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The largest and the smallest fixed points of permutations

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ABSTRACT

We give a new interpretation of the derangement numbers d_n as the sum of the values of the largest fixed points of all non-derangements of length $n-1$. We also show that the analogous sum for the smallest fixed points equals the number of permutations of length n with at least two fixed points. We provide analytic and bijective proofs of both results, as well as a new recurrence for the derangement numbers.

1. Largest fixed point

Let $[n] = \{1, 2, \ldots, n\}$, and let $\mathcal{S}_n$ denote the set of permutations of $[n]$. Throughout the paper, we will represent permutations using cycle notation unless specifically stated otherwise. Recall that i is a fixed point of $\pi \in \mathcal{S}_n$ if $\pi(i) = i$. Denote by $\mathcal{D}_n$ the set of derangements of $[n]$, i.e., permutations with no fixed points, and let $d_n = |\mathcal{D}_n|$. Given $\pi \in \mathcal{S}_n \setminus \mathcal{D}_n$, let $\ell(\pi)$ denote the largest fixed point of π. Let

$$a_{n,k} = |\{\pi \in \mathcal{S}_n \setminus \mathcal{D}_n : \ell(\pi) = k\}|.$$

Clearly,

$$a_{n,1} = d_{n-1} \quad \text{and} \quad a_{n,n} = (n-1)!. \tag{1}$$

It also follows from the definition that

$$a_{n,k} = d_{n-1} + \sum_{j=1}^{k-1} a_{n-1,j}, \tag{2}$$

since by removing the largest fixed point k of a permutation in $\mathcal{S}_n \setminus \mathcal{D}_n$, we get a permutation of $\{1, \ldots, k-1, k+1, \ldots, n\}$ whose largest fixed point (if any) is less than k. If in (2) we replace k by

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Table 1
The values of $a_{n,k}$ for n up to 6.

$n \setminus k$	1	2	3	4	5	6
1	1					
2	0	1				
3	1	1	2			
4	2	3	4	6		
5	9	11	14	18	24	
6	44	53	64	78	96	120

$k-1$, then by subtraction we obtain

$$a_{n,k} = a_{n,k-1} + a_{n-1,k-1} \tag{3}$$

for $k \geq 2$, or equivalently, $a_{n,k} = a_{n,k+1} - a_{n-1,k}$ for $k \geq 1$. Together with the second equation in (1), it follows that the numbers $a_{n,k}$ form Euler's difference table of the factorials (see [2–4]). Table 1 shows the values of $a_{n,k}$ for small n. The combinatorial interpretation given in [2,3] is that $a_{n,k}$ is the number of permutations of $[n-1]$ where none of $k, k+1, \dots, n-1$ is a fixed point. This interpretation is clearly equivalent to ours using the same reasoning behind Eq. (2). In fact, Eqs. (1)–(3) can also be derived from [2].

We point out that it is possible to give a direct combinatorial proof of the recurrence (3) from our definition of the $a_{n,k}$. Indeed, let $\pi \in \mathcal{S}_n$ with $\ell(\pi) = k$. If $\pi(1) = m \neq 1$, then the permutation of $[n]$ obtained from the one-line notation of π by moving m to the end, replacing 1 with $n+1$, and subtracting one from all the entries has largest fixed point $k-1$. If $\pi(1) = 1$, then removing 1 and subtracting one from the remaining entries of π we get a permutation of $[n-1]$ whose largest fixed point is $k-1$.

From (1) and (3) it follows that

$$a_{n,k} = \sum_{j=0}^{k-1} \binom{k-1}{j} d_{n-j-1}.$$

A simple combinatorial proof of this equation is obtained by observing that the number of permutations $\pi \in \mathcal{S}_n \setminus \mathcal{D}_n$ with $\ell(\pi) = k$ having exactly j additional fixed points is $\binom{k-1}{j} d_{n-j-1}$, for each $0 \leq j \leq k-1$.

Define

$$\alpha_n = \sum_{k=1}^{n} k a_{n,k} = \sum_{\pi \in \mathcal{S}_n \setminus \mathcal{D}_n} \ell(\pi). \tag{4}$$

We now state our main result, which we prove analytically and bijectively in the next two subsections.

Theorem 1.1. *For $n \geq 1$, we have*

$$\alpha_n = d_{n+1}.$$

1.1. Analytic proof

Replacing n by $n+1$, from (4) we have

$$\alpha_{n+1} = a_{n+1,1} + 2a_{n+1,2} + \cdots + n a_{n+1,n} + (n+1) a_{n+1,n+1}. \tag{5}$$

Adding (4) and (5) and taking into account (3), we obtain

$$\alpha_n + \alpha_{n+1} = a_{n+1,2} + 2a_{n+1,3} + \cdots + n a_{n+1,n+1} + (n+1)!. \tag{6}$$

Adding (6) with the equality

$$(n+1)! - d_{n+1} = a_{n+1,1} + a_{n+1,2} + \cdots + a_{n+1,n} + a_{n+1,n+1},$$

which is a special case of Eq. (2), we obtain

$$\alpha_n + \alpha_{n+1} + (n+1)! - d_{n+1} = \alpha_{n+1} + (n+1)!,$$

whence $\alpha_n = d_{n+1}$.

