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Direct Sums of Representations as Modules over Their Endomorphism Rings

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1. INTRODUCTION, TERMINOLOGY, AND BACKGROUND

Our investigation into the endo-structure of infinite direct sums $\bigoplus_{i\in I} M_i$ of indecomposable modules M_i—over a ring R with identity—is centered on the following question: If $S = \operatorname{End}_R(\bigoplus_{i\in I} M_i)$, how much pressure, in terms of the S-structure of $\bigoplus_{i\in I} M_i$, is required to force the M_i into finitely many isomorphism classes? One of the consequences of our principal result in this direction (Theorem H of Section 4) is as follows. If all of the M_i are endofinite (think, for instance, of finitely generated or generic modules over an Artin algebra) and if $(M_t)_{t\in T}$ is a transversal of the isomorphism types of the M_i, then the following conditions (1)–(4) are equivalent: (1) T is finite; (2) $\bigoplus_{i\in I} M_i$ is endo-artinian and $(M_t)_{t\in T}$

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is left semi-T-nilpotent; (3) $\bigoplus_{i\in I} M_i$ is endo-noetherian and $(M_t)_{t\in T}$ is right semi-T-nilpotent; (4) $(M_t)_{t\in T}$ is left and right semi-T-nilpotent and, for any cofinite subset $T' \subseteq T$, the endosocle of $\bigoplus_{t\in T'} M_t$ has finite support in T'. Here we call a family $(M_i)_{i\in I}$ *left semi-T-nilpotent* in case, for every sequence (i_n) of distinct indices in I, every sequence of non-isomorphisms $f_n \in \mathrm{Hom}_R(M_{i_{n+1}}, M_{i_n})$, and every finitely cogenerated factor module M_{i_1}/X of M_{i_1}, the image of $f_1 \cdots f_n$ is contained in X for $n \gg 0$. The familiar dual condition of right semi-T-nilpotence, going back to Harada (see [11]), links our results to well-known theorems addressing the exchange property of direct sums.

The mentioned equivalence applies with particular strength to Artin algebras, one of the reasons for this being the following asset (Proposition L): Given any family (M_i) of finitely generated representations of an Artin algebra, the direct sum $\bigoplus_{i\in I} M_i$ is endo-artinian if and only if it is Σ-algebraically compact (=Σ-pure injective).

Our study of Σ-algebraically compact and, more restrictively, endo-artinian direct sums hinges crucially on an analysis of endosocles. The importance of this invariant in measuring the supply of maps among the M_i emerges clearly in the following consequence of Theorem H: A finite-dimensional algebra over an algebraically closed field has finite representation type if and only if the endosocles of all direct sums of indecomposable left representations of constant finite dimension have finite supports (Corollary N).

Another point of independent interest lies in the general connections between T-nilpotence conditions and the endo-structure of direct sums exhibited in Proposition E of Section 3: A family $(M_i)_{i\in I}$ of indecomposable modules is right T-nilpotent if and only if $\bigoplus_{i\in I} M_i$ has the descending chain condition for finitely generated endo-submodules; moreover, these conditions are equivalent to the requirement that the direct sum $\bigoplus_{i\in I} M_i$ be semi-artinian over its endomorphism ring, i.e., that all endo-factor modules have essential socles. As a consequence, each Σ-algebraically compact module M is a direct sum of indecomposables M_i, $i \in I$, with the property that the family $(M_i)_{i\in I}$ is right T-nilpotent.

We add a few comments on the background of our project for motivation. It is well known that the structure of non-finitely generated representations of R, viewed as modules over their endomorphism rings, is decisive in understanding the behavior of the finitely generated representations. We point to a few specific instances of such connections. As was first observed in [20, 17], finite representation type of R occurs precisely when all (left) R-modules have finite lengths over their endomorphism rings. Moreover, the rings with vanishing left pure global dimension are characterized by certain endo-chain conditions satisfied by their modules. Along a different line, Crawley-Boevey proved that, for tame finite-dimensional algebras over an algebraically closed field, the *generic modules* (i.e., the infinite-dimensional indecomposable endofinite modules) govern the one-parameter families of finitely generated indecomposable representations in an extremely strong sense [6, 7]. (It appears plausible that the endosocles of direct sums of generic modules should reflect domestic versus non-domestic representation type; see, e.g., Example C(3), due to Ringel [21].) There is a common skein tying the listed scenarios at least loosely together, namely the fact that Σ-algebraic compactness of an R-module is tantamount to the descending chain condition for a selection of its endo-submodules. Other results in a related vein link endo-chain conditions of infinite direct sums of finitely generated representations of Artin algebras to the existence of preprojective or preinjective partitions, the existence of almost split maps, and the structure of direct products (see, e.g., [3, 13, 18, 8, 1] and our concluding remarks). These multiple bridges between endo-chain conditions on one hand, and purely representation-theoretic assets of classes of modules on the other, motivate our present investigation.

