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Shuffles of copulas

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ABSTRACT

We show that every copula that is a shuffle of Min is a special push-forward of the doubly stochastic measure induced by the copula M. This fact allows to generalize the notion of shuffle by replacing the measure induced by M with an arbitrary doubly stochastic measure, and, hence, the copula M by any copula C.

1. Introduction

Let $\mathbb{I}$ be the closed unit interval $[0,1]$ and let $\mathbb{I}^2 = \mathbb{I} \times \mathbb{I}$ be the unit square. A *two-dimensional copula* (or a *copula* for brevity) is a distribution function $C : \mathbb{I}^2 \to \mathbb{I}$ whose univariate margins are distributed uniformly on $\mathbb{I}$ [16]. During the last few years, copulas have been widely studied because of their connections with, for instance, Markov operators, doubly stochastic measures and mass transportation theory [3,7,11,12,14,18,20]. However, the primary importance of this notion lies in probability theory and statistics. In fact, by *Sklar's theorem* [21] copulas link joint distribution functions to their one-dimensional margins in the sense that, if X and Y are random variables with individual distribution functions F_X and F_Y and joint distribution function H_{XY}, then there exists a copula C_{XY} such that $H_{XY}(x, y) = C_{XY}(F_X(x), F_Y(y))$. Moreover, C_{XY} is uniquely determined on $\operatorname{Ran}(F_X) \times \operatorname{Ran}(F_Y)$, and, hence, unique when X and Y are continuous.

The copulas Π, M and W, given respectively by $\Pi(u, v) = uv$, $M(u, v) = \min\{u, v\}$ and $W(u, v) = \max\{u + v - 1, 0\}$ are of particular importance. For continuous random variables X and Y, $C_{XY} = \Pi$ if, and only if, X and Y are independent, and $C_{XY} = M$ (respectively, $C_{XY} = W$) if, and only if, each of X, Y is almost surely an increasing (respectively, decreasing) function of the other one. The copula Π distributes the probability mass uniformly on the unit square, while the copulas M and W distribute it on the segment joining the points $(0, 0)$ with $(1, 1)$ and on the segment joining $(0, 1)$ with $(1, 0)$, respectively.

In this paper, the notion of shuffle of Min is reconsidered (Section 3), by describing it in terms of measure-preserving transformations of $\mathbb{I}$ and push-forward of the doubly stochastic measure induced by the copula M. Then, the concept of shuffle of an arbitrary copula C is introduced and investigated (Section 4). In particular, the rôle of the copula Π will be analysed in detail (Section 5).

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2. Preliminaries

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, let $(\Omega_1, \mathcal{F}_1)$ be a measurable space and let $f : \Omega \to \Omega_1$ be a measurable function. We recall that a *push-forward* (also called an *image measure*) of μ under f is a set function $f * \mu$ defined, for every $A \in \mathcal{F}_1$, by

$$(f * \mu)(A) = \mu\big(f^{-1}(A)\big). \tag{2.1}$$

Obviously, the push-forward $f * \mu$ is a measure on $\mathcal{F}_1$. Moreover, if μ is a probability, then so is $f * \mu$. If $(\Omega_2, \mathcal{F}_2)$ is a third measurable space and $g : \Omega_1 \to \Omega_2$ another measurable function, then

$$(g \circ f) * \mu = g * (f * \mu). \tag{2.2}$$

We recall that a *measure-preserving transformation from* the measure space $(\Omega, \mathcal{F}, \mu)$ *to* the measure space $(\Omega_1, \mathcal{F}_1, \nu)$ is a measurable map $f : \Omega \to \Omega_1$ such that $f * \mu = \nu$, or, equivalently, such that $\mu \circ f^{-1} = \nu$. If the two measure spaces coincide, we speak about a measure-preserving transformation *of* the corresponding space. Throughout this paper, the σ-algebra of Borel subsets of $\mathbb{I}$ is denoted $\mathcal{B}(\mathbb{I})$ and λ stands for the restriction of the Lebesgue measure to this σ-algebra. A *permutation* of a set S is any bijective mapping from S onto itself. We denote by $\mathcal{T}$ the set of all measure-preserving transformations of the measure space $(\mathbb{I}, \mathcal{B}(\mathbb{I}), \lambda)$ and by $\mathcal{T}_{\mathrm{p}}$ the set of all measure-preserving permutations (automorphisms) of that space. The set $\mathcal{T}$ equipped with the composition of mappings is a semigroup and $\mathcal{T}_{\mathrm{p}}$ is a subgroup of $\mathcal{T}$.

