Computers and Mathematics with Applications 59 (2010) 933–940

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On characterizations of preinvex fuzzy mappings[☆]

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ARTICLE INFO

Article history:
Received 11 April 2009
Accepted 24 September 2009

Keywords:
Fuzzy numbers
Preinvex fuzzy mapping
Semicontinuity

ABSTRACT

Several characterizations about preinvex fuzzy mapping are obtained in this paper. Firstly, an equivalent condition of preinvex fuzzy mapping is established under certain conditions. Furthermore, the necessary and sufficient conditions for differentiable and twice differentiable preinvex fuzzy mapping are provided by using the given equivalent condition of preinvex fuzzy mapping. Finally, a new proof of some known important conclusions is offered. These results generalize and improve some known results.

1. Introduction

In [1], Zadeh introduced the concept of fuzzy number. Since then, fuzzy numbers have been extensively studied by many authors. With the development of theories of fuzzy numbers and their applications, the concept of fuzzy numbers becomes more and more important. Researchers have begun to study theories in relation to fuzzy numbers. For instance, the problems of integrability and differentiability and measurability of fuzzy mapping have been discussed by Goetschel and Voxman [2], Dubois and Prade [3]. Since Nanda and Kar [4] proposed the concept of convex fuzzy mapping in 1992, the research of convexity for fuzzy mapping and application to fuzzy optimization have been developed widely and deeply (for examples, see [5–11]). Convexity plays a key role in fuzzy optimization theory. But, the condition of convex fuzzy mapping is very strict. So the area of its application is limited. Therefore, different types of convexity and generalized convexity of fuzzy mappings are defined and their properties are studied. Especially, in [12], Noor firstly introduced the concept of fuzzy preinvex functions over the fields $\mathbf{R}$, and obtained some basic properties of fuzzy preinvex functions. Subsequently, in [13–15], Syau introduced the concepts of all kinds of generalized convexity for fuzzy mappings of one variable such as invexity, pseudoinvexity, B-vexity etc. and also discussed many important properties of these generalized convex fuzzy mappings. Mishra et al. [16] introduced semistrictly preinvex fuzzy mappings. In [17,18], Wu and Xu introduce the concepts of fuzzy pseudoconvex, fuzzy invex, fuzzy pseudoinvex and fuzzy preinvex mapping from $\mathbf{R}^n$ to the set of fuzzy numbers, and obtained the relation between fuzzy vector optimization and fuzzy variational inequality.

Motivated by the recent work going on in these fields, on the basis of the concept of parameterized triples of fuzzy numbers, we discuss several important characterizations about preinvex fuzzy mapping. By making use of Weierstrass Theorem, we give a new proof of some known important conclusions. It is worth pointing out that the results obtained here generalize and improve the corresponding ones in the references.

☆ Project supported by the National Natural Science Foundation of China (No.10671207) and Chongqing Key Lab of Operations Research and System Engineering.

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doi:10.1016/j.camwa.2009.09.015

2. Preliminaries

In this section, We first quote some preliminary notations, definitions and results which will be needed in the sequel.

Definition 2.1. Let $\mathbf{R}$ denote the set of all real numbers. A fuzzy number is a mapping $\mu : \mathbf{R} \to [0, 1]$ with the following properties:

(1) μ is normal, that is, $[\mu]_1 = \{x \in \mathbf{R} : \mu(x) = 1\} \neq \emptyset$,
(2) μ is upper semicontinuous,
(3) μ is convex, that is, $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in \mathbf{R}$, $\lambda \in [0, 1]$,
(4) the support of μ, $\mathrm{supp}(\mu) = \{x \in \mathbf{R} : \mu(x) > 0\}$ and its closure $\mathrm{cl}(\mathrm{supp}\,\mu)$ is compact.

Denote by $\mathcal{F}$ the set of all fuzzy numbers on $\mathbf{R}$. Let $\alpha \in [0, 1]$, the α-level set of a fuzzy set $\mu : \mathbf{R} \to [0, 1]$, denoted by $[\mu]_\alpha$, is defined as

$$[\mu]_\alpha = \begin{cases} \{x \in \mathbf{R} : \mu(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1; \\ \mathrm{cl}(\mathrm{supp}(\mu)), & \text{if } \alpha = 0. \end{cases}$$

It is clear that the α-level set of a fuzzy number is a closed and bounded interval $[\mu_*(\alpha), \mu^*(\alpha)]$, where $\mu_*(\alpha)$ denotes the left-hand end point of $[\mu]_\alpha$ and $\mu^*(\alpha)$ denotes the right-hand endpoint of $[\mu]_\alpha$.

Also any $r \in \mathbf{R}^1$ can be considered as a fuzzy number $\tilde{r}$ defined by as

$$\tilde{r}(x) = \begin{cases} 1, & x = r, \\ 0, & x \neq r. \end{cases}$$

Thus a fuzzy number μ can be identified by a parameterized triples

$$\{(\mu_*(\alpha), \mu^*(\alpha), \alpha) : \alpha \in [0, 1]\}.$$

This leads to the following characterization of a fuzzy number in terms of the two end point functions $\mu_*(\alpha)$ and $\mu^*(\alpha)$.

