A Perron type theorem for functional differential equations

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Abstract

A Perron type theorem about the existence of the strict Lyapunov exponents of the solutions of retarded functional differential equations is established.

1. Introduction

Given $r > 0$, let $C = C([-r, 0], \mathbb{C}^n)$ denote the Banach space of continuous functions from $[-r, 0]$ into $\mathbb{C}^n$ with the supremum norm $|\phi| = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$ for $\phi \in C$.

Consider the nonlinear retarded functional differential equation

$$x'(t) = Lx_t + f(t, x_t)$$ (1.1)

as a perturbation of the linear autonomous equation

$$x'(t) = Lx_t,$$ (1.2)

where $L : C \to \mathbb{C}^n$ is a linear bounded functional and $f : [0, \infty) \times C \to \mathbb{C}^n$ is a continuous function. As usual, the symbol $x_t \in C$ is defined by $x_t(\theta) = x(t + \theta)$ for $-r \leq \theta \leq 0$.

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If the nonlinear term $f(t,x_t)$ in (1.1) is “small” as $t \to \infty$, we can expect that the solutions of (1.1) have similar asymptotic properties as the solutions of the unperturbed Eq. (1.2). In the special case of ordinary differential equations, a variety of asymptotic results of this type is available (see [3,5,7,9]). The following Perron type theorem, presented in a form given by Coppel [5], is of particular importance.

**Theorem 1.1** [5, Chapter IV, Theorem 5]. Consider the ordinary differential equation

$$x' = Bx + g(t,x),$$

(1.3)

where $B \in \mathbb{C}^{n \times n}$ is an $n \times n$ constant matrix and $g : [\sigma_0, \infty) \times \mathbb{C}^n \to \mathbb{C}^n$ is a continuous function. Let $x$ be a solution of (1.3) on $[\sigma_0, \infty)$ such that

$$|g(t,x(t))| \leq \gamma(t)||x(t)||, \quad t \geq \sigma_0,$$

(1.4)

where $\gamma : [\sigma_0, \infty) \to [0, \infty)$ is a continuous function satisfying

$$\int_{\sigma_0}^{t+1} \gamma(s) \, ds \to 0 \quad \text{as} \quad t \to \infty.$$

(1.5)

Then either

(i) the limit

$$\mu = \lim_{t \to \infty} \frac{\log |x(t)|}{t}$$

(1.6)

exists and is equal to the real part of one of the eigenvalues of the matrix $B$, or

(ii) $x(t) = 0$ for all large $t$.

Clearly, condition (1.5) on $\gamma$ holds if $\gamma(t) \to 0$ as $t \to \infty$, or $\int_{\sigma_0}^{\infty} \gamma^p(s) \, ds < \infty$ for some $p \in [1, \infty)$. The quantity $\mu$ defined by the limit (1.6) (if it exists) is sometimes called the strict Lyapunov exponent of the solution $x$.

A weaker form of Theorem 1.1 was obtained by Perron [15]. Perron’s result was improved by Lettenmeyer [13] and later by Hartman and Wintner [10]. For other variants and proofs of Theorem 1.1, see [3, Chapter 13, Theorem 4.3] or [9, Chapter X, Theorem 11.2].

Our aim in this paper is to extend Theorem 1.1 to Eq. (1.1). The paper is organized as follows. After introducing the notations and preliminaries in Section 2, in Section 3, we study the existence of the strict Lyapunov exponents of those solutions of (1.1) which satisfy the hypothesis

$$|f(t,x_t)| \leq \gamma(t)|x_t|, \quad t \geq \sigma_0$$

(1.7)

(the analogue of hypothesis (1.4) of Theorem 1.1), where $\gamma : [\sigma_0, \infty) \to [0, \infty)$ is a continuous function with property (1.5). It should be mentioned that there is an important difference between Eqs. (1.1) and (1.3) which is a consequence of the fact that the phase space $C$ for (1.1) is infinite-dimensional. Namely, in contrast to the ordinary differential
equation (1.3), Eq. (1.1) may have solutions tending to zero faster than any exponential (known as small solutions) which are not identically zero on any interval $[\sigma, \infty)$. In Theorem 3.1, we prove that with the possible exception of the small solutions the strict Lyapunov exponents

$$\mu = \mu(x) = \lim_{t \to \infty} \frac{\log |x_t|}{t}$$

exist and are equal to the real parts of the eigenvalues of Eq. (1.2). By an eigenvalue of (1.2), we mean a root of the characteristic equation

$$\det \Delta(\lambda) = 0, \quad \Delta(\lambda) = \lambda I - \int_{-r}^{0} e^{\lambda \theta} d\eta(\theta),$$

where $\eta: [-r, 0] \to \mathbb{C}^{n \times n}$ is the matrix function of bounded variation from the Riesz representation of $L$,

$$L\phi = \int_{-r}^{0} d\eta(\theta) \phi(\theta), \quad \phi \in C,$$

normalized so that $\eta$ is left continuous on $(-r, 0)$ and $\eta(0) = 0$.

Besides the extension of Theorem 1.1 to Eq. (1.1), we establish the following two properties of the solutions of (1.1) with a given finite strict Lyapunov exponent $\mu(x) = \mu$. First, for $t \to \infty$, these solutions are tangential to the generalized eigenspace associated with the eigenvalues of (1.2) having real part $\mu$ (see Theorem 3.4 below). Second, for the above solutions, the ratio $|x_{t+r}|/|x_t|$ is uniformly positive for $t \to \infty$ (see Proposition 3.5). The latter two properties will play an important role in the cognate paper [16], where we shall discuss some corollaries and further related results concerning the asymptotic behavior and the oscillation of the solutions of (1.1).

2. Notations and preliminaries

In this section, we introduce the notations and recall some facts from the theory of linear autonomous functional differential equations and perturbed linear systems which will be used in our proofs. For more details and proofs, see [8, Chapters 6 and 7].

The linear autonomous equation (1.2) generates in $C$ a strongly continuous semigroup $(T(t))_{t \geq 0}$, where the solution operator $T(t)$ is defined by $T(t)\phi = x_t(\phi)$ for $t \geq 0$ and $\phi \in C$, $x_t(\phi)$ being the unique solution of (1.2) with initial value $\phi$ at zero. The domain of the infinitesimal generator $A$ of this semigroup given by $D(A) = \{ \phi \in C \mid \phi' \in C, \quad \phi(0) = L\phi \}$ is dense in $C$ and $A\phi = \phi'$ for $\phi \in D(A)$. The spectrum of the operator $A: D(A) \to C$ is a point spectrum and consists of the eigenvalues of (1.2). In each strip $|\text{Re } z| \leq M$, $M > 0$, do not lie more than finitely many eigenvalues of (1.2). The stability modulus $d$ of (1.2) defined by

$$d = \sup \{ \text{Re } \lambda \mid \det \Delta(\lambda) = 0 \}$$

(2.1)
is finite and for every \( \varepsilon > 0 \) there exists \( M_1 = M_1(\varepsilon) > 0 \) such that
\[
|T(t)\phi| \leq M_1 e^{(d+\varepsilon)t} |\phi|, \quad t \geq 0, \ \phi \in C.
\] (2.2)