An important subclass of $\mathcal{T}_{\mathrm{p}}$ is that of the well-known *interval exchange transformations* [2]. The following particular class of such transformations will be useful for our purposes. Let $\bar{n} = \{1, 2, \ldots, n\}$ and let $\{J_{1,i}\}_{i \in \bar{n}}$ be a collection of disjoint non-degenerate intervals $J_{1,i} = [a_{1,i}, b_{1,i}[$ in $\mathbb{I}$ for all $i \in \overline{n-1}$ and the singleton $J_{1,n} = \{1\}$. Let $\{J_{2,i}\}_{i \in \bar{n}}$ be another such collection and suppose that $\lambda(J_{1,i}) = \lambda(J_{2,i})$ for all $i \in \bar{n}$. Then one among the interval exchange transformations which map $J_{1,i}$ linearly onto $J_{2,i}$ for all $i \in \overline{n-1}$ is given, for all $x \in \mathbb{I}$, by

$$T(x) = \begin{cases} x - a_{1,1} + a_{2,1} & \text{if } x \in J_{1,i,}, \\ \lambda((\mathbb{I} \setminus \bigcup_{i \in \bar{n}} J_{1,i}) \cap [0, x]) + \sum_{i \in \bar{n}} (b_{2,i} - a_{2,a}) \mathbf{1}_{[a_2,1]}(x) & \text{otherwise,} \end{cases} \tag{2.3}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Now we briefly recall some facts about copulas that will be used in the sequel; for more details we refer to the extensive treatments given by Schweizer and Sklar [19] and by Nelsen [16]. Given a copula C, a measure μ_C is defined on the semi-ring $\mathcal{R}$ of rectangles $[a, b] \times [c, d] \subseteq \mathbb{I}^2$ via

$$\mu_C\big([a, b] \times [c, d]\big) = C(b, d) - C(b, c) - C(a, d) + C(a, c).$$

Through standard measure-theoretic techniques, μ_C can be extended from the semi-ring $\mathcal{R}$ to the σ-algebra $\mathcal{B}(\mathbb{I}^2)$ of the Borel subsets of the unit square. For example, μ_Π is the restriction of the two-dimensional Lebesgue measure λ_2 to $\mathcal{B}(\mathbb{I}^2)$. For every copula C, the measure μ_C is *doubly stochastic* in the sense that $\mu_C(A \times \mathbb{I}) = \mu_C(\mathbb{I} \times A) = \lambda(A)$ for every Borel set $A \subseteq \mathbb{I}$. Conversely, to every doubly stochastic measure μ there corresponds a copula C given, for all u and v in $\mathbb{I}$, by

$$C(u, v) = \mu\big([0, u] \times [0, v]\big). \tag{2.4}$$

This one-to-one correspondence between copulas and doubly stochastic measures allows to translate some measure-theoretic concepts into the language of copulas. For example, a copula C is *absolutely continuous* if μ_C is absolutely continuous with respect to λ_2, that is $\lambda_2(A) = 0$ for a Borel set $A \subseteq \mathbb{I}^2$ implies $\mu_C(A) = 0$.

There is a correspondence between copulas and special random vectors. We say that a random vector (X, Y) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is distributed according to a copula C (and we write $(X, Y) \sim C$), whenever $(X, Y) * \mathbb{P} = \mu_C$. Conversely, given a copula C there exists a probability space and a random vector (X, Y) defined on it, such that $(X, Y) \sim C$. Obviously, if a random vector is distributed according to a copula, its components are uniformly distributed on $\mathbb{I}$ and vice versa.