Theorem 2.1 (*See [2]*). *Assume that $\mu_*(\alpha) : [0, 1] \to \mathbf{R}$ and $\mu^*(\alpha) : [0, 1] \to \mathbf{R}$ satisfy the following conditions:*

(1) *$\mu_*(\alpha) : [0, 1] \to \mathbf{R}$ is a nondecreasing function,*
(2) *$\mu^*(\alpha) : [0, 1] \to \mathbf{R}$ is a nonincreasing function,*
(3) *$\mu_*(1) \leq \mu^*(1)$,*
(4) *$\mu_*(\alpha)$ and $\mu^*(\alpha)$ are bounded and left continuous on $(0, 1]$, and right continuous at $\alpha = 0$.*

Then $\mu : \mathbf{R} \to [0, 1]$ defined by

$$\mu(x) = \sup\{\alpha : \mu_*(\alpha) \leq x \leq \mu^*(\alpha)\}$$

is a fuzzy number with parameterization given by $\{(\mu_*(\alpha), \mu^*(\alpha), \alpha) : \alpha \in [0, 1]\}$. Moreover, if $\mu : \mathbf{R} \to [0, 1]$ is a fuzzy number with parameterization given by $\{(\mu_*(\alpha), \mu^*(\alpha), \alpha) : \alpha \in [0, 1]\}$, then functions $\mu_*(\alpha)$ and $\mu^*(\alpha)$ satisfy conditions (1)–(4).

Definition 2.2. Let $\mu, \nu \in \mathcal{F}$ represented parametrically by

$$\{(\mu_*(\alpha), \mu^*(\alpha), \alpha) : \alpha \in [0, 1]\} \quad \text{and} \quad \{(\nu_*(\alpha), \nu^*(\alpha), \alpha) : \alpha \in [0, 1]\},$$

respectively. We say that $\mu \preceq \nu$, if for each $\alpha \in [0, 1]$, $\mu_*(\alpha) \leq \nu_*(\alpha)$ and $\mu^*(\alpha) \leq \nu^*(\alpha)$. If $\mu \preceq \nu$ and $\nu \preceq \mu$, then $\mu = \nu$. We say that $\mu \prec \nu$, if $\mu \preceq \nu$ and there exists $\alpha_0 \in [0, 1]$ such that $\mu_*(\alpha_0) < \nu_*(\alpha_0)$ or $\mu^*(\alpha_0) < \nu^*(\alpha_0)$.

Note that $\preceq$ is a partial order on $\mathcal{F}$. Sometimes we may write $\nu \succeq \mu$ instead of $\mu \preceq \nu$.

For fuzzy numbers μ and ν parameterized by

$$\{(\mu_*(\alpha), \mu^*(\alpha), \alpha) : \alpha \in [0, 1]\} \quad \text{and} \quad \{(\nu_*(\alpha), \nu^*(\alpha), \alpha) : \alpha \in [0, 1]\},$$

respectively, and each nonnegative real number k, we define the addition $\mu \tilde{+} \nu$ and nonnegative scalar multiplication $k\mu$ as follows:

$$\mu \tilde{+} \nu = \{(\mu_*(\alpha) + \nu_*(\alpha), \mu^*(\alpha) + \nu^*(\alpha), \alpha) : \alpha \in [0, 1]\}, \tag{2.1}$$

$$k\mu = \{(k\mu_*(\alpha), k\mu^*(\alpha), \alpha) : \alpha \in [0, 1]\}. \tag{2.2}$$

It is known that (see [2]) the above defined addition and nonnegative scalar multiplication on $\mathcal{F}$ are equivalent to those derived from the usual extension principle, and that $\mathcal{F}$ is closed under the addition and nonnegative scalar multiplication. Obviously, for each real number r,

$$\mu \tilde{+} r = \{(\mu_*(\alpha) + r, \mu^*(\alpha) + r, \alpha) : \alpha \in [0, 1]\}. \tag{2.3}$$

Definition 2.3. Let $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$ and $A = (a_1, a_2, \dots, a_n) \in \mathcal{F}^n$ be an n-dimensional real vector and an n-dimensional fuzzy vector, respectively. We define the product of a fuzzy vector with a real vector as $Ax^{\mathrm{T}} = \sum_{i=1}^{n} a_i x_i$, which is a fuzzy number.

Definition 2.4. Let $A = (a_{ij})$ be an $n \times n$ fuzzy matrix in which the entries $a_{ij}, i, j = 1, 2, \dots n$ are all fuzzy numbers. A is said to be a positive definite fuzzy matrix, if for each $\alpha \in [0, 1]$, the left-hand and the right-hand α-level matrices (a_{ij*}) and $(a_{ij}{}^*)$ (where $a_{ij} = \{(a_{ij*}, a_{ij}{}^*, \alpha) : \alpha \in [0, 1]\}$), are all positive definite. Similarly, A is said to be a positive semi-definite fuzzy matrix, if for each $\alpha \in [0, 1]$, the left-hand and the right-hand α-level matrices (a_{ij*}) and $(a_{ij}{}^*)$ (where $a_{ij} = \{(a_{ij*}, a_{ij}{}^*, \alpha) : \alpha \in [0, 1]\}$) are all positive semi-definite.

In this paper, a mapping $F : K \subset \mathbf{R}^n \to \mathcal{F}$ is said to be a fuzzy mapping. For any $\alpha \in [0, 1]$, the α-cut of F denoted by $F(x)[\alpha] = [F_*(\alpha, x), F^*(\alpha, x)]$.

Below we have accepted the fuzzy differentiability concept due to Buckley–Feuring [19,20], see also [7].

