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Edge-disjoint spanners of complete bipartite graphs

Christian Laforest[a], Arthur L. Liestman[b,*,1], Thomas C. Shermer[b,1], Dominique Sotteau[c]

[a] *Laboratoire de Méthodes Informatiques, Université d'Evry, Cours Monseigneur Roméro, 91025 Evry Cedex, France*
[b] *School of Computing Science, Simon Fraser University, Burnaby, BC, Canada V5A 1S6*
[c] *Laboratoire de Recherche en Informatique, Bât. 490, Université de Paris-Sud, 91405 Orsay Cedex, France*

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Abstract

A spanning subgraph $S=(V,E')$ of a connected simple graph $G=(V,E)$ is an $(x+c)$-*spanner* if for any pair of vertices u and v, $d_S(u,v)\leqslant d_G(u,v)+c$ where d_G and d_S are the usual distance functions in graphs G and S, respectively. The parameter c is called the *delay* of the spanner. We investigate the number of edge-disjoint spanners of a given delay that can exist in complete bipartite graphs. We determine the exact number of such edge-disjoint spanners of delay 4 or larger. For delay 2, we obtain many exact values of and some general bounds on this number. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Interconnection networks (the topological structure of parallel and distributed systems) are generally modeled as graphs. Consequently, researchers have been investigating those structural properties of graphs that correspond to useful properties of interconnection networks. One such property that has recently been investigated is the existence of *spanners* in a graph.

A spanner is a spanning subgraph in which the distance between any pair of vertices approximates the distance in the original graph. They were introduced by Peleg and Ullman [16], who used them for efficient simulation of synchronous distributed systems on asynchronous ones. A first systematic presentation of spanners and their basic properties in a variety of graph classes was given by Peleg and Schäffer [15]. Spanners

* Corresponding author.
E-mail address: art@cs.sfu.ca (A.L. Liestman).
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were recognized early on as the underlying graph structure in various constructions in distributed systems and communication networks, such as locality-preserving covers, partitions, regional matchings and compact routing schemes [3], and were used for a variety of applications such as efficient broadcast [2] and network design [14]. Richards and Liestman [17] suggested the use of spanners as network topologies: if one has an expensive desired topology, often a sparse (and, therefore, less expensive) spanner can be substituted, retaining a similar network structure for only a slight increase in communication costs. In a series of papers, Liestman and Shermer [9–13] continued the study of spanners as network topologies, and introduced a more general definition of spanner. Heydemann et al. [5] also investigated spanners as network topologies. Kortsarz and Peleg [6,7] studied approximation techniques for spanner constructions. The most efficient algorithm for constructing sparse, light weight spanners for general graphs is presented and analyzed in [1,4].

In this paper we continue the investigation of edge-disjoint spanners of particular graphs which we began in [8]. One possible use of edge-disjoint spanners is to partition a parallel computer for several independent users or processes without significantly decreasing the performance of each 'virtual computer'; this would be most useful when the network is asynchronous (making timeslicing on the edges of the network difficult) or when the speed of communication is much lower than the speed of the processors. Another situation in which this type of partition could be useful for allowing several simultaneous independent processes on a network that uses wormhole routing — the communication paths set up by one process would not be interfered with by the operation of the other processes. In [8], we considered edge-disjoint spanners of complete graphs. Here, we show that one can still find a large number of good edge-disjoint spanners in a class of much sparser graphs.

2. Definitions and preliminary results

We characterize a spanner by its *delay*, a number which represents how closely it models its underlying graph. A network is represented by a connected simple graph G. We use $d_G(u,v)$ to denote the distance from vertex u to vertex v in graph G. In [10], Liestman and Shermer introduced a general definition of graph spanner: A spanning subgraph S of a connected simple graph G is an $f(x)$-*spanner* if for any pair of vertices u and v, $d_S(u,v) \leqslant f(d_G(u,v))$. We call $d_S(u,v) - d_G(u,v)$ the *delay between vertices u and v in S*. For an $f(x)$-spanner S, we refer to $f(x) - x$ as the *delay* of the spanner. Note that $f(x) - x$ is an upper bound (but not necessarily a tight bound) on the maximum delay in S between any pair of vertices at distance x in G. A path in S of length $d_S(u,v)$ between u and v is called a *replacement path*, because it replaces the path in G of length $d_G(u,v)$.

We are interested in constructing multiple edge-disjoint $(x+c)$-spanners of a graph G, for appropriate constants c. Let $\mathrm{EDS}(G,c)$ denote the maximum number of edge-disjoint $(x+c)$-spanners of G for $c \geqslant 0$. Note that for all graphs G, $\mathrm{EDS}(G,0) = 1$.