Associated with (1.2) is the \textit{transposed equation}
\[
y'(t) = - \int_{-r}^{0} y(t - \theta) d[\eta(\theta)],
\] (2.3)

where \( y(t) \) is an \( n \)-dimensional row vector. The phase space for (2.3) is \( C' = C([0, r], \mathbb{C}^n) \), where \( \mathbb{C}^n \) is the space of row \( n \)-vectors. For \( \psi \in C', \phi \in C \), we define the bilinear form
\[
(\psi, \phi) = \psi(0)\phi(0) - \int_{-r}^{0} \int_{0}^{\theta} \psi(\theta - \tau) d[\eta(\tau)] \phi(\theta) d\theta.
\] (2.4)

If \( \Lambda \) is a finite set of eigenvalues of (1.2), then \( C \) is decomposed by \( \Lambda \) into a direct sum
\[
C = P_{\Lambda} \oplus Q_{\Lambda},
\] (2.5)
where \( P_{\Lambda} \) is the \textit{generalized eigenspace associated with} \( \Lambda \) (see [8, Section 7.5] for the definition) and \( Q_{\Lambda} \) is the complementary subspace of \( C \) such that \( T(t)Q_{\Lambda} \subset Q_{\Lambda} \) for \( t \geq 0 \). If \( \Phi_{\Lambda} \) is the basis for \( P_{\Lambda} \), then there exists a square matrix \( B_{\Lambda} \) such that
\[
A\Phi_{\Lambda} = \Phi_{\Lambda}B_{\Lambda}
\]
and the spectrum of \( B_{\Lambda} \) coincides with \( \Lambda \). The solutions on \( P_{\Lambda} \) can be extended to all \( t \in \mathbb{R} \) by
\[
T(t)\Phi_{\Lambda}a = \Phi_{\Lambda}e^{B_{\Lambda}t}a,
\] (2.6)
where \( a \) is a vector of the same dimension as \( \Phi_{\Lambda} \). Further, if
\[
\phi = \phi^{P_{\Lambda}} + \phi^{Q_{\Lambda}}, \quad \phi^{P_{\Lambda}} \in P_{\Lambda}, \ \phi^{Q_{\Lambda}} \in Q_{\Lambda}
\] (2.7)
is the decomposition of \( \phi \in C \) in \( P_{\Lambda} \oplus Q_{\Lambda} \), then
\[
\phi^{P_{\Lambda}} = \Phi_{\Lambda}(\Psi_{\Lambda}, \phi), \quad \phi^{Q_{\Lambda}} = \phi - \phi^{P_{\Lambda}},
\]
where \( \Psi_{\Lambda} \) is the basis for the generalized eigenspace \( P_{\Lambda}^T \) of the transposed Eq. (2.3) associated with \( \Lambda \) (see [8, Section 7.5] for the definition) normalized so that \( (\Psi_{\Lambda}, \Phi_{\Lambda}) = I \).

Let \( x \) be a solution of (1.1) on \([\sigma - r, \infty)\), \( \sigma \geq \sigma_0 \) with initial value \( x_{\sigma} = \phi \in C \). If we consider the term \( f(t, x_t) \) in (1.1) as a nonhomogeneity, then, according to the \textit{variation of constants formula} (see [8, Section 6.2]), we obtain
\[
x_t = T(t - \sigma)\phi + \int_{\sigma}^{t} d[K(t, s)] f(s, x_s), \quad t \geq \sigma,
\] (2.8)
where the kernel \( K(t, \cdot) [\sigma, t] \rightarrow C \) is given by
\[
K(t, s)(\theta) = \int_{\sigma}^{s} X(t + \theta - \alpha) d\alpha, \quad -r \leq \theta \leq 0,
\]
and $X$ is the fundamental matrix of (1.2), the unique matrix solution of (1.2) with initial values $X_0(0) = I$ and $X_0(\theta) = 0$ for $-r \leq \theta < 0$. If we make the decomposition (2.5) in (2.8), we obtain an equivalent system

\[
x_t = x^P_t + x^Q_t, \tag{2.9}
\]

\[
x^P_t = T(t - \sigma)\phi^P + \int_{\sigma}^{t} T(t - s)X_0^P f(s, x_s) \, ds, \tag{2.10}
\]

\[
x^Q_t = T(t - \sigma)\phi^Q + \int_{\sigma}^{t} d[K(t, s)Q]\ f(s, x_s) \tag{2.11}
\]

for $t \geq \sigma$, where

\[
X_0^P = \Phi_A \Psi_A(0) \quad \text{and} \quad K(t, s)^Q = K(t, s) - \Phi_A(\Psi_A, K(t, s)). \tag{2.12}
\]

Suppose that $\Lambda$ has the form

\[
\Lambda = \Lambda(c) = \{ \lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0, \ \text{Re} \lambda \geq c \}, \tag{2.13}
\]

where $c$ is a given real number. Then for every sufficiently small $\varepsilon > 0$ there exists $M_2 = M_2(\varepsilon) > 0$ such that

\[
|T(t)\phi^Q| \leq M_2 e^{(c-\varepsilon)t} |\phi^Q|, \quad t \geq 0, \ \phi \in \mathbb{C}, \tag{2.14}
\]

\[
|T(t)\phi^P| \leq M_2 e^{(c+\varepsilon)t} |\phi^P|, \quad t \leq 0, \ \phi \in \mathbb{C}, \tag{2.15}
\]

and

\[
|T(t)X_0^P| \leq M_2 e^{(c-\varepsilon)t}, \quad t \geq 0. \tag{2.16}
\]

If, in addition, we assume that (1.2) has no eigenvalue on the vertical line $\text{Re} \lambda = c$, then Eq. (1.2) has an exponential dichotomy with respect to $c$, that is, (2.15) can be replaced with the stronger condition

\[
|T(t)\phi^P| \leq M_2 e^{(c+\varepsilon)t} |\phi^P|, \quad t \leq 0, \ \phi \in \mathbb{C}, \tag{2.17}
\]

provided $\varepsilon > 0$ is sufficiently small.

In order to estimate the norm of the solution $x_t$ of (1.1) and its projections onto $P_A$ and $Q_A$, we shall frequently use the following estimates for the integrals in (2.8), (2.10) and (2.11) holding for $t \geq \sigma \geq \sigma_0$:

\[
\left| \int_{\sigma}^{t} d[K(t, s)] f(s, x_s) \right| \leq M_3 \int_{\sigma}^{t} e^{(d+\varepsilon)(t-s)} \left| f(s, x_s) \right| \, ds, \tag{2.18}
\]

\[
\left| \int_{\sigma}^{t} T(t - s)X_0^P f(s, x_s) \, ds \right| \leq M_3 \int_{\sigma}^{t} e^{(d+\varepsilon)(t-s)} \left| f(s, x_s) \right| \, ds, \tag{2.19}
\]

\[
\left| \int_{\sigma}^{t} d[K(t, s)Q] f(s, x_s) \right| \leq M_3 \int_{\sigma}^{t} e^{(c-\varepsilon)(t-s)} \left| f(s, x_s) \right| \, ds, \tag{2.20}
\]
where \( d \) and \( c \) have the meaning from (2.1) and (2.13), \( \varepsilon > 0 \) is sufficiently small and \( M_3 = M_3(\varepsilon) \) is a positive constant independent of the solution \( x \). The above estimates follow from (2.2), (2.14) and the representations of \( K(t, s) \) and \( K(t, s)^Q\Lambda \).