Definition 2.5 (*See [7]*). Let $F : K \to \mathcal{F}$ be a fuzzy mapping, where $K \subset \mathbf{R}^n$ is an open set. Let $x = (x_1, x_2, \dots, x_n) \in K$. Let $D_{x_i}, i = 1, 2, \dots, n$, stand for the "partial differentiation" with respect to the ith variable x_i. Assume that for all $\alpha \in [0, 1]$, $F_*(\alpha, x)$, $F^*(\alpha, x)$ have continuous partial derivatives so that $D_{x_i}F_*(\alpha, x)$, $D_{x_i}F^*(\alpha, x)$ are continuous. Define

$$D_{x_i}F(x)[\alpha] = [D_{x_i}F_*(\alpha, x), D_{x_i}F^*(\alpha, x)], \quad \text{for } i = 1, 2, \dots, n,\ \alpha \in [0, 1].$$

If for each $i, j = 1, 2, \dots, n$, $D_{x_i}F(x)[\alpha]$ defines the α-cut of a fuzzy number, then we will say that F is differentiable at x, and we write

$$\tilde{\nabla}F(x) = (D_{x_1}F(x), D_{x_2}F(x), \dots, D_{x_n}F(x)).$$

We call $\tilde{\nabla}F(x)$, the gradient of the fuzzy function F at x.

Note that $\tilde{\nabla}F(x)$ is an n-dimensional fuzzy vector. For the gradient of a fuzzy mapping we use the symbol $\tilde{\nabla}$, whereas for the gradient of a real-valued function we use the symbol ∇.

Definition 2.6 (*See [7]*). Let $F : K \to \mathcal{F}$ be a fuzzy mapping, where $K \subset \mathbf{R}^n$ is an open set. Let $x = (x_1, x_2, \dots, x_n) \in K$. Let $D_{x_i x_j}, i, j = 1, 2, \dots, n$, stand for the "second-order partial" with respect to the ith variable x_i and jth variable x_j. Assume that $\tilde{\nabla}F(x)$ exists and for all $\alpha \in [0, 1]$, $F_*(\alpha, x)$, $F^*(\alpha, x)$ have continuous second-order partial derivatives so that $D_{x_i x_j}F_*(\alpha, x)$, $D_{x_i x_j}F^*(\alpha, x)$ are continuous. Define

$$D_{x_i x_j}F(x)[\alpha] = [D_{x_i x_j}F_*(\alpha, x), D_{x_i x_j}F^*(\alpha, x)], \quad \text{for } i, j = 1, 2, \dots, n,\ \alpha \in [0, 1].$$

If for each $i, j = 1, 2, \dots, n$, $D_{x_i x_j}F(x)[\alpha]$ defines the α-cut of a fuzzy number, then we define the Hessian of the fuzzy function (in the matrix notation) as follows:

$$\tilde{\nabla}^2 F(x) = (D_{x_i x_j}F(x))_{i,j=1,2,\dots,n}.$$

We will say that F is twice differentiable at x, if the Hessian of the fuzzy function exists and both $F_*(\alpha, x)$, $F^*(\alpha, x)$ are twice differentiable at x.

Definition 2.7 (*See [9]*). Let $F : K \subset \mathbf{R}^n \to \mathcal{F}$ be a fuzzy mapping parameterized by

$$F(x) = \{(F_*(\alpha, x), F^*(\alpha, x), \alpha) : \alpha \in [0, 1]\}, \quad \forall x \in K.$$

(1) F is said to be upper semicontinuous at $x_0 \in K$, if both $F_*(\alpha, x)$ and $F^*(\alpha, x)$ are upper semicontinuous at x_0 uniformly in $\alpha \in [0, 1]$. F is upper semicontinuous on K, if it is upper semicontinuous at each point of K.

(2) F is said to be lower semicontinuous at $x_0 \in K$, if both $F_*(\alpha, x)$ and $F^*(\alpha, x)$ are lower semicontinuous at x_0 uniformly in $\alpha \in [0, 1]$. F is lower semicontinuous on K, if it is lower semicontinuous at each point of K.

Note that the concept of upper and lower semicontinuity introduced here is essentially same as that given by Bao and Wu [10].

Theorem 2.2 (*See [21]*). *Let X be a nonempty closed and bounded subset of $\mathbf{R}^n$. A real-valued function $f : \mathbf{R}^n \to \mathbf{R}^1$ is upper (respectively, lower) semicontinuous on X attains its maximum (respectively, minimum) on X.*

3. Some characterizations of preinvex fuzzy mapping

In this section, we will discussed several new characterizations of preinvex fuzzy mapping in view of complex analysis.

Definition 3.1. Let $y \in K(\subset \mathbf{R}^n)$, we say K is invex at y with respect to (w.r.t. in short) a function $\eta : K \times K \longrightarrow \mathbf{R}^n$, if for each $x \in K$, $\lambda \in [0, 1]$, $y + \lambda\eta(x, y) \in K$.$K$ is said to be an invex set w.r.t.η, if K is invex at each $y \in K$.

Definition 3.2. Let $K(\subset \mathbf{R}^n)$ be an invex set w.r.t. η. A fuzzy mapping $F : K \to \mathcal{F}$ is said to be preinvex on K, if $F(y+\lambda\eta(x,y)) \preceq \lambda F(x) \tilde{+} (1-\lambda)F(y)$ holds for all $x, y \in K$ and any $\lambda \in [0,1]$.