Section 2 is devoted to exploring *endosocle series* of direct sums. In Section 3, we compare T-nilpotence conditions for families of modules with endo-chain conditions for their direct sums, and in Section 4, we follow with our main results, applications, and examples.

Prerequisites

Recall from [23, 24] that a *p-functor* on R-Mod is a subfunctor of the forgetful functor $R\text{-Mod} \to \mathbb{Z}\text{-Mod}$ which commutes with direct products; here R-Mod stands for the category of all left R-modules. Special instances of p-functors can be described as follows: A *pointed matrix* over R consists of a row-finite matrix $\mathcal{A} = (a_{ij})_{i\in I,\, j\in J}$ of elements in R, paired with a column index $\alpha \in J$. Given a pointed matrix $(\mathcal{A}, \alpha)$, we call the following p-functor $[\mathcal{A}, \alpha]$ on R-Mod a *matrix functor*: For any R-module M, the subgroup $[\mathcal{A}, \alpha]M$ is defined to be the αth projection of the solution set in M^J of the homogeneous system

$$\sum_{j\in J} a_{ij} X_j = 0 \quad \text{for all } i \in I.$$

In other words,

$$[\mathcal{A}, \alpha]M = \big\{m \in M \mid \exists \text{ a solution } (m_j) \in M^J \text{ of the above system with } m_\alpha = m\big\}.$$

Following [24], we call $[\mathcal{A}, \alpha]M$ a *matrix subgroup* of M, and a *finite matrix subgroup* in case I and J are finite. We refer the reader to [24] or [14] for more information on p-functors.

One of the most salient reasons for our present interest in p-functors lies in the following equivalent description of Σ-algebraically compact modules, i.e., of the modules M with the property that all direct sums of copies of M are algebraically compact (=pure injective): Namely, M is Σ-algebraically compact if and only if every descending chain of p-functorial subgroups of M becomes stationary, a condition which is in turn equivalent to the DCC for finite matrix subgroups of M (see [23, 9, 24, 10]); here we call a subgroup U of the abelian group underlying M *p-functorial* in case there exists a p-functor P: R-Mod $\to \mathbb{Z}$-Mod with $P(M) = U$. Since, clearly, every p-functorial subgroup of M is an endo-submodule, this is a well-known instance of a link between the R- and endo-structures of M. While it guarantees that endo-artinian modules are Σ-algebraically compact, the converse fails in general; see, e.g., [5] or [15, Theorem 5]. By contrast, M is endo-noetherian if and only if M has the ascending chain condition for p-functorial subgroups, since every finitely generated endo-submodule of M is a matrix subgroup. In case M is a direct sum of finitely presented modules, the ACC for *finite* matrix subgroups—not equivalent to the ACC for arbitrary matrix subgroups in general—already suffices to guarantee endo-noetherianness (Observation 8 of [17]). In light of the fact that p-functors automatically commute with direct sums (as arbitrary subfunctors of the forgetful functor R-Mod $\to \mathbb{Z}$-Mod do), one deduces the following observation which will prove useful in connection with dualities.

Observation. For any family $(M_i)_{i \in I}$ of finitely presented modules, the following conditions are equivalent:

(1) $\bigoplus_{i \in I} M_i$ is endo-noetherian.

(2) $\bigoplus_{i \in I} M_i$ satisfies the ACC for (finite) matrix subgroups.

(3) $\prod_{i \in I} M_i$ satisfies the ACC for (finite) matrix subgroups.

(4) $\prod_{i \in I} M_i$ is endo-noetherian.

2. ENDOSOCLE SERIES OF DIRECT SUMS OF INDECOMPOSABLE MODULES

The following elementary observations will be very useful to us in the rest of this paper.

LEMMA A. *Every Σ-algebraically compact R-module M is semi-artinian over its endomorphism ring S, meaning that every S-factor module of M has essential socle.*

Proof. Suppose that M is a Σ-algebraically compact R-module. As we pointed out above, M then satisfies the descending chain condition for matrix subgroups which, in particular, yields the DCC for finitely generated S-submodules of M. But this latter condition clearly entails our claim: Indeed, given S-submodules $U \subsetneqq V$ of M, there exists a minimal S-submodule W of V not contained in U, and any such choice of W gives rise to a simple submodule $(W+U)/U \subseteq V/U$. ∎

The converse of Lemma A fails in general: Let R be any left perfect ring which fails to be left Σ-algebraically compact (for examples see [25]); then all left R^{op}-modules have essential socles, and therefore the regular left R-module is semi-artinian over its endomorphism ring.

