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Note

An algebraic characterization of observational equivalence

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Abstract

We show that observational equivalence can be characterized by saturating homomorphisms (with respect to Hennessy–Milner logic), thus bringing together results developed independently by Castellani and by Arnold and Dicky on characterizations of transition system equivalences. We take this opportunity to compare Castellani's abstraction homomorphisms and Arnold–Dicky's saturating homomorphisms. It turns out that they are very similar notions: their difference in formulation is partly due to the fact that abstraction homomorphisms were defined on a restricted class of transition systems.

1. Introduction

A characterization of Milner's weak bisimulation equivalence – also called observational equivalence in that it abstracts away from internal actions – by means of particular transition system homomorphisms called *abstraction homomorphisms* was given in [7]. A similar approach was taken in [2, 3] to characterize logical equivalences on transition systems: here the relevant notion is that of *saturating homomorphism* with respect to a given logic. In both cases transition systems are shown to be equivalent if and only if they have a common image under surjective homomorphisms.

Related notions of morphisms on automata and transition systems had been considered previously in the literature, e.g. by Gourlay et al. [11], Park [16], Sifakis [17], Boudol [4], but had not been explicitly connected with bisimulation or logical equivalences (although the idea was somehow present in [16] and [4]). More recently, morphisms of a similar kind have been used in [9], under the name transition preserving homomorphisms. We should mention also the similitude with the zig-zag morphisms of

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[18], which was pointed out in [14]. This last work, together with [6], constitute to our knowledge the most recent developments on characterizing bisimulation equivalences via morphisms, for models more general than transition systems.

The notion of abstraction homomorphism was first put forward in [8] for a class of labelled event structures. In [7] abstraction homomorphisms were given a simple reformulation for a class of transition systems modelling CCS processes, and formally related to the notion of observational equivalence. In [5] the results of [7] are recast in categorical terms and applied to general automata. The approach of [7] has also been resumed in [15], where it is extended to computation trees labelled by partial orders.

The Arnold – Dicky approach [2, 3], on the other side, has been applied to a number of logically defined equivalences, such as strong bisimulation equivalence, the generalized transition system bisimulation of [13] (characterized by the "Future Perfect" logic), and branching bisimulation (characterized by the "until" Hennessy–Milner logic of [10]). However, the case of weak bisimulation equivalence was not treated so far.

This note aims at bringing together the results of [7] and [2, 3], by showing that observational equivalence can also be characterized by saturating homomorphisms (with respect to Hennessy–Milner logic). We take this opportunity to compare abstraction homomorphisms and saturating homomorphisms. It turns out that they are very similar notions: their difference in formulation is partly due to the fact that abstraction homomorphisms were defined in [7] on a restricted class of acyclic transition systems, lying between Milner's synchronisation trees and general transition systems.

2. Background

In this section, we give some basic definitions and recall the necessary results from [12] and [2].

2.1. Transition systems

Given a set A (whose elements represent the actions a process may perform), a labelled transition system $\mathscr{A}$ over A is defined by a set S of states, and a set $T \subseteq S \times A \times S$ of transitions. In the sequel a labelled transition system will be just named transition system.

As usual, a transition (s_1, a, s_2) will be denoted as $s_1 \xrightarrow{a} s_2$, so that for any a in A, $\xrightarrow{a}$ is a binary relation on S. A path of $\mathscr{A}$ is a sequence of successive transitions, and will be denoted as $s_0 \xrightarrow{a_1} \cdots \xrightarrow{a_n} s_n$. A transition system $\mathscr{A}$ is finite if both S and T are finite. It is *image-finite* if for any action a and state s the set $\xrightarrow{a}(s) =_{def} \{s' | s \xrightarrow{a} s'\}$ is finite.

Given two transition systems $\mathscr{A} = (A, S, T)$ and $\mathscr{A}' = (A, S', T')$, a transition system *homomorphism* $h: \mathscr{A} \longrightarrow \mathscr{A}'$ is a mapping of S into S' such that $h(T) \subseteq T'$, where $h(T) = \{h(q_1) \xrightarrow{a} h(q_2) \mid q_1 \xrightarrow{a} q_2 \in T\}$. The homomorphism h is said to be *surjective* if $h(S) = S'$.

2.2. Observable actions

Let us assume that A contains a special *unobservable action* denoted by τ; then $A_0 = A - \{\tau\}$ is the set of *observable actions.*

Given a transition system $\mathscr{A}$ over A, we define the new relations $\stackrel{a}{\Rightarrow}$ for $a \in A$ by

- $\stackrel{\tau}{\Rightarrow}$ is equal to $(\stackrel{\tau}{\to})^*$, the reflexive and transitive closure of $\stackrel{\tau}{\to}$,
- for a in A_0, $\stackrel{a}{\Rightarrow} \; = \; \stackrel{\tau}{\Rightarrow} \cdot \stackrel{a}{\to} \cdot \stackrel{\tau}{\Rightarrow}$.