3. Main results

The following theorem is the analogue of Theorem 1.1 for Eq. (1.1).

**Theorem 3.1.** Let \( x \) be a solution of (1.1) on \([\sigma_0 - r, \infty)\) such that (1.7) holds with a continuous function \( \gamma : [\sigma_0, \infty) \rightarrow [0, \infty) \) satisfying (1.5). Then either

(i) the limit (1.8) exists and is equal to the real part of one of the eigenvalues of (1.2), or
(ii) for each \( b \in \mathbb{R} \), we have that \( \lim_{t \to \infty} e^{bt} x(t) = 0 \).

We shall call the quantity \( \mu = \mu(x) \) defined by the limit (1.8) (if it exists) the strict Lyapunov exponent of the solution \( x \). Solutions \( x \) which satisfy conclusion (ii) of the above theorem are known as small solutions.

**Remark.** Evidently, any solution \( x \) which satisfies conclusion (ii) of Theorem 1.1 is a small solution. However, as noted in Section 1, in contrast to the ordinary differential equation (1.3), Eq. (1.1) may have small solutions which are not identically zero on any interval \([\sigma, \infty)\). An example of such solutions is the solution \( x(t) = e^{-t^2} \) of the scalar equation

\[
x'(t) = -2te^{1-2r}x(t-1)
\]

on the interval \([-1, \infty)\) (see [8, p. 97]).

**Remark.** In conclusion (i) of Theorem 3.1 it is important that the solution \( x \) is interpreted in \( C \) in the sense that \( x_t \) in (1.8) cannot be replaced with \( x(t) \). This is shown by the following simple example. Consider the scalar equation

\[
x'(t) = -x \left( t - \frac{5\pi}{2} \right),
\]

\[ (3.1) \]
a special case of (1.1) when \( n = 1, r = 5\pi/2, L\phi = -\phi(-5\pi/2) \) and \( f(t, \phi) \equiv 0 \). For the solution \( x(t) = \cos t \) of (3.2) the strict Lyapunov exponent \( \mu(x) \) equals zero, the real part of the eigenvalue \( \lambda = i \) of (3.2). However, the limit \( \lim_{t \to \infty} t^{-1} \log |x(t)| \) does not exist.

**Remark.** It is easily seen that if the strict Lyapunov exponent \( \mu(x) \) exists, then

\[
\mu(x) = \limsup_{t \to \infty} \frac{\log |x(t)|}{t}.
\]

**Remark.** In the geometric theory of functional differential equations, qualitative results similar to Theorem 3.1 are widely used. It can be shown that under certain additional assumptions alternative (ii) of Theorem 3.1 cannot occur, i.e., there are no nontrivial small
solutions. Then it is possible to relate the solutions close to an equilibrium to the eigensolutions of the linear variational equation at the equilibrium (see [1,2,4,11,12,14]) for details).

Before we present the proof of Theorem 3.1, we establish two lemmas. The first lemma is a simple consequence of Gronwall’s inequality.

**Lemma 3.2.** Let \( x \) be a solution of (1.1) satisfying the hypotheses of Theorem 3.1. Then for every \( \varepsilon > 0 \) there exist constants \( C_1, C_2 > 0 \) such that for all \( t \geq \sigma_1 \geq \sigma_0 \),

\[
|x_t| \leq C_1|x_{\sigma_1}|e^{(d+\varepsilon)(t-\sigma_1)} \exp \left( C_2 \int_{\sigma_1}^{t} \gamma(s) \, ds \right),
\]

where \( d \) is the stability modulus of (1.2) given by (2.1). In particular, there exists \( C_3 > 0 \) such that for all integers \( k \geq \sigma_0/r \), we have that

\[
C_3^{-1}|x((k+1)r)| \leq |x_t| \leq C_3|x_{kr}|, \quad kr \leq t \leq (k+1)r.
\]

**Proof.** By the variation of constants formula (2.8), we have for \( t \geq \sigma_1 \geq \sigma_0 \),

\[
x_t = T(t - \sigma_1)x_{\sigma_1} + \int_{\sigma_1}^{t} \mathbf{d} \left[ K(t, s) \right] f(s, x_s).
\]

By virtue of (1.7), (2.2) and (2.18), we have for \( t \geq \sigma_1 \geq \sigma_0 \),

\[
|x_t| \leq M_1 e^{(d+\varepsilon)(t-\sigma_1)} |x_{\sigma_1}| + M_2 \int_{\sigma_1}^{t} e^{(d+\varepsilon)(t-s)} \gamma(s) |x_s| \, ds
\]

and hence

\[
e^{-((d+\varepsilon)t)} |x_t| \leq M_1 e^{-(d+\varepsilon)\sigma_1} |x_{\sigma_1}| + M_2 \int_{\sigma_1}^{t} \gamma(s) e^{-(d+\varepsilon)s} |x_s| \, ds.
\]

By Gronwall’s lemma, the last inequality implies that (3.3) holds with \( C_1 = M_1 \) and \( C_2 = M_2 \). By virtue of (1.5), we have that

\[
S = \sup_{t \geq \sigma_0} \int_{t}^{t+r} \gamma(s) \, ds < \infty.
\]

Consequently, (3.3) implies that (3.4) holds with

\[
C_3 = C_1 \exp(C_2 S) \max_{0 \leq \tau \leq r} e^{(d+\varepsilon)\tau}.
\]

The following lemma will play a key role in the proof of Theorem 3.1.

**Lemma 3.3.** Let \( x \) be a solution of (1.1) satisfying the hypotheses of Theorem 3.1 and such that \( |x_t| > 0 \) for \( t \geq \sigma_0 \). Let \( C = P_{A} \oplus Q_{A} \) be the decomposition of \( C \) by the set of
eigenvalues $\Lambda = \Lambda(c)$ defined by (2.13), where $c \in \mathbb{R}$ is chosen such that Eq. (1.2) has no eigenvalue on the vertical line $\text{Re}\lambda = c$. Then either

$$\limsup_{t \to \infty} \frac{\log |x_t|}{t} < c$$

and

$$x_{kr}^P = o\left(|x_{kr}^Q|\right) \quad \text{as} \quad k \to \infty,$$

or

$$\liminf_{t \to \infty} \frac{\log |x_t|}{t} > c$$

and

$$x_{kr}^Q = o\left(|x_{kr}^P|\right) \quad \text{as} \quad k \to \infty,$$

where $k$ in the asymptotic relations (3.6) and (3.8) is an integer.