Finally, we mention a correspondence between copulas and measure-preserving transformations of the unit interval [4,10,17,23]. For all $f, g \in \mathcal{T}$, a copula $C_{f,g}$ is defined, for all u and v in $\mathbb{I}$, via

$$C_{f,g}(u, v) = \lambda\big(f^{-1}[0, u] \cap g^{-1}[0, v]\big). \tag{2.5}$$

Conversely, for every copula C, there exist f and g in $\mathcal{T}$ such that $C = C_{f,g}$. This representation is not unique; for example, $C_{f,g} = C_{f \circ \varphi, g \circ \varphi}$ for every $\varphi \in \mathcal{T}$. Observe that this representation is just a special case of the correspondence mentioned in the previous paragraph as the pair (f, g) is always a random vector on the probability space $(\mathbb{I}, \mathcal{B}(\mathbb{I}), \lambda)$ provided $f, g \in \mathcal{T}$.

3. Shuffling the minimum copula

We present here the original definition of shuffle of Min [13, Definition 2.1].

Definition 1. A copula C is a *shuffle of Min* if there is a natural number n, two partitions

$$0 = s_0 < s_1 < \cdots < s_n = 1 \quad \text{and} \quad 0 = t_0 < t_1 < \cdots < t_n = 1$$

of $\mathbb{I}$, and a permutation σ of $\{1, \dots, n\}$ such that each $[s_{i-1}, s_i] \times [t_{\sigma(i)-1}, t_{\sigma(i)}]$ is a square in which C distributes a mass $s_i - s_{i-1}$ uniformly spread along one of the diagonals.

An interesting probabilistic interpretation of this construction has been provided in terms of piece-wise continuous bijections [13, Corollary 2.1 and remark] where a function is *piece-wise continuous* if it is defined on a non-degenerate interval and has at most *finitely* many discontinuities, all of them being jumps.

Theorem 2. *Let (X, Y) be a random vector distributed according to the copula C. Then the following statements are equivalent*:

(a) *C is a shuffle of Min*;
(b) *a bijective piece-wise continuous function f exists such that*

$$\mathbb{P}(Y = f \circ X) = 1.$$

Besides this characterization, also a geometric and intuitive way of constructing a shuffle of Min has been formulated [13]. Namely, a shuffle of Min is a copula whose mass distribution can be obtained performing the following procedure:

(1_{ShM}) placing the mass for the minimum copula M on the unit square $\mathbb{I}^2$;
(2_{ShM}) cutting the unit square into a finite number of strips;
(3_{ShM}) permuting ("shuffling") the strips with perhaps some of them flipped around their vertical axis of symmetry;
(4_{ShM}) reassembling the strips to reform the unit square.

The resulting mass distribution generates a shuffle of Min.

In what follows we characterize shuffles of Min in terms of push-forwards and provide a natural generalization. To this end, we introduce the notion of *shuffling*, which formalizes the construction (1_{ShM}–4_{ShM}) presented in Section 1.

Given a mapping $T : \mathbb{I} \to \mathbb{I}$ we define a map $S_T : \mathbb{I}^2 \to \mathbb{I}^2$ via

$$S_T(u, v) = \big(T(u), v\big) \tag{3.1}$$

for every $(u, v) \in \mathbb{I}^2$. Let J denote a (possibly degenerate) interval in $\mathbb{I}$. By the *vertical strip* (or simply *strip*) *with base* J we mean the set $J \times \mathbb{I}$. A *strip partition* is a partition of the unit square into, possibly infinitely many, vertical strips. A *shuffling* of a strip partition $\{J_i \times \mathbb{I}\}_{i \in \mathcal{I}}$ is any permutation S of the unit square that

(1_{Sh}) admits the representation $S = S_T$ for some permutation $T : \mathbb{I} \to \mathbb{I}$;
(2_{Sh}) is measure-preserving on the space $(\mathbb{I}^2, \mathcal{B}(\mathbb{I}^2), \lambda_2)$;
(3_{Sh}) the restriction $S|_{J_i \times \mathbb{I}}$ of S to every strip $J_i \times \mathbb{I}$ is continuous with respect to the standard product topology on $\mathbb{I}^2$.

Intuitively, shuffling is just a reordering of the strips. This feature is captured by the condition (1_{Sh}), which represents the shuffling by a single transformation T of the unit interval. Because of (2_{Sh}) the single strips maintain their measure after shuffling. Finally, condition (3_{Sh}) is just a technical tool for ensuring that, during shuffling, the integrity of strips is preserved.

Below in Theorem 4 we characterize shuffles of Min. A preliminary result will be needed first.