In other words, $s \stackrel{\tau}{\Rightarrow} s'$ iff $s = s'$ or there is a path $s = s_0 \stackrel{\tau}{\to} \cdots \stackrel{\tau}{\to} s_n = s'$, and for $a \neq \tau$, $s \stackrel{a}{\Rightarrow} s'$ iff there is a path $s = s_0 \stackrel{\tau}{\to} \cdots s_i \stackrel{a}{\to} s_{i+1} \cdots \stackrel{\tau}{\to} s_n = s'$, with $0 \leqslant i$ and $i + 1 \leqslant n$.

In what follows, we shall always assume, unless otherwise stated, that transition systems are image-finite w.r.t. weak transitions, that is such that $\stackrel{a}{\Rightarrow}(s)$ is finite for any action a and state s.

2.3. Hennessy–Milner logic

The formulas of the Hennessy–Milner logic [12] are built up from

- the constants **1** et **0** (*true* and *false*), and the usual logical operators $\vee$, $\wedge$, $\neg$,
- a unary operator $\langle a \rangle$, for each letter a of the alphabet A.

For any formula F, the set of states of a given transition system $\mathscr{A}$ that satisfy F is the set $F_{\mathscr{A}}$ defined by structural induction on F:

- $\mathbf{1}_{\mathscr{A}} = S$, $\mathbf{0}_{\mathscr{A}} = \emptyset$,
- $(F_1 \vee F_2)_{\mathscr{A}} = (F_1)_{\mathscr{A}} \cup (F_2)_{\mathscr{A}}$, $(F_1 \wedge F_2)_{\mathscr{A}} = (F_1)_{\mathscr{A}} \cap (F_2)_{\mathscr{A}}$, $(\neg F)_{\mathscr{A}} = S - F_{\mathscr{A}}$,
- $(\langle a \rangle F)_{\mathscr{A}} = \langle a \rangle_{\mathscr{A}}(F_{\mathscr{A}})$,

where $\langle a \rangle_{\mathscr{A}}$ is the mapping from $\wp(S)$ into itself defined by

$$\langle a \rangle_{\mathscr{A}}(X) = \{s \mid \exists s' \in X : s \stackrel{a}{\Rightarrow} s'\} = (\stackrel{a}{\Rightarrow})^{-1}(X).$$

2.4. Observational equivalence

Let $\mathscr{A}$ and $\mathscr{A}'$ be two transition systems on the alphabet $A = A_0 \cup \{\tau\}$. A *weak bisimulation* between $\mathscr{A}$ and $\mathscr{A}'$ is a relation $R \subseteq S \times S'$ such that

- $\forall s \in S, \exists s' \in S' : sRs'$ and $\forall s' \in S', \exists s \in S : sRs'$,
- if $s_1 R s_1'$ then $\forall a \in A_0 \cup \{\tau\}$:
  - if $s_1 \stackrel{a}{\Rightarrow} s_2$ then there is s_2' such that $s_2 R s_2'$ and $s_1' \stackrel{a}{\Rightarrow} s_2'$,
  - if $s_1' \stackrel{a}{\Rightarrow} s_2'$ then there is s_2 such that $s_2 R s_2'$ and $s_1 \stackrel{a}{\Rightarrow} s_2$.

We say that two transition systems $\mathscr{A}$ and $\mathscr{A}'$ are *observationally equivalent* if there is a weak bisimulation between them.

2.5. Hennessy–Milner theorem

In the above framework, the Hennessy–Milner theorem [12] [1] can be stated as follows.

[1] We recall that transition systems are assumed to be image-finite w.r.t. weak transitions.

Theorem 1. *If $\mathscr{A}$ and $\mathscr{A}'$ are observationally equivalent, there is a largest weak bisimulation between them. This relation R is characterized by sRs' iff for all formulas F, $s \in F_{\mathscr{A}} \Leftrightarrow s' \in F_{\mathscr{A}'}$.*

2.6. Algebraic and logical equivalences of transition systems

Here we recall some notions introduced by Arnold and Dicky [2, 3] to deal with various equivalences of transition systems.

A logic $\mathscr{L}$ is defined by a set Ω of nonlogical operators, such that each operator $\omega \in \Omega$ has a fixed arity $n \geqslant 0$, and is given an interpretation in every transition system $\mathscr{A}$ as a mapping $\omega_{\mathscr{A}} : (\wp(S))^n \to \wp(S)$. Obviously, the interpretations of the logical operators $\vee, \wedge, \neg$ are defined by

$$X \vee_{\mathscr{A}} Y = X \cup Y,\ X \wedge_{\mathscr{A}} Y = X \cap Y,\ \neg_{\mathscr{A}} X = S - X.$$

It follows that every closed formula F is interpreted in $\mathscr{A}$ as a subset $F_{\mathscr{A}}$ of S.