Recall that the ascending socle series of an S-module M is defined as follows: $\operatorname{soc}_0 M = 0, \operatorname{soc}_1 M = \operatorname{soc}_S M$, and, if $\alpha = \beta + 1$ is a successor ordinal, $\operatorname{soc}_\alpha M$ is the preimage of $\operatorname{soc}_S(M/\operatorname{soc}_\beta M)$ under the canonical epimorphism $M \to M/\operatorname{soc}_\beta M$; for a limit ordinal α, finally, $\operatorname{soc}_\alpha M$ is defined to be the union $\bigcup_{\beta<\alpha} \operatorname{soc}_\beta M$. Accordingly, the socle length of M is the least ordinal μ with the property that $\operatorname{soc}_{\mu+1} M = \operatorname{soc}_\mu M$.

Given any left R-module M with endomorphism ring S, we will refer to the S-socle of M as the *endosocle*, denoted by $\operatorname{endosoc} M$, and to the S-socle series as the *endosocle series*; a generic term of this series will be labeled $\operatorname{endosoc}_\alpha M$. The *endosocle length* of M, finally, is the length of its S-socle series. Whenever we speak of $\operatorname{endosoc} M$ as having a property (X), we are referring to the S-structure, not the R-structure.

Since indecomposable algebraically compact modules have local endomorphism rings [15], Lemma A yields

COROLLARY A′. *Suppose that M is a Σ-algebraically compact R-module with endosocle length μ. Then* $\operatorname{endosoc}_\mu M = M$.

Hence, if the module M is moreover indecomposable with endomorphism ring S, it contains an exhaustive, well-ordered chain of S-submodules with consecutive factors of the form $(S/J(S))^{(A)}$ (i.e., a chain $(M_\alpha)_{\alpha<\tau}$ of S-submodules, where τ is an ordinal, with $M_\alpha \subseteq M_\beta$ for $\alpha < \beta < \tau$ such that $\bigcup_{\alpha<\tau} M_\alpha = M$ and all consecutive factors $M_{\alpha+1}/M_\alpha$ are direct sums of copies of $S/J(S)$).

In contrast to the preceding observation, *direct sums* of Σ-algebraically compact modules may have trivial endosocles—think of $M = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/(p^n)$, for instance, where p is prime. This can be seen as a pronounced failure of the direct sum of Σ-algebraically compact modules to inherit crucial properties implied by Σ-algebraic compactness.

Recall that any Σ-algebraically compact module M is a direct sum of submodules with local endomorphism rings. Our main interest being in families (M_i) of Σ-algebraically compact or, more restrictively, endo-artinian modules, most of our discussion will therefore focus on the situation where all of the M_i have local endomorphism rings. In this setting, we record how the endosocle of $\bigoplus_{i\in I} M_i$ relates to the endosocles of the individual M_i. In case all of the M_i are pairwise isomorphic, the connection is straightforward; otherwise, it reflects the behavior of maps among the summands M_i as follows.

LEMMA B. *Let $(M_i)_{i\in I}$ be a family of modules with local endomorphism rings and let $S = \operatorname{End}_R(\bigoplus_{i\in I} M_i)$.*

(1) *First suppose that $M_i = N$ for all $i \in I$. Then:*

• $\operatorname{endosoc}(N^{(I)}) = (\operatorname{endosoc}(N))^{(I)}$, *and the endosocle of $N^{(I)}$ consists of a single homogeneous component.*

• *The endosocle of $N^{(I)}$ is finitely generated as an endo-submodule if and only if the same is true for the endosocle of N.*

(2) *In general,* $\operatorname{endosoc}(\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I} B_i$, *where each B_i is the $\operatorname{End}_R(M_i)$-submodule of M_i consisting precisely of those elements of M_i which are annihilated by all non-isomorphisms in $\bigcup_{j\in I} \operatorname{Hom}_R(M_i, M_j)$.*
In particular, if $M_i \ncong M_j$ for $i \neq j$, then each B_i is a semisimple $\operatorname{End}_R(M_i)$-*submodule of M_i, as well as a semisimple S-submodule of the direct sum $\bigoplus_{i\in I} M_i$.*

(3) *Whenever $(M_n)_{n\in\mathbb{N}}$ is a family of pairwise non-isomorphic modules with local endomorphism rings which permits a sequence of monomorphisms $M_1 \hookrightarrow M_2 \hookrightarrow M_3 \hookrightarrow \cdots$, the direct sum $\bigoplus_{n\in\mathbb{N}} M_n$ has trivial endosocle.*

Proof. Set $M = \bigoplus_{i\in I} M_i$ and observe that each S-submodule U of M is of the form $U = \bigoplus_{i\in I}(U \cap M_i)$, where the intersection $U \cap M_i$ is an $\operatorname{End}_R(M_i)$-submodule of M_i. We deduce that each simple S-submodule of M can be written in the form Sx_i for a nonzero element x_i of some M_i.

Part (1) is immediate from these remarks.

