\alpha) \geqq 0$, and $\beta < 1$.

Further, taking $p(z) = f(z)/z$ for $f(z) \in A(n)$ in Theorem 1, we have

EXAMPLE 2. If $f(z) \in A(n)$ satisfies

$$\text{Re}\left\{(1 - \alpha)\frac{f(z)}{z} + \alpha f'(z)\right\} > \beta \qquad (z \in U), \tag{2.15}$$

then

$$\text{Re}\left\{\frac{f(z)}{z}\right\} > \beta + (1 - \beta)\left\{2\int_0^1 \frac{1}{1 + \rho^{n\text{Re}(\alpha)}}\, d\rho - 1\right\} \qquad (z \in U), \tag{2.16}$$

where $\alpha \neq 0$, $\text{Re}(\alpha) \geqq 0$, and $\beta < 1$.

Next, we derive

THEOREM 2. *Let $p(z)$ be analytic in U with $p(0) = 1$, $p'(0) = \cdots = p^{(n-1)}(0) = 0$. If*

$$\text{Re}\{p(z) + \alpha z p'(z)\} > \beta \qquad (z \in U), \tag{2.17}$$

then

$$|\arg(p(z) - \beta)| \leqq \operatorname*{Max}_{z\in U} \left|\arg\left(2\int_0^1 \frac{1}{1 + z\rho^{\alpha n}}\, d\rho - 1\right)\right| \qquad (z \in U), \tag{2.18}$$

where $\alpha \neq 0$, $\operatorname{Re}(\alpha) \geqq 0$, *and* $\beta < 1$.

Proof. In view of Theorem 1, we see that (2.8) implies

$$\frac{p(z) - \beta}{1 - \beta} \propto 2\int_0^1 \frac{1}{1 + z\rho^{\alpha n}}\, d\rho - 1. \tag{2.19}$$

It follows from (2.19) that

$$\left|\arg\left(\frac{p(z) - \beta}{1 - \beta}\right)\right| \leqq \operatorname*{Max}_{z\in U} \left|arg\left(2\int_0^1 \frac{1}{1 + z\rho^{\alpha n}}\, d\rho - 1\right)\right| \qquad (z \in U),$$

which is equivalent to (2.18).

Letting $\operatorname{Re}(\alpha) = 1/n$ in Theorem 2, we have

COROLLARY 2. *Let $p(z)$ be analytic in U with $p(0) = 1$, $p'(0) = \cdots = p^{(n-1)}(0) = 0$. If*

$$\operatorname{Re}\{p(z) + \alpha z p'(z)\} > \beta \qquad (z \in U), \tag{2.20}$$

then

$$|\arg(p(z) - \beta)| < \frac{\pi}{2} \qquad (z \in U), \tag{2.21}$$

where $\operatorname{Re}(\alpha) = 1/n$ *and* $\beta > 1$.

Proof. Note that

$$\begin{aligned}\operatorname{Re}\left\{\int_0^1 \frac{1}{1 + z\rho^{\alpha n}}\, d\rho\right\} &\geqq \operatorname{Re}\left\{\int_0^1 \frac{1}{1 + z\rho}\, d\rho\right\} \\ &> \int_0^1 \frac{1}{1 + \rho}\, d\rho \\ &= \log 2\end{aligned} \tag{2.22}$$

for $\operatorname{Re}(\alpha) = 1/n$, so that

$$\operatorname{Re}\left\{2\int_0^1 \frac{1}{1 + z\rho^{\alpha n}}\, d\rho - 1\right\} > 2\log 2 - 1 > 0 \qquad (z \in U). \tag{2.23}$$

This implies that

$$\left|\arg\left(2\int_0^1 \frac{1}{1+z\rho^{\alpha n}}\,d\rho - 1\right)\right| < \frac{\pi}{2} \qquad (z \in U). \tag{2.24}$$

Taking $p(z) = f'(z)$ for $f(z) \in A(n)$ in Corollary 2, we have

EXAMPLE 3. If $f(z) \in A(n)$ satisfies

$$\operatorname{Re}\{f'(z) + \alpha z f''(z)\} > \beta \qquad (z \in U), \tag{2.25}$$

then

$$|\arg(f'(z) - \beta)| < \frac{\pi}{2} \qquad (z \in U), \tag{2.26}$$

where $\operatorname{Re}(\alpha) = 1/n$ and $\beta < 1$.