We use $K_{p,q}$ to denote the complete bipartite graph $G=(U,V,E)$ where $U=\{u_0,u_1,\dots,u_{p-1}\}$ and $V=\{v_0,v_1,\dots,v_{q-1}\}$ are the vertex sets and $E=U\times V$. Without loss of generality, we will assume that $p\leqslant q$. In these graphs, $d_G(u,v)=1$ for any pair of vertices u and v with $u\in U$ and $v\in V$ and $d_G(a,b)=2$ for any pair of vertices a,b which are both from the same vertex set.

We begin with a few simple observations:

Observation 2.1. $\mathrm{EDS}(K_{p,q},0)=\mathrm{EDS}(K_{p,q},1)=1$.

Observation 2.2. $\mathrm{EDS}(K_{p,q},2c)=\mathrm{EDS}(K_{p,q},2c+1)=1$ *for* $c\geqslant 0$.

Observation 2.3. $\mathrm{EDS}(K_{p,q},c)\geqslant\mathrm{EDS}(K_{p,q},c')$ *where* $0\leqslant c\leqslant c'$.

By *duplicating* a vertex x in a set of edge-disjoint spanners of $K_{p,q}$, we mean adding a new vertex x' whose neighbors in each of the spanners are the same as the neighbors of x. This idea leads us to the following two bounds.

Observation 2.4. $\mathrm{EDS}(K_{p+1,q},c)\geqslant\mathrm{EDS}(K_{p,q},c)$ *for* $c\geqslant 0$.

Observation 2.5. $\mathrm{EDS}(K_{p,q+1},c)\geqslant\mathrm{EDS}(K_{p,q},c)$ *for* $c\geqslant 0$.

Theorem 2.6. *For* $p\geqslant 2$ *and* $c\geqslant 2$, $\lfloor p/2\rfloor\leqslant\mathrm{EDS}(K_{p,q},c)\leqslant p-1$.

Proof. Every vertex must have degree at least one in each spanner. Since each vertex in V has degree p in $K_{p,q}$ there can be at most p edge-disjoint spanners. If there are p edge-disjoint spanners, then each vertex of V has degree exactly one in every spanner. Since $p\geqslant 2$, such a structure would not be connected. Thus, $\mathrm{EDS}(K_{p,q},c)\leqslant p-1$.

We will now show how to construct $\lfloor p/2\rfloor$ edge-disjoint $(x+2)$-spanners $S_0,S_1,\dots,S_{\lfloor p/2\rfloor-1}$ of $K_{p,p}$. We will refer to the vertices of U and V with even indices as *even* vertices and those with odd indices as *odd* vertices. As in many constructions in this paper, each spanner S_i has a *hub* — a central set of vertices through which many of the replacement paths pass. Typically, a replacement path from x to y will consist of three sections: a subpath from x to the hub, one within the hub, and one from the hub to y.

The hub of S_i consists of the vertices $u_{2i},u_{2i+1},v_{2i},v_{2i+1}$. Vertices with index smaller than $2i$ are said to be *above* the hub while those with index larger than $2i+1$ are said to be *below* the hub. This structure is illustrated in Fig. 1 which shows S_1 in $K_{8,8}$.

The four vertices of the hub are connected as a complete bipartite graph. Each hub vertex is also connected to all vertices of opposite parity above the hub and to all vertices of the same parity below the center.

More formally, the edges of S_i are of five types:

- $E_{\mathrm{hub}}(i)=\{[u_{2i},v_{2i}],[u_{2i},v_{2i+1}],[u_{2i+1},v_{2i}],[u_{2i+1},v_{2i+1}]\}$.
- $E_1(i)=\{[u_{2i},v_{2j+1}]:0\leqslant j<i\}\cup\{[u_{2i},v_{2j}]:i<j\leqslant\lfloor\frac{p}{2}\rfloor-1\}$.
- $E_2(i)=\{[u_{2i+1},v_{2j}]:0\leqslant j<i\}\cup\{[u_{2i+1},v_{2j+1}]:i<j\leqslant\lfloor\frac{p}{2}\rfloor-1\}$.

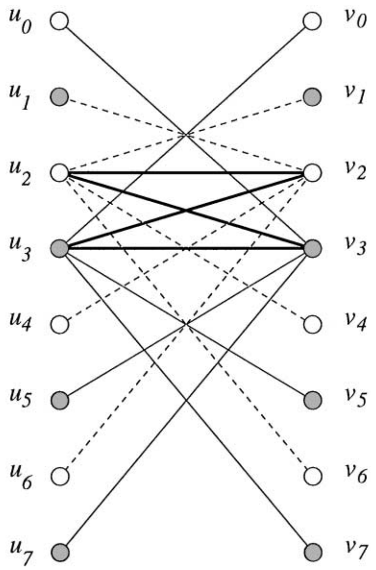

Fig. 1. spanner S_1 of $K_{8,8}$.