(2) Set $B_i = (\operatorname{endosoc} M) \cap M_i$. To see that B_i is contained in the annihilator of the set of non-isomorphisms in $\bigcup_{j\in I} \operatorname{Hom}_R(M_i, M_j)$, observe that, due to the locality of the rings $\operatorname{End}_R(M_i)$, each of the indicated non-isomorphisms belongs to the Jacobson radical of S.

For the other inclusion, fix $i \in I$ and set $I(i) = \{j \in I \mid M_j \cong M_i\}$ and $S(i) = \operatorname{End}_R(\bigoplus_{j\in I(i)} M_j)$. If $x \in M_i$ is annihilated by all non-isomorphisms in the above union, then x clearly belongs to the $\operatorname{End}_R(M_i)$-socle of M_i, and consequently $S(i)x$ is contained in the $S(i)$-socle of $\bigoplus_{j\in I(i)} M_j$ by part (1). Since $Sx = S(i)x$, this shows that $Sx \subseteq \operatorname{endosoc} M$, which yields $x \in B_i$ as required.

The final assertion under (2) is now obvious, as is (3). ∎

EXAMPLES C. (1) Let $R = \mathbb{Z}$ and p a prime. The terms of the endosocle series of $M = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/(p^n) \oplus \mathbb{Z}(p^\infty)$ are as follows: $\operatorname{endosoc}_\alpha M$ equals the copy of $\mathbb{Z}/(p^\alpha)$ inside $\mathbb{Z}(p^\infty)$ for $\alpha < \omega$, and $\operatorname{endosoc}_\alpha M = \mathbb{Z}(p^\infty)$ for $\alpha \geq \omega$.

(2) Let Λ be the Kronecker algebra. If $(M_n)_{n\in\mathbb{N}}$ is the family of all preprojective modules in R-ind, then $\operatorname{endosoc}(\bigoplus_{n\in\mathbb{N}} M_n) = 0$; in fact, $\operatorname{endosoc}(\bigoplus_{n\geq m} M_n) = 0$ for all m. On the other hand, if $M' = \bigoplus_{n\in\mathbb{N}} M'_n$ is the direct sum of all preinjective modules, with $\operatorname{length}(M'_n) = 2n-1$, the endosocle of M' equals $M'_1 \bigoplus \operatorname{soc}_\Lambda M'_2$, and, for $m \geq 2$, we have $\operatorname{endosoc}(\bigoplus_{n\geq m} M'_n) = M'_m$.

(3) The following is due to Ringel [21]: If Λ is a string algebra (i.e., a finite-dimensional monomial relation algebra which is special biserial), but fails to be of domestic representation type, then there exists a family $(M_n)_{n\in\mathbb{N}}$ of pairwise non-isomorphic generic Λ-modules allowing for consecutive embeddings $M_1 \hookrightarrow M_2 \hookrightarrow \cdots$; by Lemma B(3), we infer that the endosocle of the direct sum of the M_n's is zero.

We continue to assume that $M = \bigoplus_{i\in I} M_i$, where all M_i have local endomorphism rings. In exploring the endo-structure of M, a different sequence of iterated endosocles—not forming an ascending chain—is frequently more helpful than the series we just discussed. This is due to the fact that the traditional endosocle series may be infinite and still get stalled within a finite subsum of M_i's (see Example C(1)). By contrast, any infinite sequence of nonzero terms of the following *relative endosocle series* involves infinitely many summands M_i. This often makes it a more effective tool in studying homomorphisms among non-isomorphic M_i's.

As usual, we start with $\operatorname{endosoc}'_0 M = 0$; moreover, we set $I_0 = \varnothing$. Next, we set $\operatorname{endosoc}'_1 M = \operatorname{endosoc} M$ and denote by I_1 the support of $\operatorname{endosoc}'_1 M$ in I. Assume that the terms $\operatorname{endosoc}'_\beta M$ have already been defined for all ordinal numbers $\beta < \alpha$. In case α is a limit ordinal, we set $\operatorname{endosoc}'_\alpha M = 0$. If, on the other hand, α is a successor ordinal, say $\alpha = \beta + 1$, we let I_β be the support of $\sum_{\gamma\leq\beta} \operatorname{endosoc}'_\gamma M$ and define $\operatorname{endosoc}'_\alpha M$ to be the endosocle of the trimmed direct sum $\bigoplus_{I\setminus I_\beta} M_i$. In particular, we see that I_α is defined for each ordinal α and equals the support of $\sum_{\beta\leq\alpha} \operatorname{endosoc}'_\beta M = \bigoplus_{\beta\leq\alpha} \operatorname{endosoc}'_\beta M$. Finally, we refer to the least ordinal number μ such that $\operatorname{endosoc}'_{\mu+1} M = 0$ (or, equivalently, the least ordinal number μ with $I_{\mu+1} = I_\mu$) as the *relative endosocle length* of M.