Lemma 3. *Let $T : \mathbb{I} \to \mathbb{I}$ be a Borel measurable mapping. Consider the push-forward of a doubly stochastic measure μ under S_T. Then the following statements are equivalent*:

(a) *$S_T * \mu$ is doubly stochastic,*
(b) *T is in $\mathcal{T}$.*

Proof. The measure μ is doubly stochastic by assumption. For every Borel set $A \subseteq \mathbb{I}$ we have

$$S_T * \mu(\mathbb{I} \times A) = \mu\big(S_T^{-1}(\mathbb{I} \times A)\big) = \mu(\mathbb{I} \times A) = \lambda(A),$$
$$S_T * \mu(A \times \mathbb{I}) = \mu\big(S_T^{-1}(A \times \mathbb{I})\big) = \mu\big(T^{-1}(A) \times \mathbb{I}\big) = \lambda\big(T^{-1}(A)\big),$$

proving that the push-forward $S_T * \mu$ is doubly stochastic if, and only if, T is in $\mathcal{T}$. □

Theorem 4. *The following statements are equivalent*:

(a) *a copula C is a shuffle of Min*;
(b) *there exists a piece-wise continuous $T \in \mathcal{T}_p$ such that $\mu_C = S_T * \mu_M$.*

Proof. (b) ⇒ (a) Let Y be a random variable uniformly distributed on $\mathbb{I}$. As mentioned in Section 1, $(Y, Y) \sim M$ or, using the push-forward notation, $(Y, Y) * \mathbb{P} = \mu_M$. Thus, invoking (3.1) and (2.2) one can derive

$$(T \circ Y, Y) * \mathbb{P} = \big(S_T \circ (Y, Y)\big) * \mathbb{P} = S_T * \big((Y, Y) * \mathbb{P}\big) = S_T * \mu_M \tag{3.2}$$

for every measurable (but not necessarily measure-preserving) transformation T of the unit interval.

Now, let T be a piece-wise continuous function in $\mathcal{T}_p$. By Lemma 3, the measure $S_T * \mu_M$ is doubly stochastic, and, hence, corresponds to a copula C. By (3.2) the random vector $(T \circ Y, Y)$ is distributed according to C. Finally, by Theorem 2, C is a shuffle of Min.

(a) ⇒ (b) Let C be a shuffle of Min. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random vector (X, Y) defined on it such that $(X, Y) \sim C$. Moreover, by Theorem 2 there exists a bijective and piece-wise continuous function T for which $\mathbb{P}(X = T(Y)) = 1$. Such a T is easily seen to be Borel-measurable, and, as a consequence, also $(T \circ Y, Y)$ is a random vector. This random vector differs from (X, Y) on a set of zero $\mathbb{P}$-measure, which proves that

$$(T \circ Y, Y) * \mathbb{P} = (X, Y) * \mathbb{P} = \mu_C.$$

Thus, invoking (3.2) allows to derive the representation $\mu_C = S_T * \mu_M$.

In order to conclude the proof, it is enough to note that T is measure-preserving as a consequence of Lemma 3. □

Theorem 4 suggests how to generalize the definition of a shuffle of Min.

Definition 5. A copula C is a *generalized shuffle of Min* if $\mu_C = S_T * \mu_M$ for some $T \in \mathcal{T}_p$. Such a shuffle of Min is denoted by M_T.

In this definition, T is allowed to have *infinitely* many discontinuity points, which is a quite natural generalization of the original notion of shuffle of Min. We illustrate this fact by means of an example.

Example 6. For every $i \in \mathbb{N}$ define

$$J_{1,i} = \left[\frac{1}{i+1}, \frac{1}{i}\right[\quad \text{and} \quad J_{2,i} = \left[1 - \frac{1}{i}, 1 - \frac{1}{i+1}\right[.$$