- $E_3(i) = \{[v_{2i}, u_{2j+1}]: 0 \leqslant j < i\} \cup \{[v_{2i}, u_{2j}]: i < j \leqslant \lfloor \frac{p}{2} \rfloor - 1\}$.
- $E_4(i) = \{[v_{2i+1}, u_{2j}]: 0 \leqslant j < i\} \cup \{[v_{2i+1}, u_{2j+1}]: i < j \leqslant \lfloor \frac{p}{2} \rfloor - 1\}$.

Consider two vertices a, b in S_i. If both are in U (or in V), then each is connected to a hub vertex in V (or U) and these two hub vertices are either identical or connected by a path of length two, giving a path of length at most four between a and b. If a is in U and b is in V, then a is adjacent to a hub vertex of V (possibly b) and b is adjacent to a hub vertex of U (possibly a) and these two hub vertices are also adjacent, giving a path of length at most three. Thus, the delay of S_i is at most two.

The spanners $S_0, S_1, \ldots, S_{\lfloor p/2 \rfloor - 1}$ are edge-disjoint. Consider two spanners S_i and S_j with $i < j$ and an even vertex u_{2k} of U. If k is neither i nor j, then u_{2k} is connected to a single hub vertex in S_i and in S_j. Since the hubs of these spanners are disjoint, there are no common edges on u_{2k} in these spanners. If $k = i$, then u_{2k} is a hub vertex in S_i and not a hub vertex in S_j. Since $i < j$, in S_j, vertex u_{2k} has exactly one edge, connecting to v_{2j+1}. However, in S_i, u_{2k} is connected only to even vertices of V below the hub. Thus, there are no common edges on u_{2k} in these spanners. A similar argument holds for $k = j$. We can repeat the above argument with suitable changes for odd vertices of U, and for both even and odd vertices of V.

Thus, $\mathrm{EDS}(K_{p,p}, 2) \geqslant \lfloor p/2 \rfloor$ and from Observation 2.3, $\mathrm{EDS}(K_{p,p}, c) \geqslant \lfloor p/2 \rfloor$, for $c \geqslant 2$. With Observation 2.5, we conclude that $\mathrm{EDS}(K_{p,p}, c) \geqslant \lfloor p/2 \rfloor$ for $c \geqslant 2$. □

Theorem 2.7. $\mathrm{EDS}(K_{p,p}, c) = \lfloor p/2 \rfloor$ *for* $p \geqslant 2$ *and* $c \geqslant 2$.

Proof. Theorem 2.6 shows that $\mathrm{EDS}(K_{p,p}, c) \geqslant \lfloor p/2 \rfloor$.

Since $K_{p,p}$ contains p^2 edges and any spanner of $K_{p,p}$ must contain at least $2p-1$ edges to be connected, there can be at most $\lfloor p^2/(2p-1)\rfloor$ edge-disjoint spanners of $K_{p,p}$. Since $p\geqslant 2, \lfloor p^2/(2p-1)\rfloor = \lfloor p/2\rfloor$. $\square$

At this point, we will continue the analysis in two separate sections. In the following section, we consider the case $c\geqslant 4$ and obtain exact values for all $\mathrm{EDS}(K_{p,q}, c)$. Following that, we consider the remaining case $c=2$ and obtain many exact values of and some general bounds on $\mathrm{EDS}(K_{p,q}, 2)$.

3. Delay at least four

In this section, we establish the value of $\mathrm{EDS}(K_{p,q}, c)$ for $c\geqslant 4$.

Theorem 3.1. *For* $c\geqslant 4$, $\mathrm{EDS}(K_{p,q}, c) = \lfloor pq/(p+q-1)\rfloor$.

Proof. $K_{p,q}$ has pq edges. Each spanner must be connected and thus contain at least $p+q-1$ edges. The upper bound follows.

We now show how to construct a set of $e=\lfloor pq/(p+q-1)\rfloor$ spanners of $K_{p,q}$ with delay 4. By Observation 2.3, this will establish the lower bound for all $c\geqslant 4$. If $e=1$, then let $K_{p,q}$ be the spanner. We now assume that $e\geqslant 2$. Each spanner will have the following structure: there is a distinguished vertex $u\in U$ which we will call the *hub* of the spanner. Every vertex in U is at distance at most two from u, and every vertex in V is connected to at least one vertex in U. Such a structure is necessarily an $(x+4)$-spanner of $K_{p,q}$, as there is a path of length at most 4 through u between any pair of vertices in U, a path of length at most 5 between any vertex in U and any vertex in V, and a path of length at most 6 between any pair of vertices in V.