Clearly, the isomorphism types of the R-submodules $\operatorname{endosoc}'_\alpha M$ are isomorphism invariants of M, and the sum $\sum_{\alpha\leq\mu} \operatorname{endosoc}'_\alpha M$ is direct by construction. The second of the following observations will be used repeatedly in the next section. Both are straightforward from the definitions.

LEMMA D. *Let $M = \bigoplus_{i\in I} M_i$, where each M_i has local endomorphism ring.*

(1) *If each nontrivial subsum of M_i's has nontrivial endosocle, then $I_\mu = I$, where μ is the relative endosocle length of M.*

(2) *Let α be a successor ordinal, say $\alpha = \beta + 1$, such that $\bigoplus_{i \in I \setminus I_\beta} M_i$ has essential endosocle. If U is an endo-submodule of M which is not contained in $\bigoplus_{i \in I_\alpha} M_i$, then $(J(S)U) \cap \operatorname{endosoc}'_\alpha M \neq 0$.*

Observe that all hypotheses of Lemma D are satisfied if M is Σ-algebraically compact.

3. T-NILPOTENCE VERSUS ENDO-CHAIN CONDITIONS

Harada [11] called a family $(M_i)_{i \in I}$ of indecomposable R-modules *semi-T-nilpotent* in case, for any sequence $(i_n)_{n \in \mathbb{N}}$ of distinct indices in I, any family of non-isomorphisms $f_n \in \operatorname{Hom}_R(M_{i_n}, M_{i_{n+1}})$, and any finitely generated R-submodule X of M_{i_1}, there exists $n_0 \in \mathbb{N}$ with

$$f_{n_0} f_{n_0 - 1} \cdots f_1(X) = 0.$$

Obviously, this is a condition which can be tested "pointwise," i.e., the family (M_i) satisfies it precisely when, for each family of indices and non-isomorphisms f_n as above and any element $x \in M_{i_1}$, there exists a number n_0 with $f_{n_0} f_{n_0-1} \cdots f_1(x) = 0$. To make room for a dual concept, we will refer to the described condition as *right semi-T-nilpotence*. The following twin concept of *left semi-T-nilpotence* of a family $(M_i)_{i \in I}$ of indecomposables requires that, for any sequence $(i_n)_{n \in \mathbb{N}}$ of distinct indices in I, any family of non-isomorphisms $f_n \in \operatorname{Hom}_R(M_{i_{n+1}}, M_{i_n})$, and any finitely cogenerated factor module M_{i_1}/X of the R-module M_{i_1}, there exists a natural number n_0 such that

$$\operatorname{Im}(f_1 f_2 \cdots f_{n_0}) \subseteq X.$$

Left semi-T-nilpotence clearly implies the following elementwise condition: For each family of indices and homomorphisms f_n as specified above and for each nonzero $y \in M_{i_1}$, there exists $n_0 \in \mathbb{N}$ such that $y \notin \operatorname{Im}(f_1 f_2 \cdots f_{n_0})$. While the converse fails in general, it does hold if all of the M_i's are finitely cogenerated—so, in particular, if they have finite length—in which case either condition is equivalent to the requirement that the family of homomorphisms among the M_i's be artinian in the sense of Auslander (see below).

The stronger conditions of *right/left T-nilpotence*, finally, call for the same conclusions as the corresponding semi-T-nilpotence conditions, but on waiving the premise that the families of indices considered be free of repetitions.

We point out that, for the special case of finitely generated indecomposable modules M_i over an Artin algebra, Auslander introduced alternate terms for these nilpotence conditions (see [2]). He labeled a family of morphisms among such modules noetherian (resp., artinian) in case, for every countable sequence of non-isomorphisms (f_n) selected within the family—consecutively composable in the appropriate sense—there exists an index n_0 with $f_{n_0} \cdots f_1 = 0$ (resp., $f_1 \cdots f_{n_0} = 0$). So, in this situation, the family (M_i) is right semi-T-nilpotent if and only if it is right T-nilpotent, if and only if the family of homomorphisms among the M_i is noetherian. (We alert the reader to the fact that this condition automatically comes with endo-*artinian*ness of $\bigoplus_{i \in I} M_i$—see Proposition E below—but may fail for endo-noetherian direct sums $\bigoplus_{i \in I} M_i$.)

The second of the following known characterizations of right semi-T-nilpotent families of modules will be used freely in the rest of the paper, while the third will enter into an application of our main theorem. The implications (3) $\Longrightarrow$ (2) $\Longleftrightarrow$ (1) are due to Harada [11], and the remaining one was filled in by Huisgen-Zimmermann and Zimmermann [16]. Recall that a module M is said to have the *exchange property* in case, for every equality of the form $M' \oplus A = \bigoplus_{l \in L} A_l$ with $M' \cong M$, there exist submodules $B_l \subseteq A_l$ such that $M' \oplus A = M' \oplus \bigoplus_{l \in L} B_l$.