It is obvious that preinvexity implies convexity, but that the converse is not true (see [17,18]). In the case when $\eta(x,y) = x - y$, we obtain the definition of convex fuzzy mapping (see [4]).

To prove some results in the paper, we need the well-known Condition C introduced by Mohan and Neogy in [22] and developed in [16–18].

Condition C: We say that the function η: $\mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ satisfies Condition C, if for any $x, y \in \mathbf{R}^n$,

$$\eta(y, y+\lambda\eta(x,y)) = -\lambda\eta(x,y), \qquad \eta(x, y+\lambda\eta(x,y)) = (1-\lambda)\eta(x,y),$$

are satisfied for any $\lambda \in [0,1]$.

For example, let $K = \mathbf{R} - \{0\}$ and

$$\eta(x,y) = \begin{cases} x - y, & \text{if } xy \geq 0, \\ -y, & \text{if } xy < 0. \end{cases}$$

It is clear that K is an invex set (not a convex set) and that η satisfies Condition C. In fact, there exist many vector functions that satisfy Condition C.

Lemma 3.1. *Let K be a nonempty invex set in $\mathbf{R}^n$ w.r.t. η, and let $F : K \to \mathcal{F}$ be a fuzzy mapping parameterized by*

$$F(x) = \{(F_*(\alpha,x), F^*(\alpha,x), \alpha) : \alpha \in [0,1]\}, \quad \forall x \in K. \tag{3.1}$$

Then F is preinvex on K if and only if for each $\alpha \in [0,1]$,

$$F_*(\alpha,x) \quad \text{and} \quad F^*(\alpha,x) \text{ are preinvex w.r.t } \eta \text{ on } K. \tag{3.2}$$

Proof. Assume that for each $\alpha \in [0,1]$, $F_*(\alpha,x)$ and $F^*(\alpha,x)$ are preinvex w.r.t η on K. Let $\alpha \in [0,1]$ be given. From (3.2), we have

$$F_*(\alpha, y+\lambda\eta(x,y)) \leq \lambda F_*(\alpha,x) + (1-\lambda)F_*(\alpha,y)$$

and

$$F^*(\alpha, y+\lambda\eta(x,y)) \leq \lambda F^*(\alpha,x) + (1-\lambda)F^*(\alpha,y)$$

for all $x, y \in K$ and $\lambda \in [0,1]$. Then, by (3.1), (2.1) and (2.2), we obtain

$$\begin{aligned} F(y+\lambda\eta(x,y)) &= \{(F_*(\alpha, y+\lambda\eta(x,y)), F^*(\alpha, y+\lambda\eta(x,y)), \alpha) : \alpha \in [0,1]\} \\ &\preceq \{(\lambda F_*(\alpha,x), \lambda F^*(\alpha,x), \alpha) : \alpha \in [0,1]\} \tilde{+} \{((1-\lambda)F_*(\alpha,y), (1-\lambda)F^*(\alpha,y), \alpha) : \alpha \in [0,1]\} \\ &= \lambda F(x) \tilde{+} (1-\lambda)F(y) \end{aligned}$$

for all $x, y \in K$ and $\lambda \in [0,1]$. Hence F is preinvex on K.

Conversely, let F is preinvex on K. Then for all $x, y \in K$ and $\lambda \in [0,1]$, we have $F(y+\lambda\eta(x,y)) \preceq \lambda F(x) \tilde{+} (1-\lambda)F(y)$. From (3.1), we have

$$F(y+\lambda\eta(x,y)) = \{(F_*(\alpha, y+\lambda\eta(x,y)), F^*(\alpha, y+\lambda\eta(x,y)), \alpha) : \alpha \in [0,1]\}, \tag{3.3}$$

for all $x, y \in K$ and $\lambda \in [0,1]$. From (3.1), (2.1) and (2.2), we obtain

$$\lambda F(x) \tilde{+} (1-\lambda)F(y) = \{(\lambda F_*(\alpha,x) + (1-\lambda)F_*(\alpha,y), \lambda F^*(\alpha,x) + (1-\lambda)F^*(\alpha,y), \alpha) : \alpha \in [0,1]\},$$

for all $x, y \in K$ and $\lambda \in [0,1]$. Then, by (3.3) and the preinvexity of F, we have for every $x, y \in K$ and $\lambda \in [0,1]$,

$$F_*(\alpha, y+\lambda\eta(x,y)) \leq \lambda F_*(\alpha,x) + (1-\lambda)F_*(\alpha,y)$$

and

$$F^*(\alpha, y+\lambda\eta(x,y)) \leq \lambda F^*(\alpha,x) + (1-\lambda)F^*(\alpha,y)$$

for each $\alpha \in [0,1]$. This completes the proof. □

Remark 3.1. Let $\eta(x,y) = x - y$, Lemma 3.1 is reduced to Theorem 3.1 in [11].

In [6, Theorem 4.6], Yan and Xu gave the following result for convex fuzzy mapping.

Theorem A. *Let $f(x)$ define on a convex set $K \subset \mathbf{R}^n$. For any $x, y \in K$, let $\varphi(\lambda) = f(\lambda x + (1-\lambda)y)$, $\forall \lambda \in [0,1]$. Then $f(x)$ is convex on K if and only if $\varphi(\lambda)$ is convex on* $[0,1]$.