Clearly, the indexed collections $\{J_{1,i}\}_{i\in\mathbb{N}}$ and $\{J_{2,i}\}_{i\in\mathbb{N}}$ consist of nonoverlapping intervals and $\lambda(J_{1,i}) = \lambda(J_{2,i})$ for every natural i. Let $\widetilde{T}$ be the interval exchange transformation given by (2.3). Also $T(x) = \widetilde{T}(1 - x)$ belongs to $\mathcal{T}_p$. Indeed, T is a composition of $\widetilde{T}$ and $x \mapsto 1 - x$, which are both in $\mathcal{T}_p$. Clearly, T has countably many discontinuities. Therefore, M_T is a shuffle of Min in the sense of Definition 5. Closer inspection reveals that

$$M_T = \left(\left\langle \frac{1}{i+1}, \frac{1}{i}, W \right\rangle\right)_{i\in\mathbb{N}},$$

so that M_T is an ordinal sum of countably many copies of W, proving that M_T is not a shuffle of Min in the sense of Definition 1. □

Now, let $T \in \mathcal{T}_p$, and consider the shuffle $C = M_T$. Easy calculations show that, for every $(u, v) \in \mathbb{I}^2$

$$C(u, v) = \mu_C\big([0, u] \times [0, v]\big) = S_T * \mu_M\big([0, u] \times [0, v]\big) = \mu_M\big(T^{-1}[0, u] \times [0, v]\big) = \lambda\big(T^{-1}[0, u] \cap [0, v]\big).$$

Therefore, following the notation of (2.5), a copula C is a shuffle of Min if, and only if, $C = C_{T,\mathrm{id}_{\mathbb{I}}}$ for some $T \in \mathcal{T}_p$.

4. Shuffling an arbitrary copula

By Lemma 3 not only shuffling of μ_M but in fact shuffling of *every* doubly stochastic measure leads again to a doubly stochastic measure. This suggests to generalize the idea of shuffle of Min by replacing M with an arbitrary copula C.

Definition 7. Let C be a copula. A copula D is a *shuffle of C* if there exists $T \in \mathcal{T}_p$ such that $\mu_D = S_T * \mu_C$. In this case, D is also called the *T-shuffle of C* and denoted by C_T.

In terms of measure-preserving transformations, a shuffle of a copula may be represented in the following way.

Theorem 8. *Let $C_{f,g}$ be a copula as in* (2.5) *for f and g in $\mathcal{T}$ and let $T \in \mathcal{T}_{\mathrm{p}}$. Then*

$$(C_{f,g})_T = C_{T\circ f,g}. \tag{4.1}$$

Proof. The proof will be carried out for the induced measures. One has

$$S_T * \mu_{C_{f,g}} = S_T * \big((f,g)*\lambda\big) = \big(S_T \circ (f,g)\big) * \lambda = (T \circ f, g) * \lambda = \mu_{C_{T\circ f,g}},$$

which concludes the proof. □

Example 9. Let $T \in \mathcal{T}_{\mathrm{p}}$ be the transformation given by (2.3) which interchanges the intervals $[0, \frac{1}{2}[$ and $[\frac{1}{2}, 1[$. Then, for every copula C,

$$C_T(u,v) = C\left(\min\left\{u + \frac{1}{2}, 1\right\}, v\right) - C\left(\frac{1}{2}, v\right) + C\left(\max\left\{u - \frac{1}{2}, 0\right\}, v\right).$$

For instance, if C is the arithmetic mean of M and W, its T-shuffle C_T is the copula of circular uniform distribution [16, Section 3.1.2]. This particular situation can be treated in a different way; it can be easily shown that the shuffle of a convex combination of two copulas is the convex combination of corresponding shuffles. Therefore the copula of the circular uniform distribution can be expressed as $\frac{1}{2}(M_T + W_T)$.

Observe that the mapping which assigns to every $T \in \mathcal{T}_{\mathrm{p}}$ and to every copula C the corresponding shuffle C_T defines an *action* of the group $\mathcal{T}_{\mathrm{p}}$ on the set of all copulas. The *orbit* of a copula C with respect to this action is the set $\mathcal{T}_{\mathrm{p}}(C) = \{C_T \mid T \in \mathcal{T}_{\mathrm{p}}\}$ constituted by all shuffles of C. The general theory of group actions guarantees that the classes of type $\mathcal{T}_{\mathrm{p}}(C)$ form a partition of the set of all copulas. A particular interesting result holds for the orbit of Π.

Theorem 10. *For a copula C the following statements are equivalent*:

(a) $C = \Pi$;
(b) $\mathcal{T}_{\mathrm{p}}(C) = \{C\}$.