**Proof.** Under the hypotheses of the lemma, the semigroup $(T(t))_{t \geq 0}$ generated by Eq. (1.2) has an exponential dichotomy with respect to $c$. That is, estimates (2.14) and (2.17) hold with suitable positive constants $\varepsilon$ and $M_2$. Write $P_\Lambda = P$ and $Q_\Lambda = Q$ for brevity. Define

$$\|\phi\| = \sup_{t \geq 0} e^{-(c-\varepsilon)t} \left| T(t)\phi \right| + \sup_{t \leq 0} e^{-(c+\varepsilon)t} \left| T(t)\phi \right|, \quad \phi \in C.$$  

(3.9)

It is easily verified (see also [6, Chapter VIII, Exercises 5.2 and 5.3]) that $\| \cdot \|$ is a norm on $C$ which is equivalent to the original norm and for which estimates (2.14) and (2.17) hold with $M_2 = 1$. More precisely, there exists $q > 1$ such that

$$|\phi| \leq \|\phi\| \leq q|\phi|, \quad \phi \in C,$$

(3.10)

and, for all $\phi \in C$, we have that

$$\|\phi\| = \|\phi^P\| + \|\phi^Q\|,$$

(3.11)

$$\left| T(t)\phi^P \right| \leq e^{(c+\varepsilon)t} \|\phi^P\|, \quad t \leq 0,$$

(3.12)

$$\left| T(t)\phi^Q \right| \leq e^{(c-\varepsilon)t} \|\phi^Q\|, \quad t \geq 0.$$

(3.13)

The group property of $T(t)$ on $P$ and (3.12) yield for $t \geq 0$ and $\phi \in C$,

$$\|\phi^P\| = \left| T(-t)T(t)\phi^P \right| \leq e^{-(c+\varepsilon)t} \left| T(t)\phi^P \right|.$$  

Hence

$$\left| T(t)\phi^P \right| \geq e^{(c+\varepsilon)t} \|\phi^P\|, \quad t \geq 0, \quad \phi \in C.$$

(3.14)

Let $k \geq \sigma_0/r$ be a fixed integer. By the variation of constants formula, we have for $t \geq kr$,

$$x_t^P = T(t-kr)x_{kr}^P + \int_{kr}^t T(t-s)X_0^P f(s, x_s) \, ds,$$

(3.15)

$$x_t^Q = T(t-kr)x_{kr}^Q + \int_{kr}^t d[K(t,s)^Q] f(s, x_s).$$

(3.16)
From (3.15), using (3.10) and (3.14), we find for $t \geq kr$,
\[
\|x_t^P\| \geq \|T(t-kr)x_{kr}^P\| - \left\| \int_{kr}^{t} T(t-s)X_0^P f(s,x_s) \, ds \right\|
\geq e^{(c+\epsilon)(t-kr)} \|x_{kr}^P\| - q \left\| \int_{kr}^{t} T(t-s)X_0^P f(s,x_s) \, ds \right\|
\]
and hence (by (1.7) and (2.19))
\[
\|x_t^P\| \geq e^{(c+\epsilon)(t-kr)} \|x_{kr}^P\| - qM_3 \int_{kr}^{t} e^{(d+\epsilon)(t-s)} \gamma(s)|x_s| \, ds. \tag{3.17}
\]
From (3.16), it follows by similar estimates (using (2.20) and (3.13) instead of (2.19) and (3.14), respectively) for $t \geq kr$,
\[
\|x_t^Q\| \leq e^{(c-\epsilon)(t-kr)} \|x_{kr}^Q\| + qM_3 \int_{kr}^{t} e^{(c-\epsilon)(t-s)} \gamma(s)|x_s| \, ds. \tag{3.18}
\]
From (3.17) and conclusion (3.4) of Lemma 3.2, we obtain for $kr \leq t \leq (k+1)r$,
\[
\|x_t^P\| \geq e^{(c+\epsilon)(t-kr)} \|x_{kr}^P\| - qM_3C_3|x_{kr}|\gamma_k \max_{0 \leq \tau \leq r} e^{(d+\epsilon)\tau},
\]
where
\[
\gamma_k = \int_{kr}^{(k+1)r} \gamma(s) \, ds.
\]
From this, (3.10) and (3.11), we find for $kr \leq t \leq (k+1)r$,
\[
\|x_t^P\| \geq e^{(c+\epsilon)(t-kr)} \|x_{kr}^P\| - D_1\gamma_k(\|x_{kr}^P\| + \|x_{kr}^Q\|), \tag{3.19}
\]
where $D_1 = qM_3C_3 \max_{0 \leq \tau \leq r} e^{(d+\epsilon)\tau}$. From (3.18), we obtain by similar estimates for $kr \leq t \leq (k+1)r$,
\[
\|x_t^Q\| \leq e^{(c-\epsilon)(t-kr)} \|x_{kr}^Q\| + D_2\gamma_k(\|x_{kr}^P\| + \|x_{kr}^Q\|), \tag{3.20}
\]
where $D_2 = qM_3C_3 \max_{0 \leq \tau \leq r} e^{(c-\epsilon)\tau}$. Inequalities (3.19) and (3.20) imply for all integers $k \geq \sigma_0/r$,
\[
\|x_{(k+1)r}^P\| \geq \alpha \|x_{kr}^P\| - D\gamma_k(\|x_{kr}^P\| + \|x_{kr}^Q\|), \tag{3.21}
\]
\[
\|x_{(k+1)r}^Q\| \leq \beta \|x_{kr}^Q\| + D\gamma_k(\|x_{kr}^P\| + \|x_{kr}^Q\|), \tag{3.22}
\]
where $D = D_1 + D_2$,
\[
\alpha = e^{(c+\epsilon)r} \quad \text{and} \quad \beta = e^{(c-\epsilon)r}; \quad 0 < \beta < \alpha. \tag{3.23}
\]
From (1.5), it follows that
\[ \gamma_k \to 0 \quad \text{as} \quad k \to \infty. \] (3.24)

We claim that either
\[ \| x_{Pkr} \| \leq \| x_{Qkr} \| \quad \text{for all large} \quad k, \] (3.25)
or
\[ \| x_{Qkr} \| < \| x_{Pkr} \| \quad \text{for all large} \quad k. \] (3.26)

We shall prove the above claim by showing that if (3.25) fails, then (3.26) must hold. Suppose that (3.25) does not hold. Then
\[ \| x_{Qkr} \| < \| x_{Pkr} \| \quad \text{for infinitely many} \quad k. \] (3.27)