Consider a vertex $v\in V$. Vertex v must have degree at least one in each spanner. Since v is of degree p in $K_{p,q}$, there are $p-e$ 'extra' edges allocated to v. These extra edges will be used in 0, 1, or 2 of the spanners. The vertex v is called a *connector* in spanner S if all of its extra edges are included in S, a *partial connector* in S if some of its extra edges are included in S, and a *leaf* in S otherwise.

We describe how to construct S_i for $0\leqslant i\leqslant e-1$. We choose u_i as the hub vertex in S_i.

If $i>0$ and a vertex v was a partial connector used to finish S_{i-1} then we include the edge from v to u_i. We also include k edges from v to the first k vertices of $U\backslash\{u_{i-1}, u_i\}$, where k is the number of extra edges remaining to be allocated to v.

We now repeat the following step. If there are at least $p-e$ vertices of U not connected to u_i by a path of length 2, we introduce a new connector vertex v connected to u_i and the first $p-e$ such vertices of U. If there are fewer than $p-e$ such vertices (but at least one of them), we finish the spanner by introducing a new partial connector vertex v connected to u_i and those vertices of U not yet connected to u_i by a path of length 2.

Let v be the partial connector used to finish S_{i-1}. Then, in S_{i-1}, v is connected to u_{i-1} and to the last k vertices u_j where $p-k \leqslant j \leqslant p-1$. Observe that $i-1 < e-1 \leqslant p-k$, so u_{i-1} is not among these k vertices. In S_i, v is connected to u_i and to some $(p-e)-k$ other vertices u_j where $0 \leqslant j \leqslant (p-e)-k+1$ (not including u_{i-1} or u_i). Since $p-k > e$ and $e \geqslant 2$, no edge from v is used in both S_i and S_{i-1}.

We now consider each vertex $v \in V$ in turn. If v was used as a connector or partial connector in k of the spanners ($k=1$ or 2), then v has at least $e-k$ edges remaining to be allocated. If v was a leaf, then it has p edges remaining to be allocated. We allocate one of these edges of each spanner in which v is neither a connector nor a partial connector.

This construction works provided that q is large enough that we can choose an unused vertex from V whenever we need a new connector or partial connector. In each spanner we have allocated $p-1$ 'extra' edges, one to each vertex of U except the hub. Over all spanners, we have used $(p-1)e$ extra edges. Each vertex used as a connector or partial connector in some spanner has all $p-e$ of its extra edges used, except perhaps for the partial connector that finishes S_e. Hence, the construction works if $q \geqslant \lceil (p-1)e/(p-e) \rceil$.

Since $e = \lfloor pq/(p+q-1) \rfloor$, we have $e \leqslant pq/(p+q-1)$, which implies $pe+qe-e \leqslant pq$ and thus $q \geqslant \lceil (pe-e)/(p-e) \rceil$. □

4. Delay 2

We now turn to the specific case of delay 2. We begin with two lemmas to establish a general upper bound on $\mathrm{EDS}(K_{p,q}, 2)$.

Lemma 4.1. *Let $\mathscr{S}$ be a set of edge-disjoint $(x+2)$-spanners of $K_{p,q}$. If some spanner in $\mathscr{S}$ is a tree, then $|\mathscr{S}| = 1$.*

Proof. Suppose that there is an $(x+2)$-spanner S of $K_{p,q}$ that is a tree. S must have a leaf $v \in V$ with neighbor $u \in U$. As the delay of S is at most 2, the eccentricity of v is either 3 or 4. If the eccentricity is 3, then u must be adjacent to all vertices of V in S leaving it no edges for a second spanner.

If the eccentricity of v is 4, the tree S must be composed of v, u, and three other sets of vertices D_2, D_3, D_4 where D_i represents the vertices at distance i from v in S. If D_2 contains only one vertex v' of V, then v' must be connected to all of the vertices of U leaving it no edges for a second spanner. If there are at least 2 vertices D_2, there must be vertices v' in D_2, u' in D_3, and v'' in D_4 such that (v', u') and (u', v'') are edges in S. If a second vertex in D_2 has a neighbor u'' in D_3, the distance between v'' and u'' in S must be 5, giving delay at least 4, a contradiction. Hence, no vertices of $D_2 \backslash \{v'\}$ have neighbors in D_3 and v' must be connected to all of the vertices of U leaving it no edges for a second spanner. □

Lemma 4.2. $\mathrm{EDS}(K_{p,q},2) \leqslant \lfloor pq/(p+q+1) \rfloor$ *for* $q > p \geqslant 4$.