In the following, we shall generalize the above Theorem A to the case of preinvex fuzzy mapping and establish an equivalent condition for preinvex fuzzy mapping under certain assumptions. From the blow Theorem 3.1, we can see that the preinvexity can be characterized by convexity of a fuzzy function defined on [0, 1] equivalently.

Theorem 3.1. *Let K be a nonempty invex set in $\mathbf{R}^n$ w.r.t. η where η satisfies Condition C. Assume that $F : K \to \mathcal{F}$ is a fuzzy mapping and satisfies $F(y + \eta(x, y)) \preceq F(x)$ for all $x, y \in K$. Then F is a preinvex on K if and only if, for any $x, y \in K$, $\varphi(\lambda) = F(y + \lambda\eta(x, y))$ is convex on [0,1].*

Proof. Suppose that F is preinvex on K, for arbitrary fixed $x, y \in K$ and $\forall\lambda, \alpha_1, \alpha_2 \in [0, 1]$, if $\alpha_1 = \alpha_2$, then the result is obvious.

If $\alpha_1 > \alpha_2$, then $\alpha_1 - \alpha_2 > 0$ and $\alpha_2 \neq 1$, thus we have

$$0 < (\alpha_1 - \alpha_2)/(1 - \alpha_2) \leq 1.$$

From Condition C, for $\forall x, y \in K, \alpha \in [0, 1]$, we have

$$\begin{aligned}\eta(y + \alpha\eta(x, y), y) &= \eta(y + \alpha\eta(x, y), y + \alpha\eta(x, y) - \alpha\eta(x, y)) \\ &= \eta(y + \alpha\eta(x, y), y + \alpha\eta(x, y) + \eta(y, y + \alpha\eta(x, y))) \\ &= -\eta(y, y + \alpha\eta(x, y)) = \alpha\eta(x, y),\end{aligned} \tag{3.4}$$

again from Condition C and (3.4), we have

$$\begin{aligned}\eta(y + \alpha_1\eta(x, y), y + \alpha_2\eta(x, y)) &= \eta(y + \alpha_2\eta(x, y) + (\alpha_1 - \alpha_2)\eta(x, y), y + \alpha_2\eta(x, y)) \\ &= \eta(y + \alpha_2\eta(x, y) + [(\alpha_1 - \alpha_2)/(1 - \alpha_2)]\eta(x, y + \alpha_2\eta(x, y)), y + \alpha_2\eta(x, y)) \\ &= [(\alpha_1 - \alpha_2)/(1 - \alpha_2)]\eta(x, y + \alpha_2\eta(x, y)) = (\alpha_1 - \alpha_2)\eta(x, y).\end{aligned} \tag{3.5}$$

By (3.5) and the preinvexity of F, thus for arbitrary fixed $x, y \in K$,

$$\begin{aligned}\varphi(\alpha_2 + \lambda(\alpha_1 - \alpha_2)) &= F(y + \alpha_2\eta(x, y) + \lambda(\alpha_1 - \alpha_2)\eta(x, y)) \\ &= F(y + \alpha_2\eta(x, y) + \lambda\eta(y + \alpha_1\eta(x, y), y + \alpha_2\eta(x, y))) \\ &\preceq \lambda F(y + \alpha_1\eta(x, y))\tilde{+}(1 - \lambda)F(y + \alpha_2\eta(x, y)) \\ &= \lambda\varphi(\alpha_1)\tilde{+}(1 - \lambda)\varphi(\alpha_2).\end{aligned} \tag{3.6}$$

If $\alpha_1 < \alpha_2$, by a similar way, then we have

$$\varphi(\alpha_2 + \lambda(\alpha_1 - \alpha_2)) \preceq \lambda\varphi(\alpha_1)\tilde{+}(1 - \lambda)\varphi(\alpha_2). \tag{3.7}$$

Hence, by (3.6), (3.7) we conclude that $\varphi(\lambda)$ is a convex function on [0, 1].

Next, we prove the sufficiency of this theorem. Since $\varphi(\lambda) = F(y+\lambda\eta(x, y))$ is convex on [0,1], and $F(y+\eta(x, y)) \preceq F(x)$, then for any $\lambda \in [0, 1]$ and arbitrary fixed $x, y \in K$, we have

$$\begin{aligned}F(y + \lambda\eta(x, y)) &= \varphi(\lambda) = \varphi(\lambda \cdot 1 + (1 - \lambda) \cdot 0) \preceq \lambda\varphi(1)\tilde{+}(1 - \lambda)\varphi(0) \\ &= \lambda F(y + \eta(x, y))\tilde{+}(1 - \lambda)F(y) \preceq \lambda F(x)\tilde{+}(1 - \lambda)F(y).\end{aligned}$$

This completes the proof. □

In [7, Theorem 4.6], Panigrahi and Nanda gave the following result for convex fuzzy mapping.

Theorem B. *Let $\tilde{f}$ be a twice differentiable fuzzy mapping on an open convex set $\Omega \subseteq \mathbf{R}^n$ to $\mathcal{F}$. $\tilde{f}$ is convex on Ω if and only if for each $x \in \Omega$, $\tilde{\nabla}^2\tilde{f}$ is a positive semi-definite fuzzy matrix.*

Obviously, let $\tilde{f} : (a, b) \to \mathcal{F}$ be twice differentiable. Then $\tilde{f}$ is convex on (a,b) if and only if $\tilde{\nabla}^2\tilde{f}(x) \succeq \tilde{0}$ for any $x \in (a, b)$.