As a consequence, Π is the unique copula invariant under any shuffling. Before proving this result, the following technical lemma will be needed.

Lemma 11. *Let $f : \mathbb{I} \to \mathbb{R}$ be a continuous function with $f(0) = 0$. If*

$$f(a_1) - f(a_2) = f(b_1) - f(b_2)$$

holds for any a_1, a_2, b_1, b_2 in $\mathbb{I}$ with $a_2 - a_1 = b_2 - b_1$, then there exists a positive constant v such that $f(x) = vx$.

Proof. It will be shown that f behaves like a linear function on the rationals of the unit interval. Since these are dense in $\mathbb{I}$ the assertion follows by the continuity of f.

Obviously

$$f\left(\frac{m}{n}\right) = f\left(\frac{m}{n}\right) - f\left(\frac{0}{n}\right) = \sum_{k=1}^{m}\left[f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right)\right] \tag{4.2}$$

for all natural numbers m and n with $m \leqslant n$. By assumption, all the terms of the sum are equal and, since $f(0) = 0$, we have $f(\frac{m}{n}) = mf(\frac{1}{n})$. The choice $m = n$ yields $v = nf(\frac{1}{n})$ or $\frac{v}{n} = f(\frac{1}{n})$, so that $f(\frac{m}{n}) = v\frac{m}{n}$ where $v = f(1)$. □

Proof of Theorem 10. (a) ⇒ (b) Every shuffle preserves the restriction λ_2 to $\mathcal{B}(\mathbb{I}^2)$ of the 2-dimensional Lebesgue measure, which is μ_Π (recall (2_{Sh})). Therefore $S_T * \mu_\Pi = \mu_\Pi$ for every $T \in \mathcal{T}_{\mathrm{p}}$.

(b) ⇒ (a) Let the copula C be invariant under every $T \in \mathcal{T}_{\mathrm{p}}$. Then, for every $T \in \mathcal{T}_{\mathrm{p}}$ and for every $R = [u_1, u_2] \times [v_1, v_2] \subseteq \mathbb{I}^2$,

$$S_T * \mu_C(R) = \mu_C\big(T^{-1}[u_1, u_2] \times [v_1, v_2]\big) = \mu_C(R).$$

Now, fix $v \in [0,1]$ and consider the v-horizontal section φ_v of C, that is $\varphi_v(t) : [0,1] \to [0,v]$ defined by $\varphi_v(t) = C(t,v)$. As a consequence of the properties of a copula, the function φ_v is increasing, continuous and satisfies $\varphi_v(0) = 0$ and $\varphi_v(1) = v$. For all u_1 and u_2 in $\mathbb{I}$ with $u_1 \leqslant u_2$, one has

$$\mu_C\big([u_1, u_2] \times [0, v]\big) = \varphi_v(u_2) - \varphi_v(u_1).$$

For every v_1 and v_2 in $\mathbb{I}$ such that $v_2 - v_1 = u_2 - u_1$, let T be an interval exchange transformation that sends $[v_1, v_2]$ into $[u_1, u_2]$ (for example, that one given by (2.3)). Thus, one has

$$\varphi_v(u_2) - \varphi_v(u_1) = \mu_C\big([u_1, u_2] \times [0, v]\big) = \mu_C\big(T^{-1}[u_1, u_2] \times [0, v]\big) = \mu_C\big([v_1, v_2] \times [0, v]\big) = \varphi_v(v_2) - \varphi_v(v_1),$$

whence, $\varphi_v(a_2) - \varphi_v(a_1) = \varphi_v(b_2) - \varphi_v(b_1)$ for all a_1, a_2, b_1 and b_2 in $[0, 1]$ with $a_2 - a_1 = b_2 - b_1$. Therefore, in view of Lemma 11, φ_v is linear and satisfies $\varphi_v(u) = uv$; as a consequence, $C = \Pi$. □

Notice that, in general, the orbit of an absolutely continuous copula contains just absolutely continuous elements, in view of the following result.