1.2. Bijective proof

To find a bijective proof of Theorem 1.1, we first construct a set whose cardinality is α_n. Let $\mathcal{M}_n \subset (\mathcal{S}_n \setminus \mathcal{D}_n) \times [n]$ be the set of pairs (π, i) where $\pi \in \mathcal{S}_n \setminus \mathcal{D}_n$ and $i \le \ell(\pi)$. We underline the number i in π to indicate that it is marked. For example, we write $(2)(3)(7)(8)(1, \underline{4}, 9)(5, 6)$ instead of the pair $((2)(3)(7)(8)(1, 4, 9)(5, 6), 4)$. It is clear that

$$|\mathcal{M}_n| = \sum_{k=1}^{n} k a_{n,k} = \alpha_n.$$

To prove Theorem 1.1, we give a bijection between $\mathcal{D}_{n+1}$ and $\mathcal{M}_n$.

Given $\pi \in \mathcal{D}_{n+1}$, we assign to it an element $\widehat{\pi} \in \mathcal{M}_n$ as follows. Write π as a product of cycles, starting with the one containing $n+1$, say

$$\pi = (n+1, i_1, i_2, \ldots, i_r)\, \sigma.$$

Let q be the largest index, $1 \le q \le r$, such that $i_1 < i_2 < \cdots < i_q$. We define

$$\widehat{\pi} = \begin{cases} (i_1)(i_2)\ldots(\underline{i_r})\, \sigma & \text{if } q = r, \\ (i_1)(i_2)\ldots(i_q)(\underline{i_{q+1}}, i_{q+2}, \ldots, i_r)\, \sigma & \text{if } q < r. \end{cases}$$

Now we describe the inverse map. Given $\widehat{\pi} \in \mathcal{M}_n$, let its unmarked fixed points be $i_1 < i_2 < \cdots < i_q$, and let j_1 be the marked element. We can write $\widehat{\pi} = (i_1)\ldots(i_q)(\underline{j_1}, j_2, \ldots, j_t)\, \sigma$. Notice that $t = 1$ if the marked element is a fixed point. Define

$$\pi = (n+1, i_1, i_2, \ldots, i_q, j_1, j_2, \ldots, j_t)\, \sigma.$$

Here are some examples of the bijection between $\mathcal{D}_{n+1}$ and $\mathcal{M}_n$:

$$\pi = (12, 2, 4, 9, 7, 5, 6)(1, 3)(8, 11, 10) \leftrightarrow \widehat{\pi} = (2)(4)(9)(\underline{7}, 5, 6)(1, 3)(8, 11, 10),$$
$$\pi = (10, 2, 3, 7, 8)(1, 4, 9)(5, 6) \leftrightarrow \widehat{\pi} = (2)(3)(7)(\underline{8})(1, 4, 9)(5, 6),$$
$$\pi = (10, 2, 7, 8, 3)(1, 4, 9)(5, 6) \leftrightarrow \widehat{\pi} = (2)(7)(8)(\underline{3})(1, 4, 9)(5, 6),$$
$$\pi = (10, 2, 3, 7, 8, 4, 9, 1)(5, 6) \leftrightarrow \widehat{\pi} = (2)(3)(7)(8)(\underline{4}, 9, 1)(5, 6).$$

2. Smallest fixed point

In a symmetric fashion to the statistic $\ell(\pi)$, we can define $s(\pi)$ to be the smallest fixed point of $\pi \in \mathcal{S}_n \setminus \mathcal{D}_n$. Let

$$b_{n,k} = |\{\pi \in \mathcal{S}_n \setminus \mathcal{D}_n : s(\pi) = k\}|.$$

The numbers $b_{n,k}$ appear in [1, pp. 174–176, 185] as $R_{n,k}$ (called *rank*). Define

$$\beta_n = \sum_{k=1}^{n} k b_{n,k} = \sum_{\pi \in \mathcal{S}_n \setminus \mathcal{D}_n} s(\pi). \tag{7}$$

It is not hard to see that, by symmetry,

$$b_{n,k} = a_{n,n+1-k}. \tag{8}$$

Indeed, one can use the involution $\pi \mapsto \pi'$ on S_n where $\pi'(i) = n+1-\pi(n+1-i)$. This involution is equivalent to replacing each entry i in the cycle representation of π with $n+1-i$; for example, (183)(2)(4975)(6) is mapped to (927)(8)(6135)(4).