Choose \( \delta > 0 \) such that \( \delta < \alpha / (2D) \) and \( \delta < (\alpha - \beta) / (4D) \). In view of (3.23) such a \( \delta \) certainly exists. By virtue of (3.24), there exists \( k_1 \) such that \( \gamma_k < \delta \) for \( k \geq k_1 \). From this, (3.21) and (3.22), we find for \( k \geq k_1 \),
\[ \| x_{Pkr} \| \geq (\alpha - D\delta) \| x_{kr}^P \| - D\delta \| x_{kr}^Q \|, \] (3.28)
\[ \| x_{Qkr} \| \leq (\beta + D\delta) \| x_{kr}^Q \| + D\delta \| x_{kr}^P \|. \] (3.29)

By virtue of (3.27), there exists \( k_2 \geq k_1 \) such that \( \| x_{Qkr} \| < \| x_{Pkr} \| \). We shall show by induction on \( k \) that \( \| x_{Qkr} \| < \| x_{Pkr} \| \) holds for all \( k \geq k_2 \). Suppose for induction that \( \| x_{Qkr} \| < \| x_{Pkr} \| \) for some \( k \geq k_2 \). The last inequality, together with (3.28) and (3.29), implies that
\[ \| x_{Qkr} \| \leq (\beta + 2D\delta) \| x_{kr}^P \|. \]

Hence
\[ \| x_{Qkr} \| \leq \frac{\beta + 2D\delta}{\alpha - 2D\delta} \| x_{Pkr} \| < \| x_{Pkr} \|. \]

by the choice of \( \delta \). This proves that \( \| x_{Qkr} \| < \| x_{Pkr} \| \) for all \( k \geq k_2 \). Thus, we have shown that if (3.25) fails, then (3.26) holds. As a consequence, we have the following two possible cases corresponding to alternatives (3.25) and (3.26), respectively.

**Case 1.** Supposing that (3.25) holds. We shall show that in this case conclusions (3.5) and (3.6) hold. Let \( \eta > 0 \) be given. Choose \( k_0 \) so large that both \( \gamma_k < \eta \) and \( \| x_{kr}^P \| \leq \| x_{kr}^Q \| \) hold for \( k \geq k_0 \). Using the last two inequalities in (3.22), we find for \( k \geq k_0 \),
\[ \| x_{Qkr} \| \leq (\beta + 2D\eta) \| x_{kr}^Q \| \]
which implies by easy induction on \( k \) that
\[ \| x_{kr}^Q \| \leq K_1 (\beta + 2D\eta)^k, \quad k \geq k_0, \]
where \( K_1 = (\beta + 2D\eta)^{-k_0} \| x_{Qkr} \| \). This, together with conclusion (3.4) of Lemma 3.2, (3.10) and (3.11), yields for \( k \geq k_0 \) and \( kr \leq t \leq (k + 1)r \),
\[ |x_t| \leq C_3 |x_{kr}| \leq C_3 \| x_{kr} \| = C_3 \left( \| x_{kr}^P \| + \| x_{kr}^Q \| \right) \leq 2C_3 \| x_{kr}^Q \| \leq 2C_3 K_1 (\beta + 2D\eta)^k. \]
The last inequality implies that in the case when \( \beta + 2D\eta \geq 1 \), we have
\[
|x_t| \leq 2C_3 K_1 (\beta + 2D\eta)^{t/r}, \quad t \geq k_0 r,
\]
while, in case \( \beta + 2D\eta < 1 \), we have
\[
|x_t| \leq 2C_3 K_1 (\beta + 2D\eta)^{(t-r)/r}, \quad t \geq k_0 r.
\]
In both cases, we have that
\[
\limsup_{t \to \infty} \frac{\log |x_t|}{t} \leq \frac{\log(\beta + 2D\eta)}{r}.
\]
Since \( \eta > 0 \) was arbitrary, we obtain
\[
\limsup_{t \to \infty} \frac{\log |x_t|}{t} \leq \frac{\log \beta}{r} = c - \varepsilon < c.
\]
Thus, (3.5) holds. We now prove (3.6). In view of (3.10), it is enough to show (3.6) for the norm defined by (3.9). Note that \( \|x^{Q}_{kr}\| > 0 \) for all large \( k \). (Otherwise, (3.11) and (3.25) lead to \( \|x_{kr}\| = \|x^{P}_{kr}\| + \|x^{Q}_{kr}\| \leq 2\|x^{Q}_{kr}\| = 0 \) for infinitely many \( k \), contradicting the hypothesis that \( \|x_{t}\| \geq |x_t| > 0 \) for \( t \geq \sigma_0 \).) Define
\[
S = \limsup_{k \to \infty} \frac{\|x^{P}_{kr}\|}{\|x^{Q}_{kr}\|}.
\]
By virtue of (3.25), \( 0 \leq S \leq 1 \). Using (3.25) in (3.22), we obtain for all large \( k \),
\[
\|x^{Q}_{(k+1)r}\| \leq (\beta + 2D\gamma_k)\|x^{Q}_{kr}\|.
\]
This, together with (3.21), yields for all large \( k \),
\[
\frac{\|x^{P}_{(k+1)r}\|}{\|x^{Q}_{(k+1)r}\|} \geq \frac{\alpha - D\gamma_k}{\beta + 2D\gamma_k} \|x^{P}_{kr}\| - \frac{D\gamma_k}{\beta + 2D\gamma_k}
\]
and hence
\[
\frac{\|x^{P}_{kr}\|}{\|x^{Q}_{kr}\|} \leq \frac{\beta + 2D\gamma_k}{\alpha - D\gamma_k} \left[ \frac{\|x^{P}_{(k+1)r}\|}{\|x^{Q}_{(k+1)r}\|} + \frac{D\gamma_k}{\beta + 2D\gamma_k} \right].
\]
Taking the limsup on both sides and using (3.24), we obtain \( S \leq (\beta/\alpha)S \). Since \( \beta/\alpha < 1 \) (see (3.23)), this implies that \( S = 0 \). Thus, (3.6) holds.

**Case 2.** Suppose now that (3.26) holds. We shall show that in this case conclusions (3.7) and (3.8) hold. Choose \( \eta \) such that \( 0 < \eta < \alpha/(2D) \). Find \( k_0 \) such that both \( \gamma_k \leq \eta \) and \( \|x^{Q}_{kr}\| < \|x^{P}_{kr}\| \) hold for \( k \geq k_0 \). Using the last two inequalities in (3.21), we find for \( k \geq k_0 \),
\[
\|x^{P}_{(k+1)r}\| \geq (\alpha - 2D\eta)\|x^{P}_{kr}\|
\]
and hence
\[
\|x^{P}_{kr}\| \geq K_2(\alpha - 2D\eta)^{k},
\]
where $K_2 = (\alpha - 2D\eta)^{-k_0} \|x^P_{kr}\| > 0$. Conclusion (3.4) of Lemma 3.2, together with (3.10) and (3.11), yields for $k \geq k_0$ and $kr \leq t \leq (k+1)r$,

$$|x_t| \geq C_3^{-1} |x_{(k+1)r}| \geq C_3^{-1} q^{-1} \|x_{(k+1)r}\| = C_3^{-1} q^{-1} (\|x^P_{(k+1)r}\| + \|x^Q_{(k+1)r}\|) \geq C_3^{-1} q^{-1} \|x^P_{(k+1)r}\| \geq C_3^{-1} q^{-1} K_2 (\alpha - 2D\eta)^{k+1}.$$ 