Proposition 12. *If C is absolutely continuous then so are all its shuffles.*

Proof. Let C be an absolutely continuous copula, let T belong to $\mathcal{T}_p$ and let A be a Borel set of the unit square with $\lambda_2(A) = 0$. Then

$$\lambda_2\big(S_T^{-1}(A)\big) = S_T * \lambda_2(A) = \lambda_2(A) = 0$$

and, by the absolute continuity of C,

$$S_T * \mu_C(A) = \mu_C\big(S_T^{-1}(A)\big) = 0.$$

Thus $S_T * \mu_C$ is absolutely continuous, as asserted. □

A copula is said to be *symmetric* if $C(u, v) = C(v, u)$ for every u, v in $\mathbb{I}$. Several recent investigations have been concerned with the construction of (absolutely continuous) copulas that are not symmetric [5,6,9,15]. Clearly, one cannot expect that symmetry is preserved under shuffling. More can be said in this respect.

Theorem 13. *Every copula C other than Π has a non-symmetric shuffle.*

Proof. We define $\Delta : \mathbb{I}^2 \to \mathbb{I}^2$ to be the reflection of the unit square in its main diagonal, i.e., $\Delta(x, y) = (y, x)$. Observe, that the symmetry of a copula C is equivalent to $\mu_C(A) = \mu_C(\Delta(A))$ for every Borel set $A \subseteq \mathbb{I}^2$. Let J_{1x}, J_{1y}, J_{2x}, J_{2y} be arbitrary but fixed intervals of type $[a, b[\subseteq \mathbb{I}$ with

$$J_{1x} \cap J_{2x} = J_{1y} \cap J_{2y} = \emptyset \quad \text{and} \quad \lambda(J_{1x}) = \lambda(J_{1y}) = \lambda(J_{2x}) = \lambda(J_{2y}).$$

Define the squares $R_1 = J_{2x} \times J_{1y}$, $R_2 = J_{1x} \times J_{2y}$, and $R_* = J_{2y} \times J_{1y}$. Further, let T be the interval exchange transformation defined by (2.3) which sends J_{ix} onto J_{iy} for $i = 1, 2$. Observe that

$$S_T^{-1}(R_*) = R_1 \quad \text{and} \quad S_T^{-1}\big(\Delta(R_*)\big) = R_2.$$

Let C be a copula such that every shuffle of C is symmetric. Then, we have

$$\mu_C(R_1) = S_T * \mu_C(R_*) = S_T * \mu_C\big(\Delta(R_*)\big) = \mu_C(R_2).$$

To summarize it, if two squares of the same size have disjoint projections along the x- and the y-axis, then they are of the same C-measure.

Now, let us fix a natural number $n \geqslant 3$ and define, for $i, j \in \{0, 1, \ldots, n-1\}$,

$$I_n = \left\{ \frac{m}{n} \;\middle|\; m = 0, 1, 2, \ldots, n \right\} \quad \text{and} \quad R_{i,j} = \left[\frac{i}{n}, \frac{i+1}{n}\right[\times \left[\frac{j}{n}, \frac{j+1}{n}\right[.$$

According to the previous paragraph, $\mu_C(R_{i,j}) = \mu_C(R_{k,l})$ whenever $i \neq k$ and $j \neq l$. For $n \geqslant 3$, this is enough to conclude that all $R_{i,j}$ are of the same C-measure. Since the squares $R_{i,j}$ form a partition of the unit square, the C-measure of each of them is n^{-2}. As a consequence

$$C\left(\frac{i}{n}, \frac{j}{n}\right) = \mu_C\left(\left[0, \frac{i}{n}\right[\times \left[0, \frac{j}{n}\right[\right) = \frac{ij}{n^2}$$

which can be written alternatively as $C|_{I_n \times I_n} = \Pi|_{I_n \times I_n}$. As this can be proved for any arbitrary natural number $n \geqslant 3$, we have $C = \Pi$. □

5. Approximation of copulas by means of shuffles

Shuffles of Min are dense in the space of all copulas, endowed with the sup norm, or, equivalently, with the L^∞-norm [13]. We recall that pointwise and uniform convergence are equivalent in the class of all copulas. It follows at once that the joint distribution function of any pair of continuous random variables (X, Y) can be approximated uniformly (or, equivalently, in the L^∞-norm), by the joint distribution function of another such pair (U, V), for which $F_U = F_X$ and $F_V = F_Y$, but where each of U, V is now almost surely an invertible function of the other. In particular, any pair of independent random variables can be approximated by a pair of random variables that are functionally dependent. This intriguing fact was discovered earlier by Kimeldorf and Sampson [8] and, with a different technique, by Vitale [22].