Next, we will establish the necessary and sufficient conditions for a twice differentiable preinvex fuzzy mapping by making use of Theorem 3.1.

Theorem 3.2. *Let K be a nonempty open invex set in $\mathbf{R}^n$ w.r.t. η where η satisfies Condition C. Assume that $F : K \to \mathcal{F}$ is a twice differentiable fuzzy mapping and satisfies $F(y + \eta(x, y)) \preceq F(x)$ for all $x, y \in K$. Then $F(x)$ is preinvex fuzzy mapping w.r.t. η if and only if, for any $x, y \in K$, $\eta(x, y)\tilde{\nabla}^2F(y)\eta(x, y)^{\mathrm{T}} \succeq \tilde{0}$.*

Proof. Necessity. Suppose that $F(x)$ is twice differentiable preinvex fuzzy mapping w.r.t. η. From Theorem 3.1, for any $x, y \in K$, $\varphi(\lambda) = F(y + \lambda\eta(x, y))$ is twice differentiable convex on $(0, 1)$. By Theorem B, we have for any $\lambda \in (0, 1)$, $\tilde{\nabla}^2\varphi(\lambda) \succeq \tilde{0}$. Since

$$\begin{aligned}&\tilde{\nabla}\varphi(\lambda) = \tilde{\nabla}F(y + \lambda\eta(x, y))\eta(x, y)^{\mathrm{T}}, \\ &\tilde{\nabla}^2\varphi(\lambda) = \eta(x, y)\tilde{\nabla}^2F(y + \lambda\eta(x, y))\eta(x, y)^{\mathrm{T}},\end{aligned}$$

thus, we can obtain

$$\tilde{\nabla}^2\varphi(\lambda) = \eta(x, y)\tilde{\nabla}^2 F(y + \lambda\eta(x, y))\eta(x, y)^{\mathrm{T}} \succeq \tilde{0}. \tag{3.8}$$

Let $\lambda \to 0^+$ in (3.8), then

$$\eta(x, y)\tilde{\nabla}^2 F(y)\eta(x, y)^{\mathrm{T}} \succeq \tilde{0}.$$

Sufficiency. For any $x, y \in K$, $\eta(x, y)\tilde{\nabla}^2 F(y)\eta(x, y)^{\mathrm{T}} \succeq \tilde{0}$. Then, for $\forall\lambda \in (0, 1)$, $y + \lambda\eta(x, y) \in K$ and

$$\eta(x, y + \lambda\eta(x, y))\tilde{\nabla}^2 F(y + \lambda\eta(x, y))\eta(x, y + \lambda\eta(x, y))^{\mathrm{T}} \succeq \tilde{0}.$$

From Condition C, thus

$$(1 - \lambda)^2\eta(x, y)\tilde{\nabla}^2 F(y + \lambda\eta(x, y))\eta(x, y)^{\mathrm{T}} \succeq \tilde{0}.$$

This means, since $\lambda \in (0, 1)$,

$$\eta(x, y)\tilde{\nabla}^2 F(y + \lambda\eta(x, y))\eta(x, y)^{\mathrm{T}} \succeq \tilde{0}.$$

Hence we can get $\tilde{\nabla}^2\varphi(\lambda) \succeq \tilde{0}$. Again by Theorem B and Theorem 3.1, then $F(x)$ is preinvex w.r.t. η. □

Theorem 3.3. *Let K be a nonempty open invex set in $\mathbf{R}^n$ w.r.t. η that which satisfy Condition C. Assume that $F : K \to \mathcal{F}$ is differentiable on K. Then $F(x)$ is preinvex on K if and only if, for any $x, y \in K$,*

$$F(x) \succeq \tilde{\nabla}F(y)\eta(x, y)^{\mathrm{T}}\tilde{+}F(y). \tag{3.9}$$

Proof. Assume that F is differentiable and preinvex on K. By Lemma 3.1, for every $x, y \in K$ and $\lambda \in [0, 1]$, then

$$F_*(\alpha, y + \lambda\eta(x, y)) \le \lambda F_*(\alpha, x) + (1 - \lambda)F_*(\alpha, y), \tag{3.10}$$

$$F^*(\alpha, y + \lambda\eta(x, y)) \le \lambda F^*(\alpha, x) + (1 - \lambda)F^*(\alpha, y), \tag{3.11}$$

for each $\alpha \in [0, 1]$. According to the mean-valued theorem, for each fixed $\alpha \in [0, 1]$, we have

$$F_*(\alpha, y + \lambda\eta(x, y)) = F_*(\alpha, y) + \lambda\nabla F_*(\alpha, \xi)\eta(x, y)^{\mathrm{T}}, \tag{3.12}$$

$$F^*(\alpha, y + \lambda\eta(x, y)) = F^*(\alpha, y) + \lambda\nabla F^*(\alpha, \zeta)\eta(x, y)^{\mathrm{T}}, \tag{3.13}$$

where $\xi = y + \theta_1\lambda\eta(x, y)$, $\zeta = y + \theta_2\lambda\eta(x, y)$, $0 < \theta_1, \theta_2 < 1$. From (3.10), (3.12), and taking the limit as $\theta_1 \to 0^+$, we can obtain

$$F_*(\alpha, x) \ge \nabla F_*(\alpha, y)\eta(x, y)^{\mathrm{T}} + F_*(\alpha, y).$$

By the same way as $\theta_2 \to 0^+$ gives

$$F^*(\alpha, x) \ge \nabla F^*(\alpha, y)\eta(x, y)^{\mathrm{T}} + F^*(\alpha, y).$$

From the above two inequalities, the desired conclusion (3.9) is obtained.