Consequently, if $\alpha - 2D\eta \geq 1$, then

$$|x_t| \geq C_3^{-1} q^{-1} K_2 (\alpha - 2D\eta)^{t/r}, \quad t \geq k_0r,$$

while, in case $\alpha - 2D\eta < 1$, we have that

$$|x_t| \geq C_3^{-1} q^{-1} K_2 (\alpha - 2D\eta)^{(t+r)/r}, \quad t \geq k_0r.$$

In both cases, we have that

$$\liminf_{t \to \infty} \frac{\log |x_t|}{t} \geq \frac{\log (\alpha - 2D\eta)}{r}.$$

From this, letting $\eta \to 0$, we obtain

$$\liminf_{t \to \infty} \frac{\log |x_t|}{t} \geq \frac{\log \alpha}{r} = c + \varepsilon > c.$$

Thus, (3.7) holds. We now prove (3.8). As noted before, it is enough to show (3.8) for the equivalent norm given by (3.9). Define

$$R = \limsup_{k \to \infty} \frac{\|x^Q_{kr}\|}{\|x^P_{kr}\|}.$$

In view of (3.26), $0 \leq R \leq 1$. Using (3.26) in (3.21), we find for all large $k$,

$$\|x^P_{(k+1)r}\| \geq (\alpha - 2D\gamma_k) \|x^P_{kr}\|.$$

This, together with (3.22), yields for all large $k$,

$$\frac{\|x^Q_{(k+1)r}\|}{\|x^P_{(k+1)r}\|} \leq \frac{\beta + D\gamma_k}{\alpha - 2D\gamma_k} \frac{\|x^Q_{kr}\|}{\|x^P_{kr}\|} + \frac{D\gamma_k}{\alpha - 2D\gamma_k}.$$

Taking the limsup on both sides and using (3.24), we obtain that $R \leq (\beta/\alpha) R$. In view of (3.23), this implies that $R = 0$. Thus, (3.8) holds. \qed

We are now in a position to give the proof of Theorem 3.1.

**Proof of Theorem 3.1.** Let $x$ be a solution of (1.1) satisfying the hypotheses of the theorem. If $|x_\sigma| = 0$ for some $\sigma_1 \geq \sigma_0$, then conclusion (3.3) of Lemma 3.2 implies that $x(t) = 0$ for all $t \geq \sigma_1$ and hence conclusion (ii) of the theorem holds. We exclude this case from now on. Suppose that $|x_t| > 0$ for all $t \geq \sigma_0$. Let $\{\mu_j\}_{j=1}^N$, $1 \leq N \leq \infty$ be the sequence of all distinct real parts of the eigenvalues of (1.2) ordered so that $\mu_1 > \mu_2 > \mu_3 > \cdots$. Choose a sequence of real numbers $\{c_j\}_{j=1}^N$ such that $\mu_{j+1} < c_j < \mu_j$ for $1 \leq j < N$ and $c_N < \mu_N$ if $N < \infty$. Applying Lemma 3.3 to each of the spectral sets $\Lambda(c_j)$ defined by (2.13) with $c = c_j$, we conclude that one of the two cases below may occur.
Case 1. For each $j$, we have that
\[ \limsup_{t \to \infty} \frac{\log |x_t|}{t} < c_j. \] (3.30)

Case 2. There exists an index $j$ such that
\[ \liminf_{t \to \infty} \frac{\log |x_t|}{t} > c_j. \] (3.31)

Since $\Lambda(c_j) = \Lambda(\mu_j)$ for each $j$, the corresponding subspaces $P_{\Lambda(c_j)}$ and $Q_{\Lambda(c_j)}$ from Lemma 3.3 and hence conditions (3.30) and (3.31) are independent of the choice of $c_j$ in the open interval $(\mu_{j+1}, \mu_j)$.

Consider Case 1. As noted in Section 2, the eigenvalues of (1.2) have no finite accumulation points. Consequently, if $N = \infty$, then $\mu_j$ and hence $c_j$ tend to $-\infty$ as $j \to \infty$. Letting $j \to \infty$ in (3.30), we conclude that $x$ satisfies conclusion (ii) of the theorem.

If $N < \infty$, then taking into account that $c_N < \mu_N$ can be chosen arbitrarily and letting $c_N \to -\infty$ in (3.30), we obtain again that conclusion (ii) holds.

Consider now Case 2. Let $m$ be the least index with the property
\[ \liminf_{t \to \infty} \frac{\log |x_t|}{t} > c_m. \]

As noted before, $c_m$ can be chosen arbitrarily in the interval $(\mu_{m+1}, \mu_m)$. Letting $c_m \to \mu_m$, we find that
\[ \liminf_{t \to \infty} \frac{\log |x_t|}{t} \geq \mu_m. \] (3.32)

We now show that
\[ \limsup_{t \to \infty} \frac{\log |x_t|}{t} \leq \mu_m. \] (3.33)

If $m = 1$, then conclusion (3.3) of Lemma 3.2 yields for $t \geq \sigma_0$,
\[ \frac{\log |x_t|}{t} \leq \frac{\log(C_1|x_{\sigma_0}|)}{t} + \frac{(d + \epsilon)(t - \sigma_0)}{t} + \frac{C_2}{t} \int_{\sigma_0}^{t} \gamma(s) \, ds. \]

Letting $t \to \infty$, we obtain
\[ \limsup_{t \to \infty} \frac{\log |x_t|}{t} \leq d + \epsilon, \]

since $t^{-1} \int_{\sigma_0}^{t} \gamma(s) \, ds \to 0$ as $t \to \infty$ by (1.5). Since $\epsilon > 0$ is arbitrary (see Lemma 3.2) and $d = \mu_1$, (3.33) holds with $m = 1$. Consider now the case when $m > 1$. From Lemma 3.3 and the minimality of $m$, it follows that
\[ \limsup_{t \to \infty} \frac{\log |x_t|}{t} < c_{m-1}, \]

where $c_{m-1} \in (\mu_m, \mu_{m-1})$ is arbitrary. Letting $c_{m-1} \to \mu_m$ in the last inequality, we obtain (3.33). Finally, (3.32) and (3.33) imply that (1.8) holds with $\mu = \mu_m$. \qed
Theorem 3.1 applies to all solutions of Eq. (1.1) provided
\[ |f(t, \phi)| \leq \gamma(t)|\phi|, \quad t \geq \sigma_0, \quad \phi \in C, \] (3.34)
where \( \gamma: [\sigma_0, \infty) \to [0, \infty) \) is a continuous function satisfying (1.5). In particular, it applies to all solutions of the asymptotically autonomous linear equation
\[ x'(t) = Lx_t + M(t)x_t, \] (3.35)
where each \( M(t): C \to \mathbb{C}^n \), \( t \geq \sigma_0 \) is a bounded linear functional such that
\[ \|M(t)\| \to 0 \quad \text{as} \quad t \to \infty, \] (3.36)
or, more generally,
\[ \int_{t}^{t+1} \|M(s)\| \, ds \to 0 \quad \text{as} \quad t \to \infty, \] (3.37)
\( \| \cdot \| \) being the operator norm. If
\[ f(t, \phi) = o(|\phi|) \quad \text{as} \quad t \to \infty \quad \text{and} \quad |\phi| \to 0, \] (3.38)
then Theorem 3.1 applies to all solutions of (1.1) which tend to zero as \( t \to \infty \).