Proof. Let $\mathscr{S}$ be a set of edge-disjoint $(x+2)$-spanners of $K_{p,q}$. If every spanner in $\mathscr{S}$ has at least $p+q+1$ edges then $|\mathscr{S}| \leqslant \lfloor pq/(p+q+1) \rfloor$. Otherwise, there is some spanner $S \in \mathscr{S}$ with at most $p+q$ edges. If S has $p+q-1$ edges, then by Lemma 4.1, $|\mathscr{S}|=1$. Therefore, we consider the remaining case that S has exactly $p+q$ edges, i.e., S contains exactly one cycle C. As $K_{p,q}$ is bipartite, C must be of even length and we denote the vertices of C as $u_1, v_1, u_2, v_2, \ldots, u_k, v_k$. Any vertex x not on the cycle has a unique closest vertex on the cycle, which we call the *root* of x. We examine several cases based on the value of k.

If $k \geqslant 5$, then C is a cycle of size at least 10. Such a cycle contains a pair of vertices a, b at distance at least 5. Due to the structure of S, the only paths between a and b are along the cycle. Thus, a and b are at distance at least 5 in S, and at most 2 in $K_{p,q}$, so S is not an $(x+2)$-spanner.

If $k=4$, then C is a cycle of size 8. Since $q \geqslant 5$, there is at least one vertex of S which is not on the cycle. Thus, there is a vertex $x \notin C$ that is adjacent to (without loss of generality) either u_1 or v_1 in S. If x is adjacent to v_1, then $d_S(x, v_3)=5$ and, if it is adjacent to u_1, then $d_S(x, u_3)=5$. In either case the delay of S is at least four, a contradiction.

If $k=3$, then C is a cycle of size 6. There is no vertex at distance two from the cycle, as this vertex would have distance five (delay four) to some vertex in the cycle. Since $q \geqslant 5$, we may without loss of generality assume that u_1 is adjacent to some vertex $v \notin C$. As $d_S(v, v_2)=4$, the vertex v_2 can have no neighbors that are not in the cycle. Thus, every vertex of $U \backslash \{u_1, u_2, u_3\}$ is adjacent to either v_1 or v_3. So, $\deg_S(v_1)+\deg_S(v_3)=(p-3)+4=p+1$. If $\deg_S(v_1) \neq \deg_S(v_3)$ then without loss of generality $\deg_S(v_1) \geqslant \lceil (p+2)/2 \rceil$. Then, as v_1 must have at least one edge in every other spanner in $\mathscr{S}$, $|\mathscr{S}| \leqslant \lfloor (p-2)/2 \rfloor + 1$. On the other hand, if $\deg_S(v_1)=\deg_S(v_3)$, then, as $p \geqslant 4$, each of v_1 and v_3 are adjacent to some vertex not in C. This implies that neither u_2 nor u_3 is adjacent to a vertex not in C. Thus, the only element of V not adjacent to u_1 is v_2, and $|\mathscr{S}| \leqslant 2$.

If $k=2$, then C is a cycle of size 4. There is no vertex at distance three from the cycle as this vertex would have distance five to some vertex in the cycle. Suppose that there is a vertex a at distance two from the cycle. Without loss of generality, we may assume that the root of a is u_1 or v_1.

Consider the case that u_1 is the root of a. If another vertex b is at distance two from the cycle then the root of b must also be u_1 as otherwise $d_S(a,b) \geqslant 5$. Furthermore, if u_2 has a neighbor $b \notin C$, then $d_S(a,b)=5$. Now consider a vertex $v \in V$ that is not in the cycle. The root of v cannot be u_2 as u_2 has no non-cycle neighbors. Neither can it be v_1 or v_2 as the only vertices rooted at v_1 or v_2 are at distance one and, hence, in U. So, v must be adjacent to u_1, implying that all elements of V are adjacent to u_1. Thus, u_1 has no edges remaining for any other spanners and $|\mathscr{S}|=1$. The case that the root of a is v_1 is handled similarly.

What remains is the situation when no vertex is at distance two from the cycle. In this case, vertices v_1 and v_2 must together be adjacent to all vertices of U, so $\deg_S(v_1) + \deg_S(v_2) = p + 2$. Without loss of generality $\deg_S(v_1) \geqslant \lceil (p+2)/2 \rceil$ and $|\mathscr{S}| \leqslant \lfloor (p-2)/2 \rfloor + 1$.