Known Facts. Suppose that all M_i have local endomorphism rings. Then the following conditions are equivalent:

(1) The family $(M_i)_{i \in I}$ is right semi-T-nilpotent.

(2) The Jacobson radical of $\mathrm{End}_R(\bigoplus_{i \in I} M_i)$ coincides with the set of those endomorphisms f which have the property that all compositions $\mathrm{pr}_j \cdot f \cdot \mathrm{in}_i$ are non-isomorphisms (here pr_j and in_i denote the canonical projections and injections, respectively).

(3) The direct sum $\bigoplus_{i \in I} M_i$ has the exchange property.

We start by relating right semi-T-nilpotence to the endo-semi-artinian condition we encountered in Lemma A. These connections will not only feed into the proof of our main result, but are of interest as supplements. Part (3) of the following proposition is a variant of [17, Proposition 4]. Moreover, we note that the implications (a) $\Longrightarrow$ (c) $\Longrightarrow$ (b) hold without the blanket hypothesis under (2).

PROPOSITION E. *Suppose that $(M_i)_{i \in I}$ is a family of indecomposable left R-modules and set $M = \bigoplus_{i \in I} M_i$.*

(1) *If (M_i) is a right T-nilpotent family, then all of the M_i have local endomorphism rings.*

(2) *If all M_i have local endomorphism rings, the following conditions are equivalent:*

(a) *The family (M_i) is right T-nilpotent.*

(b) *The direct sum M is semi-artinian over its endomorphism ring.*

(c) *M satisfies the DCC for cyclic endo-submodules (or, equivalently, the DCC for finitely generated endo-submodules).*

As a consequence, every Σ-algebraically compact R-module M is a direct sum ranging over a right T-nilpotent family of indecomposable modules.

(3) *The following condition is sufficient for right semi-T-nilpotence of the family (M_i): Each M_i has local endomorphism ring, and every descending chain of matrix functors on R*-Mod *becomes uniformly stationary on almost all of the M_i's.*

Proof. Set $S = \mathrm{End}_R(M)$.

(1) Suppose that M_i belongs to a right T-nilpotent family and let $f \in \mathrm{End}_R(M_i)$ be a non-isomorphism. Then f fails to be a monomorphism, for, given any element $x \in M$, there exists $m \in \mathbb{N}$ with $f^m(x) = 0$. To see that, for any $g \in \mathrm{End}_R(M_i)$, the map $\mathrm{id}_{M_i} - gf$ is an isomorphism, it thus suffices to check injectivity. To that end, we observe that $x = gf(x)$ implies $x = (gf)^n(x)$ for all $n \in \mathbb{N}$, and hence $x = 0$ by our T-nilpotence hypothesis.

(2) Start by recalling that the descending chain condition for cyclic submodules is equivalent to that for finitely generated submodules, due to Bjørk [4]. The canonical embeddings and projections coming with the direct sum $\bigoplus_{i\in I} M_i$ will be denoted by in_i and pr_i, respectively.

To back up (a) $\Longrightarrow$ (c), suppose (a) holds.

First we show that, for any element $m = (m_i)_{i\in I} \in M$, the cyclic S-module $Sm/J(S)m$ is semisimple. Indeed, right semi-T-nilpotence of (M_i) guarantees that the Jacobson radical $J(S)$ of S contains all endomorphisms f with the property that the compositions $\mathrm{pr}_i\, f\, \mathrm{in}_j$ are non-isomorphisms for arbitrary $i, j \in I$. We will make the assumption that $M_i \ncong M_j$ whenever $i \neq j$; this does not affect the generality of our argument, for every endo-submodule U of a direct sum of powers $\bigoplus_{i\in I} M_i^{(L_i)}$ is of the form $\bigoplus_{i\in I} U_i^{(L_i)}$ for a suitable endo-submodule U_i of M_i. Consequently,

$$Sm/J(S)m = \bigoplus_{\text{finite}} \mathrm{End}_R(M_i)m_i/J(\mathrm{End}_R(M_i))m_i,$$

where each of the summands on the right is an S-module, the S-structure of which coincides with the pertinent $\mathrm{End}_R(M_i)/J(\mathrm{End}_R(M_i))$-structure.