Conversely, for $x, y \in K$, $\lambda \in (0, 1)$, let $z = y + \lambda\eta(x, y)$, then $z \in K$. By the assumption and Condition C, we have for any $x, y \in K$,

$$F(x) \succeq \tilde{\nabla}F(z)\eta(x, z)^{\mathrm{T}}\tilde{+}F(z) = (1 - \lambda)\tilde{\nabla}F(z)\eta(x, y)^{\mathrm{T}}\tilde{+}F(z) \tag{3.14}$$

and

$$F(y) \succeq \tilde{\nabla}F(z)\eta(y, z)^{\mathrm{T}}\tilde{+}F(z) = -\lambda\tilde{\nabla}F(z)\eta(x, y)^{\mathrm{T}}\tilde{+}F(z). \tag{3.15}$$

Multiplying inequality (3.14) by λ plus multiplying inequality (3.15) by $1 - \lambda$, we can get for $x, y \in K$,

$$\lambda F(x)\tilde{+}(1 - \lambda)F(y) \succeq F(y + \lambda\eta(x, y)),$$

then F is a preinvex fuzzy mapping on K. □

Corollary 3.1. *Let K be a nonempty open invex set in $\mathbf{R}^n$ w.r.t. η that which satisfy Condition C. Assume that $F : K \to \mathcal{F}$ is differentiable on K. Then $F(x)$ is preinvex on K if and only if, for any $x, y \in K$,*

$$\tilde{\nabla}F(y)\eta(x, y)^{\mathrm{T}}\tilde{+}\tilde{\nabla}F(x)\eta(y, x)^{\mathrm{T}} \preceq \tilde{0}.$$

Corollary 3.2. *Let K be a nonempty open invex set in $\mathbf{R}^n$ w.r.t. η and $x_0 \in K$. Assume that $F : K \to \mathcal{F}$ is a differentiable preinvex fuzzy mapping on K. If for $\forall x \in K$, $\tilde{\nabla}F(x_0)\eta(x, x_0)^{\mathrm{T}} \succeq \tilde{0}$, then $F(x)$ achieves its global minimum at x_0.*

Remark 3.2. Corollaries 3.1 and 3.2 directly follow from Theorem 3.3. Let $\eta(x, y) = x - y$, Corollary 3.1 is reduced to Theorem 4.4 in [7] and Corollary 3.2 is reduced to Theorem 4.6 in [14].

4. Semicontinuity and preinvex fuzzy mapping

In [18, Theorem 3.1], Wu and Xu proved the following important results which revealed the relationships between semicontinuity and preinvexity of fuzzy mappings.

Theorem 4.1. *Let $K \subset \mathbf{R}^n$ be an open invex set w.r.t. η that which satisfy Condition C. Let $F : K \to \mathcal{F}$ be an upper semicontinuous fuzzy mapping that satisfies $F(y + \eta(x, y)) \preceq F(x)$, $\forall x, y \in K$. If there exists a $t \in (0, 1)$ such that*

$$F(y + t\eta(x, y)) \preceq tF(x)\tilde{+}(1 - t)F(y), \quad \forall x, y \in K,$$

then F is a preinvex fuzzy mapping on K.

Based on the established conclusions, we shall omit the condition that K is open set on the above Theorem 4.1 and obtain an improved conclusion in view of Theorem 2.2 and Lemma 3.1 in the following statements.

Theorem 4.2. *Let $K \subset \mathbf{R}^n$ be an invex set w.r.t. η that which satisfy Condition C. Let $F : K \to \mathcal{F}$ be an upper semicontinuous fuzzy mapping that satisfies $F(y + \eta(x, y)) \preceq F(x)$, $\forall x, y \in K$. If there exists a $t \in (0, 1)$ such that*

$$F(y + t\eta(x, y)) \preceq tF(x)\tilde{+}(1 - t)F(y), \quad \forall x, y \in K,$$

then F is a preinvex fuzzy mapping on K.

Proof. By contradiction, suppose that F is not preinvex on K. Hence, there exist $x, y \in K$ and $\bar{\lambda} \in (0, 1)$ such that

$$F(y + \bar{\lambda}\eta(x, y)) \npreceq \bar{\lambda}F(x)\tilde{+}(1 - \bar{\lambda})F(y),$$

i.e., there exist $x, y \in K$ and $\bar{\lambda} \in (0, 1)$, for some $\alpha_0 \in [0, 1]$ such that

$$F_*(\alpha_0, y + \bar{\lambda}\eta(x, y)) > \bar{\lambda}F_*(\alpha_0, x) + (1 - \bar{\lambda})F_*(\alpha_0, y), \tag{4.1}$$

or

$$F^*(\alpha_0, y + \bar{\lambda}\eta(x, y)) > \bar{\lambda}F^*(\alpha_0, x) + (1 - \bar{\lambda})F^*(\alpha_0, y). \tag{4.2}$$

Without loss of generality, suppose that (4.1) holds. Now let

$$g(\lambda) = F_*(\alpha_0, y + \lambda\eta(x, y)) - \lambda F_*(\alpha_0, x) - (1 - \lambda)F_*(\alpha_0, y).$$