In each of the above cases, we have either arrived at a contradiction or shown that $|\mathscr{S}|$ is at most $\lfloor p/2 \rfloor$. Let $q = p + q'$ where $q' \geqslant 1$. Then, $\lfloor pq/(p+q+1) \rfloor = \lfloor p(p+q')/(2p+q'+1) \rfloor = \lfloor p(p+q')/2(p+q') + (1-q') \rfloor$. As $1 - q' \leqslant 0$, this number is at least $\lfloor p(p+q')/2(p+q') \rfloor = \lfloor p/2 \rfloor$. It follows that $|\mathscr{S}| \leqslant \lfloor pq/(p+q+1) \rfloor$. $\square$

The following technical lemma will be used in the construction of spanners below.

Lemma 4.3. *Let M be a $p \times (p-h+2)$ matrix with $h \geqslant 3$ such that $M_{i,j} = 1$ if $i \in \{j, j+1, \ldots, j+h-1 \,(\mathrm{mod}\, p)\}$ and $M_{i,j} = 0$, otherwise. For any rows $i_1, i_2, \ldots, i_{h-1}$ where $0 \leqslant i_1 < i_2 < \cdots < i_{h-1} \leqslant p-1$ and column j_1 with $0 \leqslant j_1 \leqslant p-h+1$, there exists a one-to-one and onto function $\rho : \{0, 1, 2, \ldots, p-1\} \backslash \{i_1, i_2, \ldots, i_{h-1}\} \to \{0, 1, 2, \ldots, p-h+1\} \backslash \{j_1\}$ such that $M_{i,\rho(i)} = 1$ for all i.*

Proof. For convenience, we define $i_0 = -1$ and $i_{h+1} = p - 1$.

When $j_1 < i_1$,

$$\rho(i) = \begin{cases} p-h+1 & \text{for } i = 0, \\ i-1 & \text{for } i \leqslant i < j_1 + 1, \\ i & \text{for } j_1 + 1 \leqslant i < i_1, \\ i-k & \text{for } i_k + 1 \leqslant i < i_{k+1} \quad \text{for any } k,\ 1 \leqslant k \leqslant h-1. \end{cases}$$

Otherwise, $i_l \leqslant j_1 + l < i_{l+1}$, for some $l, 0 \leqslant l \leqslant h$, and we let

$$\rho(i) = \begin{cases} i-k & \text{for } i_k < i < i_{k+1} \text{ with } k < l, \\ i-l & \text{for } i_l < i < j_1 + l, \\ i-l+1 & \text{for } j_1 + l \leqslant i < i_{l+1}, \\ i-k+1 & \text{for } i_k < i < i_{k+1} \text{ with } k \geqslant l+1. \end{cases} \qquad \square$$

In the following proofs we will construct a particular type of $(x+2)$-spanner of $K_{p,q}$ which we call an *h-hub spanner*. Such a spanner has h distinguished vertices in U called *hubs*. Each hub vertex is connected by a path of length 2 to each vertex of U (including the other hubs). Furthermore, each vertex of V is adjacent to at least one hub vertex. An h-hub spanner has delay 2: there is a path of length at most 4 between any pair of vertices in U, a path of length at most 3 between any vertex of U and any vertex of V, and a path of length at most 4 between any pair of vertices in V.

For $h \geqslant 3$, we construct sets of edge-disjoint h-hub spanners of $K_{p,q}$ with overlapping hubs. In particular, we construct spanners $S_0, S_1, \ldots, S_{p-h+1}$ with spanner S_i having hubs $u_i, u_{i+1}, \ldots, u_{i+h-1}$, where $u_p = u_0$. Let M be a matrix with a row for each element of

U, a column for each spanner, and entries $M_{ij}=1$ if u_i is a hub in spanner S_j, and $M_{ij}=0$ otherwise. Note that M is a matrix of the type described in Lemma 4.3.

In each spanner of such a set, a vertex v of V will be either a *connector vertex* (a midpoint of a path of length 2 between vertices of U) or a *leaf vertex* (a degree one vertex adjacent to a hub). No vertex will be a connector vertex in more than one spanner of a set. In any spanner S_i, a connector vertex has degree $h-1$, leaving $p-h+1$ edges for use in the remaining $p-h+1$ spanners. Furthermore, by Lemma 4.3 we can choose any set of $h-1$ edges on a connector vertex and be assured that this vertex can be joined to a distinct hub vertex in each other spanner. Thus, we need describe only the edges from $v \in V$ in the spanner in which it is a connector, if any.

We treat three separate cases below: even $h \geqslant 4$, odd $h \geqslant 5$, and $h=3$.

Theorem 4.4. *For even* $h \geqslant 4$ *and* $p \geqslant h+1$, $\mathrm{EDS}(K_{p,q},2) \geqslant p-h+2$ *when* $q \geqslant (p-h+2)(2\lceil (p-h)/(h/2)-1\rceil+2)$.