We shall say that a transition system homomorphism $h : \mathscr{A} \to \mathscr{A}'$ is *saturating* with respect to $\mathscr{L}$ if every operator $\omega \in \Omega$, of arity n, satisfies the property: $\forall (X_1, \ldots, X_n) \in \wp(S')^n$,

$$\omega_{\mathscr{A}}(h^{-1}(X_1), \ldots, h^{-1}(X_n)) = h^{-1}(\omega_{\mathscr{A}'}(X_1, \ldots, X_n)).$$

Obviously, $h^{-1}(X \cup X') = h^{-1}(X) \cup h^{-1}(X')$ and $h^{-1}(X \cap X') = h^{-1}(X) \cap h^{-1}(X')$. Moreover, if h is surjective, $h^{-1}(S' - X) = S - h^{-1}(X)$. It follows that if h is a surjective homomorphism saturating with respect to $\mathscr{L}$, then for every closed formula F, $h^{-1}(F_{\mathscr{A}'}) = F_{\mathscr{A}}$.

Given a transition system $\mathscr{A}$, say that two states s_1 and s_2 of S are *indistinguishable* with respect to $\mathscr{L}$ (notation: $s_1 \sim_{\mathscr{L}} s_2$) when they satisfy the same formulas of $\mathscr{L}$: for all formulas F, $s_1 \in F_{\mathscr{A}} \Leftrightarrow s_2 \in F_{\mathscr{A}}$. Let h be the canonical mapping of S onto the quotient set $S\,|_{\sim_{\mathscr{L}}}$. Let $\mathscr{A}\,|_{\sim_{\mathscr{L}}}$ denote the transition system whose set of states is $S\,|_{\sim_{\mathscr{L}}}$, and whose transitions are the $h(s_1) \xrightarrow{a} h(s_2)$, if $s_1 \xrightarrow{a} s_2$ is a transition of $\mathscr{A}$. By construction h is a surjective transition system homomorphism of $\mathscr{A}$ onto $\mathscr{A}\,|_{\sim_{\mathscr{L}}}$.

We shall say that the logic $\mathscr{L}$ is *fully adequate* if for every transition system $\mathscr{A}$, the canonical homomorphism of $\mathscr{A}$ onto $\mathscr{A}\,|_{\sim_{\mathscr{L}}}$ is saturating with respect to $\mathscr{L}$.

Say that transition systems $\mathscr{A}_1$ and $\mathscr{A}_2$ are *equivalent* with respect to $\mathscr{L}$ when for every formula F of $\mathscr{L}$, some state of $\mathscr{A}_1$ satisfies f iff some state of $\mathscr{A}_2$ satisfies F, i.e., $F_{\mathscr{A}_1} \neq \emptyset$ iff $F_{\mathscr{A}_2} \neq \emptyset$.

We have the following result (see [1–3]), which looks like Hennessy–Milner theorem (for finite transition systems).

Proposition 1. *If $\mathscr{L}$ is fully adequate, then two finite transition systems $\mathscr{A}_1$ and $\mathscr{A}_2$ are equivalent with respect to $\mathscr{L}$ iff there exist a transition system $\mathscr{A}$ and two surjective homomorphisms $h_1 : \mathscr{A}_1 \to \mathscr{A}$ and $h_2 : \mathscr{A}_2 \to \mathscr{A}$, saturating with respect to $\mathscr{L}$. Moreover, there exists a least $\mathscr{A}$ having this property* (*with respect to the ordering*

$\mathscr{A}' \leqslant \mathscr{A}$ iff there is a surjective homomorphism $h : \mathscr{A} \to \mathscr{A}'$) and the corresponding h_i are such that $h_1(s_1) = h_2(s_2)$ iff for all formulas F of $\mathscr{L}$, $s_1 \in F_{\mathscr{A}_1} \Leftrightarrow s_2 \in F_{\mathscr{A}_2}$.

3. Observational homomorphisms of transition systems

We are now going to show that the Hennessy–Milner theorem (for finite systems) is a special case of Proposition 1, by showing that observational equivalence can be characterized by saturating homomorphisms (see Proposition 3). Then, to get the equivalent of Theorem 1 from Proposition 1, it suffices to prove that Hennessy-Milner logic is fully adequate. This proof is indeed quite similar to the proof of Theorem 2.2 in [12].