By the upper semicontinuity of F and Definition 2.7, then $F_*(\alpha, x)$ is upper semicontinuous real-valued function. Therefore, $g(\lambda)$ also is upper semicontinuous real-valued function in interval $[0, 1]$. From Theorem 2.2, it follows that $g(\lambda)$ exists maximum $M_0 > 0$ due to (4.1), in interval [0,1]. Let $\lambda_0 = \max\{\lambda \in [0, 1] : g(\lambda) = M_0\}$. By the conditions, we have

$$g(0) = 0, \quad g(1) = F_*(\alpha_0, y + \eta(x, y)) - F_*(\alpha_0, x) \leq 0.$$

Hence $\lambda_0 \in (0, 1)$. Choose a $\delta > 0$, such that

$$(\lambda_0 - (1 - t)\delta, \lambda_0 + t\delta) \subset (0, 1).$$

Let

$$\lambda_2 = \lambda_0 - (1 - t)\delta, \qquad \lambda_1 = \lambda_0 + t\delta.$$

Obviously, $\lambda_1 = \lambda_2 + \delta$ and $\lambda_1 \neq \lambda_0$, $\lambda_2 \neq \lambda_0$. Let

$$\hat{x} = y + \lambda_2\eta(x, y), \qquad \hat{y} = y + \lambda_1\eta(x, y).$$

By the Condition C, we can get that

$$\begin{aligned}
\hat{y} + t\eta(\hat{x}, \hat{y}) &= y + \lambda_1\eta(x, y) + t\eta(y + \lambda_2\eta(x, y), y + \lambda_1\eta(x, y)) \\
&= y + \lambda_1\eta(x, y) + t(\lambda_2 - \lambda_1)\eta(x, y) \\
&= y + (\lambda_0 + t\delta)\eta(x, y) - t\delta\eta(x, y) \\
&= y + \lambda_0\eta(x, y).
\end{aligned}$$

From Lemma 3.1, we known that $F_*(\alpha, x)$ is preinvex on K. Thus,

$$\begin{aligned}
M_0 = g(\lambda_0) &= F_*(\alpha_0, y + \lambda_0\eta(x, y)) - \lambda_0 F_*(\alpha_0, x) - (1 - \lambda_0)F_*(\alpha_0, y) \\
&= F_*(\alpha_0, \hat{y} + t\eta(\hat{x}, \hat{y})) - \lambda_0 F_*(\alpha_0, x) - (1 - \lambda_0)F_*(\alpha_0, y) \\
&\leq tF_*(\alpha_0, \hat{x}) + (1 - t)F_*(\alpha_0, \hat{y}) - \lambda_0 F_*(\alpha_0, x) - (1 - \lambda_0)F_*(\alpha_0, y) \\
&= t[F_*(\alpha_0, \hat{x}) - \lambda_2 F_*(\alpha_0, x) - (1 - \lambda_2)F_*(\alpha_0, y)] \\
&\quad + (1 - t)[F_*(\alpha_0, \hat{y}) - \lambda_1 F_*(\alpha_0, x) - (1 - \lambda_1)F_*(\alpha_0, y)]
\end{aligned}$$

$$\begin{aligned} &= t[F_*(\alpha_0, y + \lambda_2\eta(x, y)) - \lambda_2 F_*(\alpha_0, x) - (1 - \lambda_2)F_*(\alpha_0, y)] \\ &\quad + (1 - t)[F_*(\alpha_0, y + \lambda_1\eta(x, y)) - \lambda_1 F_*(\alpha_0, x) - (1 - \lambda_1)F_*(\alpha_0, y)] \\ &= tg(\lambda_2) + (1 - t)g(\lambda_1) < tM_0 + (1 - t)M_0 = M_0. \end{aligned}$$

This is a contradiction. Hence F is preinvex on K w.r.t. η. $\square$

Remark 4.1. Comparing with Theorem 4.1 given by Wu and Xu in [18], we omit the hypothesis on K being an open set in our theorem. It is easy to see that there is no proof on the density for the set $A = \{\lambda \in [0, 1] : F(y + t\eta(x, y)) \preceq tF(x)\tilde{+}(1 - t)F(y), \forall x, y \in K\}$, which lead to a completely different proof with [18].

By a similar way, using Theorem 2.2 and Lemma 3.1, we can conclude the following Theorem 4.3.

Theorem 4.3. *Let $K \subset \mathbf{R}^n$ be an invex set w.r.t. η that which satisfy Condition C. Let $F : K \to \mathcal{F}$ be an lower semicontinuous fuzzy mapping that satisfies $F(y + \eta(x, y)) \preceq F(x), \forall x, y \in K$. If there exists a $t \in (0, 1)$ such that*

$$F(y + t\eta(x, y)) \preceq tF(x)\tilde{+}(1 - t)F(y), \quad x, y \in K,$$

then F is a preinvex fuzzy mapping on K.

5. Conclusion

The concept of generalized convex fuzzy mappings without differentiability has been discussed in the literature by many researchers. The objective of this paper is to give some important properties of preinvex fuzzy mappings under differentiability along convex analysis. This paper also provides a new proof of some known important conclusions, which is different from the work in [18]. Furthermore, the properties obtained here for preinvex fuzzy mapping can be derived for prequasi-invex fuzzy mapping by a similar way. Also we will try to explore these characterizations of preinvex fuzzy mappings to fuzzy optimization in the future.

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