Proof. We construct a set $\mathscr{S}$ of h-hub spanners $S_0, S_1, \ldots, S_{p-h+1}$. For a particular spanner S_i we use $2\lceil (p-h)/(h/2)-1\rceil+2$ connectors. One connector is made adjacent to hub vertices $u_i, u_{i+2}, u_{i+3}, \ldots, u_{i+h-1}$ (again where $u_p=u_0$). A second connector is made adjacent to hub vertices $u_{i+1}, u_{i+2}, \ldots, u_{i+h-1}$. These two connectors provide paths of length two between all pairs of hub vertices except u_i and u_{i+1}.

We partition the non-hub vertices of U into $\lceil (p-h)/(h/2)-1\rceil$ sets C_j of size at most $(h/2)-1$. For each C_j, we use 2 connectors, one adjacent to $u_i, u_{i+1}, \ldots, u_{i+h/2-1}$ and the other adjacent to $u_{i+h/2}, u_{i+h/2+1}, \ldots, u_{i+h-1}$. Both of these connectors are also adjacent to every element of C_j. Note that these connectors together provide paths of length 2 from each hub to the elements of C_j and the first of these connectors also provides path of length 2 from u_i to u_{i+1}, completing the hub connections.

Since we have $p-h+2$ spanners in our set, with each spanner using $2\lceil (p-h)/(h/2)-1\rceil+2$ connectors, we can complete this construction if $q \geqslant (p-h+2)(2\lceil (p-h)/(h/2)-1\rceil+2)$. $\square$

Theorem 4.5. *For odd* $h \geqslant 5$ *and* $p \geqslant h+1$, $\mathrm{EDS}(K_{p,q},2) \geqslant p-h+2$ *when* $q \geqslant (p-h+2)[(h-2)\lceil (4p-4h)/(h^2-4h+3)\rceil+2]$.

Proof. We construct a set $\mathscr{S}$ of h-hub spanners $S_0, S_1, \ldots, S_{p-h+1}$. For a particular spanner S_i we use $(h-2)\lceil (4p-4h)/(h^2-4h+3)\rceil+2$ connectors. One connector is made adjacent to hub vertices $u_i, u_{i+2}, u_{i+3}, \ldots, u_{i+h-1}$ (again where $u_p=u_0$). A second connector is made adjacent to hub vertices $u_{i+1}, u_{i+2}, \ldots, u_{i+h-1}$. These two connectors provide paths of length two between all pairs of hub vertices except u_i and u_{i+1}.

We partition the non-hub vertices of U into $\lceil (p-h)/(h^2-4h+3)/4\rceil$ sets C_j of size at most $(h^2-4h+3)/4=(h-1/2)(h-3/2)$. For each C_j, we use $h-2$ connectors. These connectors are of two types. A connector of the first type connects the first $(h-1)/2$

hub vertices $u_i, u_{i+1}, \ldots, u_{i+(h-3)/2}$ to a set of $(h-1)/2$ non-hub vertices. As there are $((h-1)/2)((h-3)/2)$ non-hub vertices in C_j, we use $(h-3)/2$ connectors of this type. A connector of the second type connects the remaining $(h+1)/2$ hub vertices to a set of $(h-3)/2$ non-hub vertices; $(h-1)/2$ of these connectors are required. Note that any connector vertex of the first type provides a path of length 2 from u_i to u_{i+1}, completing the hub connections.

Since we have $p-h+2$ spanners in our set, with each spanner using $(h-2)\lceil(4p-4h)/(h^2-4h+3)\rceil+2$ connectors, we can complete this construction if $q \geqslant (p-h+2)[(h-2)\lceil(4p-4h)/(h^2-4h+3)\rceil+2]$. □

Theorem 4.6. *Let* $p \geqslant 3$. *Then* $\mathrm{EDS}(K_{p,q}, 2) = p-1$ *iff* $q \geqslant 3(p-1)(p-2)$.

Proof. We construct a set $\mathscr{S}$ of 3-hub spanners $S_0, S_1, \ldots, S_{p-2}$. For a particular spanner S_i, we use $3p-6$ connectors. Each connector has degree two in S_i. Three of these connectors are used to give length two paths between each pair of vertices in the hub. The remaining $3p-9$ connectors are used to give length two paths between each of the three hub vertices and each of the $p-3$ non-hub vertices of U.

Since we have $p-1$ spanners in $\mathscr{S}$ with each spanner using $3p-6$ connectors, we can complete this construction if $q \geqslant (3p-6)(p-1)$.

We now show that we need to have $q \geqslant (3p-6)(p-1)$ in order to have a set $\mathscr{S}$ of $p-1$ spanners of $K_{p,q}$ with delay 2.