Now, contrary to the case of shuffles of Min, it cannot be expected that, given any two copulas C_1 and C_2, $C_2 \neq M$, C_1 can be approximated by copulas in $\mathcal{T}_\mathrm{p}(C_2)$; to this end, it suffices to consider the case $C_2 = \Pi$. However, the following remarkable fact can be proved.

Theorem 14. *For every copula C, the independence copula Π can be approximated uniformly by elements of $\mathcal{T}_\mathrm{p}(C)$.*

The proof of the next result is based on ergodic theory and uses the following Lemma (derived from a characterization provided by Walters [24, remark after Theorem 1.23]). For this, we recall that a subset D of $\mathbb{Z}_+$ is said to be of *zero density* when $\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbf{1}_D(j) = 0$.

Lemma 15. *Let $(\Omega, \mathcal{F}, \nu)$ be a measure space and let $T : \Omega \to \Omega$ be a measure-preserving transformation. Suppose that T is* weakly mixing, *i.e.*

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \left|\nu\left(T^{-j}A \cap B\right) - \nu(A)\nu(B)\right| = 0$$

for all $A, B \in \mathcal{F}$. Then there exists a subset D of $\mathbb{Z}_+$ of density zero such that

$$\lim_{\substack{n\to\infty \\ n\notin D}} \int_\Omega \left(f \circ T^n\right)(x) g(x)\, d\nu = \int_\Omega f(x)\, d\nu \int_\Omega g(x)\, d\nu \tag{5.1}$$

for all real functions f and g in $L^2(\nu)$.

Proof of Theorem 14. Given a copula C and a Borel set $B \subseteq \mathbb{I}$, $\lambda(B) > 0$, define a measure $\mu_C^B : \mathcal{B}(\mathbb{I}) \to [0, \lambda(B)]$ via

$$\mu_C^B(A) = \mu_C(A \times B)$$

for every $A \in \mathcal{B}$. Clearly, μ_C^B is absolutely continuous with respect to λ, since μ_C is doubly stochastic and, hence, $\mu_C^B(A) \leqslant \lambda(A)$ for every $A \in \mathcal{B}$. By the Radon–Nikodym theorem there exists a function $f_C^B \in L^1(\lambda)$ (unique up to equivalences) such that

$$\mu_C^B(A) = \int_A f_C^B\, d\lambda$$

for every Borel set $A \subseteq \mathbb{I}$. Moreover, one has

$$\int_A f_C^B\, d\lambda = \mu_C^B(A) \leqslant \lambda(A) = \int_A d\lambda,$$

from which

$$\int_A \left(1 - f_C^B\right) d\lambda \geqslant 0$$

for every $A \in \mathcal{B}$. The arbitrariness of $A \in \mathcal{B}$ implies $f_C^B \leqslant 1$ λ-a.e.; f_C^B is therefore bounded and, hence, in $L^2(\lambda)$, so that Lemma 15 can be applied.

Let T be a weakly mixing transformation in $\mathcal{T}_\mathrm{p}$ (it is known that such a transformation exists [1]). Now, for all $A \in \mathcal{B}(\mathbb{I})$, one has

$$\left((S_T)^n * \mu_C\right)(A \times B) = \mu_C^B\left(T^{-n}(A)\right) = \int_{T^{-n}(A)} f_C^B\, d\lambda = \int_\mathbb{I} f_C^B\left(T^n(x)\right)\mathbf{1}_A\, d\lambda,$$

by a change of variables in the last equality. By Lemma 15, there exists a set D of zero density such that

$$\lim_{\substack{n\to\infty \\ n\notin D}} \int_{\mathbb{I}} f_C^B\big(T^n(x)\big)\mathbf{1}_A\,d\lambda = \int_{\mathbb{I}} f_C^B\,d\lambda \int_{\mathbb{I}} \mathbf{1}_A\,d\lambda. \tag{5.2}$$

The desired assertion follows by setting $A = [0, u]$ and $B = [0, v]$ with u and v in $]0, 1]$. □

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