First we give a characterization of the homomorphisms that saturate all the operators $\langle a \rangle$. Note that any homomorphism h satisfies one-half of the saturation property, namely:

$$\forall a \in A, \forall X' \subseteq S',\ \langle a \rangle_{\mathscr{A}}(h^{-1}(X')) \subseteq h^{-1}(\langle a \rangle_{\mathscr{A}'}(X')).$$

In fact, $s_1 \in \langle a \rangle_{\mathscr{A}}(h^{-1}(X'))$ iff $\exists s_2 \in h^{-1}(X')$ such that $s_1 \stackrel{a}{\Rightarrow} s_2$. Since h is a homomorphism, this implies that $h(s_1) \stackrel{a}{\Rightarrow} h(s_2)$, whence $s_1 \in h^{-1}(\langle a \rangle_{\mathscr{A}'}(X'))$.

Lemma 1. *A homomorphism $h : \mathscr{A} \to \mathscr{A}'$ saturates all the operators $\langle a \rangle$ iff*

(∗) $\quad \forall a \in A, \forall s_2' \in S', \forall s_1 \in S, h(s_1) \stackrel{a}{\Rightarrow} s_2'$ *implies* $\exists s_2 : h(s_2) = s_2'$ *and* $s_1 \stackrel{a}{\Rightarrow} s_2$.

Proof. Since $\langle a \rangle_{\mathscr{A}}$, $\langle a \rangle_{\mathscr{A}'}$ and h^{-1} are additive, the condition

$$\forall a \in A, \forall X' \subseteq S', h^{-1}(\langle a \rangle_{\mathscr{A}'}(X')) \subseteq \langle a \rangle_{\mathscr{A}}(h^{-1}(X'))$$

is equivalent to

$$\forall a \in A, \forall s_2' \in S', h^{-1}(\langle a \rangle_{\mathscr{A}'}(s_2')) \subseteq \langle a \rangle_{\mathscr{A}}(h^{-1}(s_2')),$$

that is,

$$\forall a \in A, \forall s_2' \in S', \forall s_1 \in S, h(s_1) \stackrel{a}{\Rightarrow} s_2' \text{ implies } \exists s_2 : h(s_2) = s_2' \text{ and } s_1 \stackrel{a}{\Rightarrow} s_2 \qquad \Box$$

The property (∗) in the statement of Lemma 1 is the same as condition (ii) of transition-preserving homomorphisms in [9]. It is also the *zig-zag* condition of [18], and will be referred to under this name in the following.

A simple induction argument on the length of the path $h(s_1) \stackrel{a}{\Rightarrow} s_2'$ gives the simpler characterization.

Lemma 2. *A homomorphism $h : \mathscr{A} \to \mathscr{A}'$ saturates all the operators $\langle a \rangle$ iff*

$$\forall a \in A, \forall s_2' \in S', \forall s_1 \in S, h(s_1) \stackrel{a}{\to} s_2' \textit{ implies } \exists s_2 : h(s_2) = s_2' \textit{ and } s_1 \stackrel{a}{\Rightarrow} s_2.$$

Obviously, a homomorphism $h : \mathscr{A} \to \mathscr{A}'$ defines a relation included in $S \times S'$, also denoted by h. We prove

Proposition 2. *Let $h : \mathscr{A} \to \mathscr{A}'$ be a homomorphism. The relation h is a weak bisimulation iff h is surjective and saturates all the operators $\langle a \rangle$.*

Proof. In case of a relation h induced by a homomorphism, the definition of a weak bisimulation becomes:

(i) $\forall s' \in S', \exists s \in S : h(s) = s'$,

if $h(s_1) = s'_1$ then

(iia) if $s_1 \overset{a}{\Rightarrow} s_2$ then there is s'_2 such that $h(s_2) = s'_2$ and $s'_1 \overset{a}{\Rightarrow} s'_2$,

(iib) if $s'_1 \overset{a}{\Rightarrow} s'_2$ then there is s_2 such that $h(s_2) = s'_2$ and $s_1 \overset{a}{\Rightarrow} s_2$.

The point (i) is the definition of a surjective mapping, the point (iia) is always true for a homomorphism, and the point (iib) is the zig-zag property occurring in Lemma 1. Therefore, the result is a consequence of Lemma 1. □

The next result shows that observational equivalence can be algebraically characterized by means of homomorphisms saturating all the operators $\langle a \rangle$. That is why we suggest to call them *observational homomorphisms.*

Proposition 3. *$\mathscr{A}_1$ and $\mathscr{A}_2$ are observationally equivalent iff there exist $\mathscr{B}$ and two surjective homomorphisms $h_1 : \mathscr{A}_1 \to \mathscr{B}$ and $h_2 : \mathscr{A}_2 \to \mathscr{B}$, saturating all the operators $\langle a \rangle$. Moreover, if R is a weak bisimulation between $\mathscr{A}_1$ and $\mathscr{A}_2$, one can choose $\mathscr{B}$, h_1, h_2 in such a way that $R \subseteq h_1 \cdot h_2^{-1}$.*

Proof. The "if" part of this result is a consequence of Proposition 2. Let us prove the "only if" part.