To the contrary of our claim, assume that M contains a strictly descending chain $Sx_1 \supsetneqq Sx_2 \supsetneqq Sx_3 \supsetneqq \cdots$ of cyclic S-submodules. Now the induced sequence in the semisimple factor module $Sx_1/J(S)x_1$ does become stationary, yielding an integer m such that $Sx_m \subseteq Sx_{m+1} + J(S)x_1$. By possibly dropping some terms from our descending chain, we may assume that $Sx_2 \subseteq Sx_3 + J(S)x_1$. We apply the same argument to $Sx_2/J(S)x_2$ and iterate, whence an obvious induction permits us to thin out our original sequence of cyclics so as to assure $Sx_n \subseteq Sx_{n+1} + J(S)x_{n-1}$ for all $n \geq 2$. For each $n \geq 2$ we thus obtain an equality $x_n = f_{n+1}(x_{n+1}) + g_{n-1}(x_{n-1})$ with $f_{n+1} \in S$ and $g_{n-1} \in J(S)$. By substituting $x_2 = f_3(x_3) + g_1(x_1)$ into the equality $x_3 = f_4(x_4) + g_2(x_2)$, we deduce that $(1 - g_2 f_3)(x_3) = f_4(x_4) + g_2 g_1(x_1)$, and multiplication by the inverse of $1 - g_2 f_3$ yields an equality of the form $x_3 = f_4'(x_4) + h_1(x_1)$ with $h_1 \in J(S)$. In a next step, we insert this expression for x_3 into $x_4 = f_5(x_5) + g_3(x_3)$, thus obtaining an element $h_2 \in J(S)$ with the property that $x_4 = f_5'(x_5) + h_2 h_1(x_1)$ for a suitable $f_5' \in S$. Again we iterate and arrive at a sequence of elements $h_1, h_2, h_3, \ldots \in J(S)$ satisfying $x_n = f_{n+1}'(x_{n+1}) + h_{n-2} \cdots h_2 h_1(x_1)$ for suitable choices of $f_n' \in S$.

T-nilpotence of the family $(M_i)_{i \in I}$, combined with König's graph theorem, will now provide us with a natural number N such that $h_N \cdots h_2 h_1(x_1) = 0$. For, if we had $h_n \cdots h_2 h_1(x_1) \neq 0$ for all n, the following graph $\mathcal{G}$ would have paths of arbitrary length: The roots of $\mathcal{G}$ are the nonzero components $x_{1i} \in M_i$ of the element x_1—finite in number, say $x_{1,\, i(1)}, \ldots, x_{1,\, i(m)}$—and the edge paths of length l in $\mathcal{G}$ correspond to the nonzero evaluations

$$(\mathrm{pr}_{j(l)} h_l \mathrm{in}_{j(l-1)})(\mathrm{pr}_{j(l-1)} h_{l-1} \mathrm{in}_{j(l-2)}) \cdots (\mathrm{pr}_{j(1)} h_1 \mathrm{in}_{j(0)})(x_{1,\, j(0)})$$

for suitable indices $j(s) \in I$ such that $j(0) \in \{i(1), \ldots, i(m)\}$. König's graph theorem would then imply the existence of an infinite path. But this is incompatible with our T-nilpotence condition, since each of the maps $\mathrm{pr}_{j(s)} h_s \mathrm{in}_{j(s-1)}\colon M_{j(s-1)} \to M_{j(s)}$ is a non-isomorphism by our choice of the h_n.

Thus, we obtain a number N as described and infer $x_{N+2} \in Sx_{N+3}$, contrary to our assumption that the chain of Sx_n be strictly descending.

To justify the implication (c) $\Longrightarrow$ (b), carry over the argument proving Lemma A.

For (b) $\Longrightarrow$ (a), see [22, Chap. VIII].

(3) Suppose that $(i_n)_{n \in \mathbb{N}}$ is a sequence of distinct elements of I and $f_{i_n}\colon M_{i_n} \to M_{i_{n+1}}$ a non-isomorphism for $n \in \mathbb{N}$. Given $x \in M_{i_1}$, consider the following chain of principal S-submodules of M:

$$S f_{i_1} x \supseteq S f_{i_2} f_{i_1} x \supseteq S f_{i_3} f_{i_2} f_{i_1} x \supseteq \cdots.$$

All terms of this chain are matrix subgroups of the R-module M (see Section 1), say $S f_{i_n} \cdots f_{i_1} x = P_n(M)$, where P_n is a matrix functor. Since the class of matrix functors on R-Mod is closed under intersections, we can normalize to the situation where $P_1 \supseteq P_2 \supseteq P_3 \supseteq \cdots$. Assuming that this chain becomes uniformly stationary on almost all of the M_i's, we obtain a natural number N, together with a finite subset $I_0 \subseteq I$, such that $P_n(\bigoplus_{i \in I \setminus I_0} M_i) = P_N(\bigoplus_{i \in I \setminus I_0} M_i)$ for all $n \geq N$. Since the i_n are pairwise different, we can adjust N upward, if necessary, so as to guarantee that, moreover, $f_{i_n} \cdots f_{i_1} x \in \bigoplus_{i \in I \setminus I_0} M_i$ for all $n \geq N$. Denoting the R-endomorphism ring of $\bigoplus_{i \in I \setminus I_0} M_i$ by S', we infer that the chain

$$S' f_{i_n} \cdots f_{i_{N+1}} (f_{i_N} f_{i_{N-1}} \cdots f_{i_1} x), \qquad n \geq N,$$