Further, letting $p(z) = f(z)/z$ for $f(z) \in A(n)$ in Corollary 2, we have

EXAMPLE 4. If $f(z) \in A(n)$ satisfies

$$\operatorname{Re}\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right\} > \beta \qquad (z \in U), \tag{2.27}$$

then

$$\left|\arg\left(\frac{f(z)}{z} - \beta\right)\right| < \frac{\pi}{2} \qquad (z \in U), \tag{2.28}$$

where $\operatorname{Re}(\alpha) = 1/n$ and $\beta < 1$.

Finally, we derive

THEOREM 3. *Let $p(z)$ be analytic in U with $p(0) = 1$, $p'(0) = \cdots = p^{(n-1)}(0) = 0$. If*

$$\operatorname{Re}\{p(z) + \alpha z p'(z)\} < \beta \qquad (z \in U), \tag{2.29}$$

then

$$\operatorname{Re}\{p(z)\} < \beta + (1-\beta)\left\{2\int_0^1 \frac{1}{1+\rho^{n\operatorname{Re}(\alpha)}}\,d\rho - 1\right\} \qquad (z \in U), \tag{2.30}$$

where $\alpha \neq 0$, $\operatorname{Re}(\alpha) \geqq 0$, and $\beta > 1$.

Proof. Note that the condition (2.29) is equivalent to

$$p(z) + \alpha z p'(z) \propto \beta + (1 - \beta)\frac{1 - z}{1 + z}. \tag{2.31}$$

Therefore, proving the same way as in the proof of Theorem 1, we have (2.30).

Making $\mathrm{Re}(\alpha) = 1/n$ in Theorem 3, we have

COROLLARY 3. *Let $p(z)$ be analytic in U with $p(0) = 1$, $p'(0) = \cdots = p^{(n-1)}(0) = 0$. If*

$$\mathrm{Re}\{p(z) + \alpha z p'(z)\} < \beta \qquad (z \in U), \tag{2.32}$$

then

$$\mathrm{Re}\{p(z)\} < 2\beta - 1 + 2(1 - \beta)\log 2 \qquad (z \in U), \tag{2.33}$$

where $\mathrm{Re}(\alpha) = 1/n$ *and* $\beta > 1$.

If we put $p(z) = f'(z)$ for $f(z) \in A(n)$ in Theorem 3, then we have

EXAMPLE 5. If $f(z) \in A(n)$ satisfies

$$\mathrm{Re}\{f'(z) + \alpha z f''(z)\} < \beta \qquad (z \in U), \tag{2.34}$$

then

$$\mathrm{Re}\{f'(z)\} < \beta + (1 - \beta)\left\{2\int_0^1 \frac{1}{1 + \rho^{n\mathrm{Re}(\alpha)}}\,d\rho - 1\right\} \qquad (z \in U), \tag{2.35}$$

where $\alpha \neq 0$, $\mathrm{Re}(\alpha) \geqq 0$, and $\beta > 1$.

Letting $p(z) = f(z)/z$ for $f(z) \in A(n)$ in Theorem 3, we have

EXAMPLE 6. If $f(z) \in A(n)$ satisfies

$$\mathrm{Re}\left\{(1 - \alpha)\frac{f(z)}{z} + \alpha f'(z)\right\} < \beta \qquad (z \in U), \tag{2.36}$$

then

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\} < \beta + (1 - \beta)\left\{2\int_0^1 \frac{1}{1 + \rho^{n\mathrm{Re}(\alpha)}}\,d\rho - 1\right\} \qquad (z \in U), \tag{2.37}$$

where $\alpha \neq 0$, $\mathrm{Re}(\alpha) \geqq 0$, and $\beta > 1$.

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