Let $R \subseteq S_1 \times S_2$ be a weak bisimulation between $\mathscr{A}_1$ and $\mathscr{A}_2$. We may assume that the set of states of $\mathscr{A}_1$ and $\mathscr{A}_2$ are disjoint. Thus, let us consider the transition system $\mathscr{A}$ whose set of states is $S_1 \cup S_2$ and whose set of transitions is $T_1 \cup T_2$. We define on $S_1 \cup S_2$ the equivalence relation $\rho = (R \cup R^{-1})^*$ and then define $\mathscr{B}$ as the quotient of $\mathscr{A}$ by ρ. Let $h : \mathscr{A} \to \mathscr{B}$ be the canonical surjective homomorphism from $\mathscr{A}$ onto its quotient.

We show that h is saturating. Let $s \xrightarrow{a} s'$ be a transition of $\mathscr{B}$, and let $s_i \in S_i$ (for $i = 1$ or $i = 2$) be such that $h(s_i) = s$. By definition of $\mathscr{B}$, for some $j \in \{1,2\}$ there is a transition $s_j \xrightarrow{a} s'_j$ in $\mathscr{A}_j$ such that $h(s_j) = s$ and $h(s'_j) = s'$. By definition of h, $s_j \rho s_i$. Using the definition of ρ and the fact that R and R^{-1} are weak bisimulations, from $s_j \xrightarrow{a} s'_j$ we get $s_i \overset{a}{\Rightarrow} s'_i$ with $s'_j \rho s'_i$, i.e., $h(s'_i) = h(s'_j) = s'$. We have then the result by Lemma 2.

Let h_i ($i = 1,2$) be the restriction of h to S_i. Obviously, $R \subseteq h_1 \cdot h_2^{-1}$: if $s_1 R s_2$ then $s_1 \rho s_2$ and thus $h_1(s_1) = h(s_1) = h(s_2) = h_2(s_2)$. Since h is saturating, h_i is also saturating, because $s \overset{a}{\Rightarrow} s'$ in $\mathscr{A}$ and $s \in S_i$ implies $s \overset{a}{\Rightarrow} s'$ in $\mathscr{A}_i$.

It remains to prove that the h_i's are surjective. Let S be the set of states of $\mathscr{B}$. By definition of $\mathscr{B}$, $s \in S$ iff $\exists s_1 \in S_1 : s = h_1(s_1)$ or $\exists s_2 \in S_2 : s = h_2(s_2)$. Let us assume that $s \notin h_2(S_2)$. Hence, $s = h_1(s_1)$; but then there must be $s_2 \in S_2$ such that $s_1 R s_2$, thus $s = h_1(s_1) = h_2(s_2)$, a contradiction. $\square$

4. Relation with abstraction homomorphisms

Abstraction homomorphisms were defined in [7] on a class of acyclic transition systems lying between Milner's synchronisation trees and general transition systems. Because these transition systems were acyclic, they were presented as *partial orders* of states with a minimal element (the initial state). As a minor variation w.r.t. the standard presentation, these transition systems are labelled by sequences of (observable) actions on states rather than by actions on transitions. In this note we shall refer to these particular transition systems as a-transition systems.

Formally, if A^* is the set of finite sequences over A, with empty sequence ε, and $\multimap$ is the covering relation of the partial order $\leqslant$:

An a-transition system over A is of the form $(A, S, r, \leqslant, \ell)$, where $r \in S$ and:

$(S, r, \leqslant)$ is a rooted poset of states: $\forall s \in S: r \leqslant s$
$\ell : S \to A^*$ is a monotonic labelling function, satisfying:
$\ell(r) = \varepsilon$,
$s \multimap s'$ implies $\ell(s) = \ell(s')$ or $\exists a \in A - \{\tau\}$ s.t. $\ell(s') = \ell(s) \cdot a$,
$\forall \sigma \in A^*: \{s \in S \mid \ell(s) = \sigma\}$ is finite.

The last property of the labelling corresponds to an image-finiteness condition on the transition relation. It also implies (together with monotonicity of ℓ) that every state is finitely preceded, namely: $\forall s \in S: \{s' \in S \mid s' \leqslant s\}$ is finite.