Suppose that at least one eigenvalue of (1.2) has real part \( \mu \). Let
\[ \Lambda = \Lambda(\mu) = \{ \lambda \mid \det \Delta(\lambda) = 0, \quad \text{Re} \lambda \geq \mu \} \] (3.39)
and consider the decomposition of \( C \) by \( \Lambda \), \( C = P \oplus Q \), where \( P = P_\Lambda \) and \( Q = Q_\Lambda \). The subspace \( P \) can be further decomposed as \( P = P_0 \oplus P_1 \), where \( P_0 = P_{\Lambda_0} \) and \( P_1 = P_{\Lambda_1} \) are the generalized eigenspaces associated with the sets of eigenvalues
\[ \Lambda_0 = \Lambda_0(\mu) = \{ \lambda \mid \det \Delta(\lambda) = 0, \quad \text{Re} \lambda = \mu \} \] (3.40)
and
\[ \Lambda_1 = \Lambda_1(\mu) = \{ \lambda \mid \det \Delta(\lambda) = 0, \quad \text{Re} \lambda > \mu \}, \] (3.41)
respectively. The following theorem shows that those solutions \( x_t \) of (1.1) which satisfy conclusion (i) of Theorem 3.1 are tangential to the subspace \( P_0 \) as \( t \to \infty \).

**Theorem 3.4.** Let \( x \) be a solution of (1.1) satisfying the hypotheses of Theorem 3.1 with a finite strict Lyapunov exponent \( \mu(x) = \mu \). With the above notation, we have that
\[ x_t = x_t^{P_0} + x_t^{P_1} + x_t^Q, \quad t \geq \sigma_0, \] (3.42)
\[ x_t^{P_1} = o(|x_t^{P_0}|) \quad \text{as} \quad t \to \infty \] (3.43)
and
\[ x_t^Q = o(|x_t^{P_0}|) \quad \text{as} \quad t \to \infty. \] (3.44)

**Proof.** We shall use some facts from the proof of Lemma 3.3. Choose \( c < \mu \) such that Eq. (1.2) has no eigenvalues in the strip \( c \leq \text{Re} z < \mu \). In this case \( \Lambda(c) = \Lambda(\mu) = \Lambda \) (with \( \Lambda(c) \) as in (2.13)). Since
\[ \mu = \lim_{t \to \infty} \frac{\log |x_t|}{t} = \liminf_{t \to \infty} \frac{\log |x_t|}{t} > c, \]
according to Lemma 3.3, we have that
\[ x_{kr}^Q = o(|x_{kr}^P|) \quad \text{as} \quad k \to \infty, \quad (3.45) \]
where \( k \) is an integer. First we prove that
\[ x_t^Q = o(|x_t^P|) \quad \text{as} \quad t \to \infty. \quad (3.46) \]
In view of (3.10), it is enough to show (3.46) for the norm given by (3.9). Clearly, (3.45) implies (3.26). Using (3.26) in (3.19) and (3.20), we obtain for all large \( k \) and \( kr \leq t \leq (k + 1)r, \)
\[ \|x_t^P\| \geq (m_1 - 2D_1 \gamma_k)\|x_{kr}^P\| \]
and
\[ \|x_t^Q\| \leq m_2 \|x_{kr}^Q\| + 2D_2 \gamma_k \|x_{kr}^P\|, \]
where \( m_1 = \min_{0 \leq \tau \leq r} e^{(c+\varepsilon)\tau} \) and \( m_2 = \max_{0 \leq \tau \leq r} e^{(c-\varepsilon)\tau}. \) From the last two inequalities, we obtain for all large \( k \) and \( kr \leq t \leq (k + 1)r, \)
\[ \frac{\|x_t^Q\|}{\|x_t^P\|} \leq \frac{m_2}{m_1 - 2D_1 \gamma_k} \frac{\|x_{kr}^Q\|}{\|x_{kr}^P\|} + \frac{2D_2 \gamma_k}{m_1 - 2D_1 \gamma_k}. \]
Letting \( k \to \infty \) and using (3.24) and (3.45), we conclude that (3.46) holds.

Our next aim is to show that
\[ x_{kr}^P = o\left(|x_{kr}^Q|\right) \quad \text{as} \quad k \to \infty, \quad (3.47) \]
where \( k \) is an integer. Consider the decomposition of \( C \) by \( A_1. \) Choose \( c > \mu \) such that Eq. (1.2) has no eigenvalue in the strip \( \mu < \text{Re} z \leq c. \) In this case \( A(c) = A_1(\mu) = A_1 \) and hence \( P_A(c) = P_1, Q_A(c) = Q_{A_1} \equiv Q_1. \) Since
\[ \mu = \lim_{t \to \infty} \frac{\log |x_t|}{t} = \limsup_{t \to \infty} \frac{\log |x_t|}{t} < c, \]
Lemma 3.3 implies that
\[ x_{kr}^P = o\left(|x_{kr}^Q|\right) \quad \text{as} \quad k \to \infty, \quad (3.48) \]
where \( k \) is an integer. Let \( \delta > 0 \) be given. Find \( \eta \in (0, 1) \) such that \( \eta(1 + \eta)(1 - \eta^2)^{-1} < \delta. \)
By virtue of (3.48), we have for all large \( k, \)
\[ |x_{kr}^P| \leq \eta |x_{kr}^Q| = \eta |x_{kr}^P + x_{kr}^Q| \leq \eta |x_{kr}^P| + \eta |x_{kr}^Q|. \quad (3.49) \]
Further, (3.46) implies for all large \( k, \)
\[ |x_{kr}^Q| \leq \eta |x_{kr}^P| = \eta |x_{kr}^P + x_{kr}^Q| \leq \eta |x_{kr}^P| + \eta |x_{kr}^P|. \]
Using the last inequality in (3.49), we find for all large \( k, \)
\[ |x_{kr}^P| \leq \eta(1 + \eta)|x_{kr}^P| + \eta^2 |x_{kr}^P| \]
and hence
\[ |x_{kr}^P| \leq \eta(1 + \eta)(1 - \eta^2)^{-1} |x_{kr}^P| \leq \delta |x_{kr}^P|. \]
Since $\delta > 0$ was arbitrary, this proves (3.47).