To find a combinatorial interpretation of β_n, let $\mathcal{E}_{n+1}$ be the set of permutations of $[n+1]$ that have at least two fixed points. We have that

$$|\mathcal{E}_{n+1}| = (n+1)! - d_{n+1} - (n+1)d_n, \tag{9}$$

since out of the $(n+1)!$ permutations of $[n+1]$, there are d_{n+1} derangements and $(n+1)d_n$ permutations having exactly one fixed point.

The following result is the analogue of Theorem 1.1 for the statistic $s(\pi)$. We give an analytic proof based on that theorem, and a direct bijective proof as well.

Theorem 2.1. *For $n \geq 1$, we have*

$$\beta_n = |\mathcal{E}_{n+1}|.$$

2.1. Analytic proof

From the definitions of α_n and β_n, and Eq. (8), it follows that

$$\alpha_n + \beta_n = (n+1)\sum_{k=1}^{n} a_{n,k} = (n+1)(n! - d_n).$$

Using Theorem 1.1, we have

$$\beta_n = (n+1)! - (n+1)d_n - d_{n+1},$$

which by (9) is just the cardinality of $\mathcal{E}_{n+1}$ as claimed.

Note also the following identities involving β_n which follow from the known recurrence $d_n = nd_{n-1} + (-1)^n$:

$$\beta_n = (n+1)! + (-1)^n - 2(n+1)d_n,$$
$$\beta_n = (n+1)\beta_{n-1} + n(-1)^{n+1}.$$

The sequence β_n starts 0, 1, 1, 7, 31, 191, …. Using the well known fact that

$$\lim_{n\to\infty} \frac{d_n}{n!} = \frac{1}{e}, \tag{10}$$

we see that

$$\lim_{n\to\infty} \frac{\beta_n}{(n+1)!} = 1 - \frac{2}{e}.$$

2.2. Bijective proof

Let $\mathcal{M}'_n \subset (\mathcal{S}_n \setminus \mathcal{D}_n) \times [n]$ be the set of pairs (π, i) where $\pi \in \mathcal{S}_n \setminus \mathcal{D}_n$ and $i \leq s(\pi)$. As before, we underline the number i in π to indicate that it is marked. It is clear that

$$|\mathcal{M}'_n| = \sum_{k=1}^{n} k b_{n,k}.$$

We now give a bijection between $\mathcal{E}_{n+1}$ and $\mathcal{M}'_n$. Given $\pi \in \mathcal{E}_{n+1}$, let i be its smallest fixed point. We can write

$$\pi = (i)(n+1, j_2, \ldots, j_t)\,\sigma,$$

where no js appear if $n+1$ is a fixed point. Define

$$\widetilde{\pi} = (\underline{i}, j_2, \ldots, j_t)\,\sigma.$$

Note that $\widetilde{\pi} \in \mathcal{M}'_n$, because if σ has fixed points then they are all larger than i, and if it does not, then $t = 1$ and i is the smallest fixed point of $\widetilde{\pi}$. Essentially, π and $\widetilde{\pi}$ are related by conjugation by the transposition $(i, n+1)$.

Conversely, given $\widetilde{\pi} \in \mathcal{M}'_n$, let i be the marked entry. We can write

$$\widetilde{\pi} = (\underline{i}, j_2, \ldots, j_t)\,\sigma,$$

where no js appear if i is a fixed point. Then

$$\pi = (i)(n+1, j_2, \ldots, j_t)\,\sigma.$$

Roughly speaking, we replace $\underline{i}$ with $n+1$ and add i as a fixed point. Note that if $t \geq 2$ then σ must have fixed points.

Here are some examples of the bijection between $\mathcal{E}_{n+1}$ and $\mathcal{M}'_n$:

$$\pi = (3)(10, 1, 7, 2, 8)(5)(6)(4, 9) \leftrightarrow \widetilde{\pi} = (\underline{3}, 1, 7, 2, 8)(5)(6)(4, 9),$$
$$\pi = (5)(10)(6)(3, 1, 7, 2, 8)(4, 9) \leftrightarrow \widetilde{\pi} = (\underline{5})(6)(3, 1, 7, 2, 8)(4, 9).$$

3. Other remarks

3.1. A recurrence for the derangement numbers

An argument similar to the bijective proof of Theorem 1.1 can be used to prove the recurrence

$$d_n = \sum_{j=2}^{n}(j-1)\binom{n}{j}d_{n-j} \tag{11}$$

combinatorially as follows.