One may interpret the partial ordering relation $\leqslant$ as a transition relation in two different ways: in the first way, which is the one suggested in [7], we take the transition relation to be simply the covering relation $\multimap$ of $\leqslant$, namely: $s \xrightarrow{\tau} s'$ if $s \multimap s'$ and $\ell(s) = \ell(s')$, and for $a \neq \tau$, $s \xrightarrow{a} s'$ if $s \multimap s'$ and $\ell(s') = \ell(s) \cdot a$.

We shall consider here a second interpretation, for which the transition relation is given by the ordering $\leqslant$ itself, whenever the labelling of the two related states differs by at most one character: $s \xrightarrow{\tau} s'$ if $s \leqslant s'$ and $\ell(s) = \ell(s')$, and for $a \neq \tau$, $s \xrightarrow{a} s'$ if $s \leqslant s'$ and $\ell(s') = \ell(s) \cdot a$.

As an example, consider the following a-transition system (where the ordering goes from top to bottom), and the corresponding transition systems under the two interpretations (see Fig. 1).

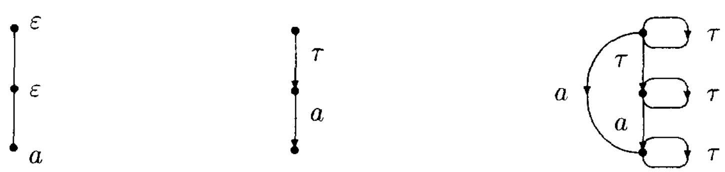

Fig. 1.

Transition systems obtained by the first interpretation only differ from the original transition systems for the labelling: they will therefore still be called a-transition systems, with a slight abuse of notation. Under the second interpretation, on the other hand, systems are completed with a strong transition $s \xrightarrow{a} s'$ whenever there is a weak transition $s \stackrel{a}{\Rightarrow} s'$ in the first interpretation. This means in particular that a self-loop $s \xrightarrow{\tau} s$ is added on every state. Apart from these tight τ-loops, the resulting transition systems are still acyclic. We shall call them ac-transition systems (for *completed*).

Let us now recall from [7] the definition of abstraction homomorphism.[2] We will use the notation $\leqslant_a(s)$ to represent the set of weak successors of state s under action a, namely $\leqslant_\tau(s) = \{s'|s \leqslant s' \ \&\ \ell(s) = \ell(s')\}$ and, for $a \neq \tau$, $\leqslant_a(s) = \{s'|s \leqslant s' \ \&\ \ell(s') = \ell(s) \cdot a\}$.

Given two a-transition systems $\mathscr{A} = (A, S, r, \leqslant, \ell)$ and $\mathscr{A}' = (A, S', r', \leqslant', \ell')$, an *abstraction homomorphism* $h: \mathscr{A} \longrightarrow \mathscr{A}'$ is a mapping of S into S' such that $\forall a \in A, \forall s \in S$:

1. $h(r) = r'$,
2. $h(\leqslant_a(s)) = \leqslant'_a(h(s))$.

For the coming discussion, it will be convenient to split property 2 in two parts:

2(a) $h(\leqslant_a(s)) \subseteq \leqslant'_a(h(s))$ (transition preserving),
2(b) $h(\leqslant_a(s)) \supseteq \leqslant'_a(h(s))$ (zig-zag condition).

It is easy to see that an abstraction homomorphism h preserves state labels, that is $\ell'(h(s)) = \ell(s)$. Note also that condition 2(b) implies forward surjectivity, namely if $s' \in h(S)$ then $\{s''|s' \leqslant' s''\} \subseteq h(S)$. Therefore, since $r' = h(r)$ and $\{s'|r' \leqslant' s'\} = S'$, abstraction homomorphisms are always *surjective*.

We shall now comment on the meaning of conditions 2(a) and 2(b) depending on how we choose to interpret $\leqslant$. Under the first interpretation, we have $\leqslant_a(s) = \stackrel{a}{\Rightarrow}(s) = \{s'|s \stackrel{a}{\Rightarrow} s'\}$, and property 2(a) amounts to weak transition preserving, while property 2(b) amounts to the zig-zag condition of Lemma 1:

$$(*) \qquad h(s_1) \stackrel{a}{\Rightarrow} s'_2 \ \Rightarrow\ (\exists s_2 : h(s_2) = s'_2 \text{ and } s_1 \stackrel{a}{\Rightarrow} s_2).$$

In fact, in this case condition 2 is equivalent to the $\langle a \rangle$-saturation property, since $s' \in h(\stackrel{a}{\Rightarrow}(s))$ iff $s \in \langle a \rangle_{\mathscr{A}}(h^{-1}(s'))$ and $s' \in \stackrel{a}{\Rightarrow}(h(s))$ iff $s \in h^{-1}(\langle a \rangle_{\mathscr{A}'}(s'))$.