Now, using (3.47), we prove (3.43). Applying the projection onto $P_1$ in the variation of constants formula, we obtain for $k \geq \sigma_0/r$ and $t \geq kr$,

$$x_t^P = T(t - kr)x_{kr}^P + \int_{kr}^{t} T(t - s)X_0^P f(s, x_s) \, ds$$

(3.50)

and hence (see (2.2) and (2.19))

$$|x_t^P| \leq M_1e^{(d+\varepsilon)(t-kr)}|x_{kr}^P| + M_3 \int_{kr}^{t} e^{(d+\varepsilon)(t-s)}\gamma(s)|x_s| \, ds.$$  

(3.51)

From this, using estimate (3.4) of Lemma 3.2, we find for $k \geq \sigma_0/r$ and $kr \leq t \leq (k+1)r$,

$$|x_t^P| \leq K_1|x_{kr}^P| + K_2\gamma_k|x_{kr}|,$$

(3.52)

where $K_1 = M_1 \max_{0 \leq \tau \leq r} e^{(d+\varepsilon)\tau}$ and $K_2 = M_3 C_3 \max_{0 \leq \tau \leq r} e^{(d+\varepsilon)\tau}$. By virtue of (3.47), we have that

$$x_{kr} = x_{kr}^0 + x_{kr}^P = O\left(|x_{kr}^0|\right) \quad \text{as } k \to \infty.$$  

This and (3.46) yield

$$x_{kr} = x_{kr}^0 + x_{kr}^P = O\left(|x_{kr}^0|\right) = O\left(|x_{kr}|\right) \quad \text{as } k \to \infty.$$  

Using the last asymptotic relation in (3.52), we obtain the existence of a positive constant $K_3$ such that for all large $k$ and $kr \leq t \leq (k+1)r$,

$$|x_t^P| \leq K_1|x_{kr}^P| + K_3\gamma_k|x_{kr}|.$$  

(3.53)

It follows by a similar argument as in the proof of (3.14) that if $\varepsilon > 0$ is sufficiently small, then

$$|T(t)\phi^P| \geq K_4 e^{(\mu-\varepsilon)t} |\phi^P|, \quad t \geq 0, \ \phi \in C$$

with a suitable positive constant $K_4 = K_4(\varepsilon)$. By replacing $P_1$ with $P_0$ in formula (3.50), applying the last estimate to the resulting integral equation for $x_t^P$ and using a similar argument as in the proof of (3.53), we obtain for all large $k$ and $kr \leq t \leq (k+1)r$,

$$|x_t^P| \geq K_5|x_{kr}^P| - K_6\gamma_k|x_{kr}|.$$  

(3.54)

where $K_5 = K_4 \min_{0 \leq \tau \leq r} e^{(\mu-\varepsilon)\tau}$ and $K_6$ is a suitable positive constant. From (3.53) and (3.54), we find for all large $k$ and $kr \leq t \leq (k+1)r$,

$$\frac{|x_t^P|}{|x_t^P|} \leq \frac{K_1}{K_5 - K_6\gamma_k} \frac{|x_{kr}^P|}{|x_{kr}|} + \frac{K_3\gamma_k}{K_5 - K_6\gamma_k}.$$  

(3.53)

Letting $k \to \infty$ in the last inequality and using (3.24) and (3.47), we conclude that (3.43) holds.
It remains to show (3.44). Let \( \delta > 0 \) be given. Choose \( \eta > 0 \) such that \( \eta(1 + \eta) < \delta \). By virtue of (3.43) and (3.46), we have that \( |x^Q_t| \leq \eta|x^P_t| \) and \( |x^P_t| \leq \eta|x^0_t| \) for all sufficiently large \( t \). Consequently, for all large \( t \),

\[
|x^Q_t| \leq \eta |x^P_t| = \eta |x^0_t + x^P_t| \leq \eta |x^0_t| + \eta |x^P_t| \leq \eta |x^0_t| + \eta^2 |x^P_t| \leq \delta |x^P_t|
\]

by the choice of \( \delta \). Since \( \delta > 0 \) was arbitrary, this implies (3.44).

We conclude this section with one more useful property of the solutions of (1.1) with finite Lyapunov exponents.

**Proposition 3.5.** Let \( x \) be a solution of (1.1) satisfying the hypotheses of Theorem 3.1 with a finite strict Lyapunov exponent \( \mu(x) = \mu \). Then there exists \( \delta > 0 \) such that

\[
\left| x_{(k+1)r} \right| \geq \delta, \quad t \geq \sigma_0.
\]

**(Proof.** As shown in the proof of Theorem 3.1, \( |x_t| > 0 \) for \( t \geq \sigma_0 \). Choose \( c < \mu \) such that (1.2) has no eigenvalue on the vertical line \( \Re \lambda = c \). Then

\[
\mu = \mu(x) = \liminf_{t \to \infty} \log \frac{|x_t|}{t} > c.
\]

By Lemma 3.3, the asymptotic relation (3.8) and hence (3.26) holds for the (equivalent) norm defined by (3.9). (We use the notation \( P = P_{\Lambda(c)} \) and \( Q = Q_{\Lambda(c)} \) from Lemma 3.3.) Using (3.26) in (3.21), we find for all large \( k \),

\[
\| x^P_{(k+1)r} \| = (\alpha - 2D\gamma_k) \| x^P_{kr} \|.
\]

By virtue of (3.8) and (3.11), we have for all large \( k \),

\[
\| x^P_{kr} \| = \left( \| x^Q_{kr} \| + \| x^P_{kr} \| \right) = \| x^Q_{kr} \| + 1 \to 1 \quad \text{as} \ k \to \infty.
\]

From this, (3.24) and (3.56), we obtain

\[
\liminf_{t \to \infty} \frac{\| x_{(k+1)r} \|}{\| x_{kr} \|} = \liminf_{t \to \infty} \frac{\| x^P_{(k+1)r} \|}{\| x^P_{kr} \|} \geq \alpha,
\]

which, together with (3.10), implies that

\[
\liminf_{t \to \infty} \frac{|x_{(k+1)r}|}{|x_{kr}|} \geq q^{-1} \alpha.
\]

Consequently, if \( \rho \in (0, q^{-1} \alpha) \), there exists an integer \( k_0 \) such that

\[
\frac{|x_{(k+1)r}|}{|x_{kr}|} \geq \rho, \quad k \geq k_0.
\]

If \( t \geq k_0 r \) and \( k \geq k_0 \) is the unique integer such that \( kr \leq t < (k+1)r \), then by the application of Lemma 3.2, we conclude that

\[
\frac{|x_{t+r}|}{|x_t|} \geq C_3^{-1} \frac{|x_{(k+2)r}|}{C_3 |x_{kr}|} = C_3^{-2} \frac{|x_{(k+2)r}|}{C_3 |x_{kr}|} \frac{|x_{(k+1)r}|}{|x_{kr}|} \geq C_3^{-2} \rho^2,
\]
the last inequality being a consequence of (3.58). Consequently, (3.55) holds with

$$\delta = \min \left\{ C_3^{-2} \rho^2, \min_{\sigma_0 \leq i \leq k_0} \frac{|x_{t+i}|}{|x_t|} \right\}. \quad \Box$$

References