Each vertex of V has degree one in $(p-2)$ of the spanners and degree one or two in the remaining spanner. As in our construction, we refer to a degree-one vertex of a spanner as a *leaf* and a degree-two vertex of a spanner as a *connector*. Similarly, we will call a vertex of U a *hub* of a spanner if in the spanner it is of distance at most two to every other vertex of U. In any spanner, each leaf of V must be adjacent to a hub vertex, in order to have distance at most 3 to every vertex of U.

In each spanner S, there is at least one leaf in V. Otherwise, every element of V is a connector and $|\mathscr{S}| = 1$. This implies that S has at least one hub vertex.

If S has exactly one hub vertex u, then consider a second spanner S' having some hub vertex u' (and possibly others). In S', there is a connector v adjacent to both u and u', or to u and some u'' if $u = u'$. In either case, v must be a leaf of S and so must be adjacent to u in S, a contradiction. Therefore, S has at least two hub vertices.

Suppose that S has exactly two hub vertices, u_1 and u_2. Neither u_1 nor u_2 can be a hub in any other spanner $S' \in \mathscr{S}$. If one of them were, then there would be a connector v adjacent to u_1 and u_2 in S'. This v must be a leaf in S and therefore adjacent to either u_1 or u_2, a contradiction. Thus, the hubs of the spanners in $\mathscr{S} \backslash \{S\}$ must come from the $p-2$ vertices of $U \backslash \{u_1, u_2\}$.

Let u_3 be a hub vertex of $S' \in \mathscr{S} \backslash \{S\}$, and u_4 be a vertex of $U \backslash \{u_1, u_2, u_3\}$. In S', there is a connector v adjacent to both u_3 and u_4. Vertex v must be a leaf vertex in each of the $p-3$ spanners of $\mathscr{S} \backslash \{S, S'\}$, but it has only $p-4$ edges to $U \backslash \{u_1, u_2\}$ remaining. This implies that v can not be adjacent to a hub in each of these spanners, a contradiction.

Table 1
Known values of (and bounds on) EDS $(K_{p,q}, 2)$

q, p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	1																
3	1	1															
4	1	1	2														
5	1	1	2	2													
6	1	2	2	2	3												
7	1	2	2	2	3	3											
8	1	2	2	2	3	3	4										
9	1	2	2	2,3	3	3	4	4									
10	1	2	2	2,3	3	3	4	4	5								
11	1	2	2	2,3	3	3,4	4	4	5	5							
12	1	2	2	3	3	3,4	4	4	5	5	6						
13	1	2	2	3	3	3,4	4	4,5	5	5	6	6					
14	1	2	2	3	3,4	3,4	4	4,5	5	5	6	6	7				
15	1	2	2	3	3,4	3,4	4,5	4,5	5	5,6	6	6	7	7			
16	1	2	2	3	3,4	3,4	4,5	4,5	5	5,6	6	6	7	7	8		
17	1	2	2	3	3,4	3,4	4,5	4,5	5,6	5,6	6	6,7	7	7	8	8	
18	1	2	2	3	3,4	3,4	4,5	4,5	5,6	5,6	6	6,7	7	7	8	8	9

Thus, every spanner in $\mathscr{S}$ contains at least three hub vertices. Since each connector has degree 2, a spanner with h hubs uses $\binom{h}{2} + h(p-h)$ connectors. As $p \geqslant h \geqslant 3$, we have $\binom{h}{2} + h(p-h) \geqslant 3p - 6$. Thus, a total of at least $(3p-6)(p-1)$ connectors are needed for the $p-1$ spanners. $\square$

Using the results from above, one can construct a table showing bounds on EDS $(K_{p,q}, 2)$ for various values of p and q. Although for some values of p and q we know EDS$(K_{p,q}, 2)$ exactly, for many p and q the bounds are not tight. The resulting bounds are shown in Table 1 for $p \leqslant q \leqslant 18$.

The diagonal entries ($p = q$) of the table come from Theorem 2.7 and hereafter we will limit our discussion to the non-diagonal entries. The values for $p = 2$ come from Theorem 2.6. All of the bounds for $p = 3$ also come from Theorem 2.6, except for the upper bounds for $q = 4, 5$ which come from Lemma 4.1. The values for $p = 4$ come from Theorems 2.6 and 4.6. For $p = 5$ and $6 \leqslant q \leqslant 11$, the values come from Theorem 2.6 and Lemma 4.2. The values for $p = 5$ and $12 \leqslant q \leqslant 18$ come from Theorem 4.4 and Lemma 4.2. The remaining entries come from Theorem 2.6 and Lemma 4.2. Although Theorems 4.4–4.6 have had little effect on this table, they will have more frequent application for larger values of p and q.

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