However in this interpretation abstraction homomorphisms are more permissive than (surjective) saturating homomorphisms, since they do not need satisfy the condition $h(T) \subseteq T'$. For instance the two mappings shown in Fig. 2 are abstraction homomorphisms but not saturating homomorphisms.

Under the second interpretation, on the other hand, $\leqslant_a(s) = \xrightarrow{a}(s) = \{s'|s \xrightarrow{a} s'\}$, and property 2(a) amounts to the transition preserving condition $h(T) \subseteq T'$, while property 2(b) amounts to a strong zig-zag condition, namely:

$$(**) \qquad h(s_1) \xrightarrow{a} s'_2 \ \Rightarrow\ (\exists s_2 : h(s_2) = s'_2 \text{ and } s_1 \xrightarrow{a} s_2).$$

[2] Since there are no operators on transition systems here, we consider the variant of abstraction homomorphisms characterizing observational equivalence, rather than observational congruence.

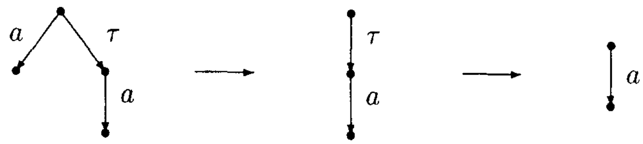

Fig. 2.

Since $(**)$ implies $(*)$, it should be clear that in this case any abstraction homomorphism is also a saturating homomorphism. To state this fact formally, let us introduce some notation: if $\mathscr{A} = (A, S, r, \leqslant, \ell)$ is an a-transition system, we denote by $\mathscr{A}_1, \mathscr{A}_2$, where $\mathscr{A}_i = (A, S, T_i)$, the corresponding transition systems under the first and second interpretation for $\leqslant$. We call $\mathscr{A}_i$ the i-interpretation of $\mathscr{A}$. So far we have established the following.

Fact 1. *Let $\mathscr{A}, \mathscr{A}'$ be a-transition systems, and $\mathscr{A}_2, \mathscr{A}'_2$ be the corresponding 2-interpretations (ac-transition systems). If the mapping $h: S \longrightarrow S'$ is an abstraction homomorphism from $\mathscr{A}$ to $\mathscr{A}'$, then h is also a saturating homomorphism from $\mathscr{A}_2$ to $\mathscr{A}'_2$.*

We show now that any surjective saturating homomorphism between (interpretations of) a-transition systems is also an abstraction homomorphism. To this end, we first prove that, when restricted to interpretations of a-transition systems, surjective saturating homomorphisms satisfy the inclusion $T' \subseteq h(T)$ (i.e. every strong transition in the second system is the image of a strong transition in the first). This is not required in general for saturating homomorphisms. For instance the two mappings in Fig. 3 are saturating homomorphisms.

Proposition 4. *Let $\mathscr{A}, \mathscr{A}'$ be a-transition systems, and for each $i \in \{1, 2\}$ let $\mathscr{A}_i, \mathscr{A}'_i$ be their corresponding i-interpretations. If $h: S \longrightarrow S'$ is a surjective saturating homomorphism from $\mathscr{A}_i$ to $\mathscr{A}'_i$, for $i = 1$ or $i = 2$, then $T'_i \subseteq h(T_i)$.*

Proof. Suppose $s'_1 \xrightarrow{a} s'_2 \in T'_i$. By the surjectivity of h there exists s_1 such that $h(s_1) = s'_1$. By the zig-zag condition $(*)$ there exists s_2 such that $h(s_2) = s'_2$ and $s_1 \overset{a}{\Rightarrow} s_2$. We distinguish two cases:

1. If $i = 2$, from $s_1 \overset{a}{\Rightarrow} s_2$ we can deduce $s_1 \xrightarrow{a} s_2$, since $\mathscr{A}_2$ is an ac-transition system. So this is the required strong transition.

2. If $i = 1$, $s'_1 \xrightarrow{a} s'_2$ implies $s'_1 \neq s'_2$, since $\mathscr{A}'_1$ is strictly acyclic: thus the case $s_1 = s_2$ (supposing $a = \tau$) is excluded because h is a function. Hence $\exists q_0, \ldots q_n$ such that $s_1 = q_0 \xrightarrow{\tau} q_1 \cdots q_i \xrightarrow{a} q_{i+1} \cdots q_{n-1} \xrightarrow{\tau} q_n = s_2$. By the transition preserving condition $h(T_1) \subseteq T'_1$ we have therefore $h(q_0) \xrightarrow{\tau} h(q_1) \cdots h(q_i) \xrightarrow{a} h(q_{i+1}) \cdots h(q_{n-1}) \xrightarrow{\tau} h(q_n)$. Now, because T'_1 is the Hasse diagram of a partial ordering, there must be $j \in \{0, \ldots, n-1\}$ such that $h(q_0) = \cdots = h(q_j) = s'_1$ and $h(q_{j+1}) = \cdots = h(q_n) = s'_2$ (in case $a \neq \tau$ this j will coincide with i, because of the label preserving condition). Then $q_j \xrightarrow{a} q_{j+1}$ is the required strong transition. $\square$

Fig. 3.