A derangement $\pi \in \mathcal{D}_n$ can be written as a product of cycles, starting with the one containing n, say

$$\pi = (n, i_1, i_2, \dots, i_r)\,\sigma.$$

Consider two cases:

- If $i_1 < i_2 < \cdots < i_{r-1}$ (this is vacuously true for $r = 1, 2$), then the number of choices for the numbers $i_1, \dots, i_r$ satisfying this condition is $r\binom{n-1}{r}$, since we can first choose an r-subset of $[n-1]$ and then decide which one is i_r. Now, the number of choices for σ is d_{n-r-1}.
- Otherwise, there is an index $1 \leq q \leq r-1$ such that $i_1 < i_2 < \cdots < i_q > i_{q+1}$. In this case, there are $q\binom{n-1}{q+1}$ choices for the numbers $i_1, \dots, i_{q+1}$, since we can first choose a $(q+1)$-subset of $[n-1]$ and then decide which element other than the maximum is i_{q+1}. Now, there are d_{n-q-1} choices for $(i_{q+1}, \dots, i_r)\,\sigma$.

The total number of choices is

$$\sum_{r=1}^{n-1} r\binom{n-1}{r}d_{n-r-1} + \sum_{q=1}^{n-1} q\binom{n-1}{q+1}d_{n-q-1} = \sum_{r=1}^{n-1} r\left(\binom{n-1}{r} + \binom{n-1}{r+1}\right)d_{n-r-1}$$
$$= \sum_{r=1}^{n-1} r\binom{n}{r+1}d_{n-r-1},$$

which equals the right hand side of (11).

Alternatively, the recurrence (11) is relatively straightforward to prove using generating functions. Indeed, let

$$D(x) = \sum_{n\geq 0} d_n \frac{x^n}{n!} = \frac{e^{-x}}{1-x}$$

be the generating function for the number of derangements. The generating function for the right hand side of (11), starting from $n = 1$, is

$$\sum_{n\geq 1}\sum_{j=2}^{n}(j-1)\binom{n}{j}d_{n-j}\frac{x^n}{n!} = \left(\sum_{i\geq 0} d_i \frac{x^i}{i!}\right)\left(\sum_{j\geq 1}(j-1)\frac{x^j}{j!}\right)$$
$$= \frac{e^{-x}}{1-x}(xe^x - e^x + 1) = -1 + \frac{e^{-x}}{1-x} = D(x) - 1.$$

3.2. *Another interpretation of* d_n

As conjectured by an anonymous referee, the derangement numbers also count permutations whose largest fixed point is greater than the first entry. We give a bijective proof of this fact.

Proposition 3.1. *For* $n \geq 1$, *we have*

$$d_n = |\{\pi \in \mathcal{S}_n \setminus \mathcal{D}_n : \ell(\pi) > \pi(1)\}|.$$

Proof. In Section 1.2 we gave a bijection between $\mathcal{D}_n$ and $\mathcal{M}_{n-1}$. Here we describe a bijection between $L_n = \{\pi \in \mathcal{S}_n \setminus \mathcal{D}_n : \ell(\pi) > \pi(1)\}$ and $\mathcal{M}_{n-1}$.

Given $\pi \in L_n$, delete $\pi(1)$ in the one-line notation $\pi(1)\pi(2)\dots\pi(n)$ and decrease by one all the entries greater than $\pi(1)$. This yields the one-line notation of a permutation $\sigma \in \mathcal{S}_{n-1} \setminus \mathcal{D}_{n-1}$. We define the image of π to be the pair $(\sigma, \pi(1)) \in \mathcal{M}_{n-1}$. Conversely, given $(\sigma, i) \in \mathcal{M}_{n-1}$, increase by one the entries greater than or equal to i in $\sigma(1)\sigma(2)\dots\sigma(n-1)$ and insert i at the beginning, recovering the one-line notation of $\pi \in L_n$. □

3.3. *Probabilistic interpretation of* Theorem 1.1

Let X_n be the random variable that gives the value of the largest fixed point of a random element of $\mathcal{S}_n \setminus \mathcal{D}_n$. Its expected value is then

$$E[X_n] = \frac{\sum_{k=1}^{n} k a_{n,k}}{|\mathcal{S}_n \setminus \mathcal{D}_n|}.$$

Theorem 1.1 is equivalent to the fact that

$$E[X_n] = \frac{d_{n+1}}{n! - d_n}. \tag{12}$$

Using (10), we get from Eq. (12) that

$$\lim_{n\to\infty} \frac{E[X_n]}{n} = \frac{1}{e-1}. \tag{13}$$

Occurrences of fixed points in a random permutation of $[n]$, normalized by dividing by n, approach a Poisson process in the interval $[0, 1]$ with mean 1 as n goes to infinity. An interpretation of Eq. (13) is that, in such a Poisson process, if we condition on the fact that there is at least one occurrence, then the largest event occurs at $1/(e-1)$ on average.

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