Proposition 5. *Let $\mathscr{A}, \mathscr{A}'$ be a-transition systems, and for each $i \in \{1,2\}$ let $\mathscr{A}_i, \mathscr{A}'_i$ be their corresponding i-interpretations. If for some $i \in \{1,2\}$ the mapping $h: S \longrightarrow S'$ is a surjective saturating homomorphism from $\mathscr{A}_i$ to $\mathscr{A}'_i$, then h is an abstraction homomorphism from $\mathscr{A}$ to $\mathscr{A}'$.*

Proof. Suppose h is a surjective saturating homomorphism from $\mathscr{A}_i$ to $\mathscr{A}'_i$, for $i = 1$ or $i = 2$. We will show that h satisfies properties 1 and 2 of abstraction homomorphisms. *Property 1.* We want to show $h(r) = r'$. Since h is surjective, there exists $s \in S$ such that $h(s) = r'$. Let $r = q_0 \xrightarrow{a_1} q_1 \cdots q_{n-1} \xrightarrow{a_n} q_n = s$ be a path from r to s in $\mathscr{A}_i$ (since s is finitely preceded in $\mathscr{A}$, this path always exists). From $h(T_i) \subseteq T'_i$ we deduce $h(q_0) \xrightarrow{a_1} h(q_1) \cdots h(q_{n-1}) \xrightarrow{a_n} h(q_n) = r'$. We distinguish now two cases:
- $i = 1$: since T'_1 is the Hasse diagram of a partial ordering with minimal element r', we have necessarily $n = 0$, that is $r = s$.
- $i = 2$: here if $s' \xrightarrow{a} r' \in T'_2$ we have necessarily $s' = r'$ and $a = \tau$. Thus $h(q_0) \xrightarrow{a_1} h(q_1) \cdots h(q_{n-1}) \xrightarrow{a_n} h(q_n) = r'$ implies either $n = 0$ and $s = r$ or $\forall i \in \{0, \ldots, n-1\}: h(q_i) = r'$ and $a_{i+1} = \tau$. Thus in particular $h(r) = h(q_0) = r'$.

Property 2. Again, we distinguish two cases:
- $i = 1$: then property 2 is equivalent to the $\langle a \rangle$-saturation property.
- $i = 2$: in this case property 2(a) is equivalent to $h(T_2) \subseteq T'_2$, which is true for all saturating homomorphisms, while property 2(b) is equivalent to the strong zig-zag condition (∗∗), which is a consequence of $T'_2 \subseteq h(T_2)$, satisfied by h by Proposition 4. □

To sum up, surjective saturating homomorphisms are a subset of abstraction homomorphisms on a-transition systems, while they exactly correspond with them on ac-transition systems.

Corollary 1. *Let $\mathscr{A}, \mathscr{A}'$ be a-transition systems, and $\mathscr{A}_2, \mathscr{A}'_2$ be the corresponding ac-transition systems. A mapping $h: S \longrightarrow S'$ is an abstraction homomorphism from $\mathscr{A}$ to $\mathscr{A}'$ if and only if it is a surjective saturating homomorphism from $\mathscr{A}_2$ to $\mathscr{A}'_2$.*

Remark. When restricted to ac-transition systems, surjective saturating homomorphisms satisfy both conditions $h(T) \subseteq T'$ and $T' \subseteq h(T)$. Therefore they coincide on this class with the *strong* saturating homomorphisms used in [2] to characterize strong bisimulation equivalence. As a consequence, the above corollary could be strenghtened as follows.

Corollary 2. *Let $\mathscr{A}, \mathscr{A}'$ be a-transition systems, and $\mathscr{A}_2, \mathscr{A}'_2$ be the corresponding ac-transition systems. A mapping $h: S \longrightarrow S'$ is an abstraction homomorphism from $\mathscr{A}$ to $\mathscr{A}'$ if and only if it is a strong surjective saturating homomorphism from $\mathscr{A}_2$ to $\mathscr{A}'_2$.*

In other words, reducibility between a-transition systems via an abstraction homomorphism amounts to reducibility between the corresponding ac-transition systems via a strong surjective saturating homomorphism. This is indeed not surprising, since ac-transition systems are maximal w.r.t. transitions, and it is well-known that on such transition systems weak bisimulation reduces to strong bisimulation.

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