is stationary. We will deduce that $f_{i_N} \cdots f_{i_1} x = 0$. Indeed, our setup yields a map $g \in S'$ with $g f_{i_{N+1}} f_{i_N} \cdots f_{i_1} x = f_{i_N} \cdots f_{i_1} x$. Clearly, we may assume that g is a map from $M_{i_{N+2}}$ to $M_{i_{N+1}}$. If $f_{i_N} \cdots f_{i_1} x$ were nonzero, Nakayama's Lemma would exclude $g f_{i_{N+1}}$ from the Jacobson radical of $\mathrm{End}_R(M_{i_N})$, and locality of this endomorphism ring would guarantee $g f_{i_{N+1}}$ to be a unit of $\mathrm{End}_R(M_{i_N})$. But this is impossible, since $M_{i_{N+2}}$ is indecomposable and $f_{i_{N+1}}$ a non-isomorphism. Hence $f_{i_N} \cdots f_{i_1} x = 0$ as claimed, and the argument is complete. ∎

Part (2) of the preceding proposition can actually be de-specified from the situation of direct sums regarded as modules over their endomorphism rings: If S is any ring and M a left S-module, then the DCC for finitely generated submodules forces M to be semi-artinian; the latter condition in turn implies that, given any $m \in M$ and any sequence $(s_1, s_2, \ldots)$ in $J(S)$, we have $s_n \cdots s_1 m = 0$ for $n \gg 0$. Moreover, these three conditions are equivalent in case $Sm/J(S)m$ is semisimple for every element $m \in M$.

4. MAIN RESULTS AND EXAMPLES

Theorem H below will zero in on our primary concern: namely, to significantly weaken conditions on the endo-structure of $\bigoplus_{i \in I} M_i$ which are known to guarantee that the M_i fall into finitely many isomorphism classes. An instance of such a condition is the endofiniteness of $\bigoplus_{i \in I} M_i$; see [7]. Our result also provides background for the following well-known characterization of the Artin algebras Λ having finite representation type, in terms of morphisms among the indecomposable finitely generated left Λ-modules, due to Auslander [2]: Namely, Λ has finite representation type precisely when every countable family of non-isomorphisms is both artinian and noetherian in the sense given at the beginning of Section 2. Related results for families of finitely generated indecomposable modules were recently obtained by Dung [8, Theorem 3.3 and Corollary 3.12]. We, too, are particularly interested in the situation where all of the M_i's have finite length over R—next to the case where the M_i's are generic—and will address it in subsequent corollaries and examples.

LEMMA F. *Let $(M_i)_{i \in I}$ be a right semi-T-nilpotent family of pairwise non-isomorphic modules with local endomorphism rings. If, for each cofinite subset I' of I, the direct sum $\bigoplus_{i \in I'} M_i$ is finitely generated over its endomorphism ring, then I is finite.*

Proof. Suppose that the endo-condition of our claim is satisfied and set $M = \bigoplus_{i \in I} M_i$. For any subset $I' \subseteq I$, we denote by $\mathrm{pr}_{I'}$ the canonical projection $M \to \bigoplus_{i \in I'} M_i$ along $\bigoplus_{i \in I \setminus I'} M_i$, preserving the convention $\mathrm{pr}_i = \mathrm{pr}_{\{i\}}$ for $i \in I$. Moreover, we set $S = \mathrm{End}_R(\bigoplus_{i \in I} M_i)$. Keeping in mind that every S-submodule U of M is of the form $\bigoplus_{i \in I}(M_i \cap U)$, where each $M_i \cap U$ is an $\mathrm{End}_R(M_i)$ submodule of M_i, we find that, given any S-generating set G of such a module U, the collection $\{\mathrm{pr}_i(x) \mid i \in I,\ x \in G\}$ of projections again belongs to U (and thus generates U).

By hypothesis, we can therefore pick a finite family of elements $(x_{l_1})_{l_1 \in L_1}$ in $\bigcup_{i \in I} M_i$ with the property that $M = \sum_{l_1 \in L_1} S x_{l_1}$; by $K_1 \subseteq I$ we denote the (finite) union of the supports of the x_{l_1}. Setting $S_1 = \mathrm{End}_R(\bigoplus_{i \in I \setminus K_1} M_i)$ and applying our hypothesis to the pared-down direct sum, we see that each of the modules $S_1 \mathrm{pr}_{I \setminus K_1}(S x_{l_1})$ can be written as a finite sum, $S_1 \mathrm{pr}_{I \setminus K_1}(S x_{l_1}) = \sum_{l_2 \in L_2} S_1 f_{l_2 l_1}(x_{l_1})$, where L_2 is a finite index set disjoint from L_1 and the $f_{l_2 l_1}$ are homomorphisms from suitable summands M_i of $\bigoplus_{k \in K_1} M_k$ to summands M_j of $\bigoplus_{k \in I \setminus K_1} M_k$; here the sources and targets of the $f_{l_2 l_1}$ may appear repeatedly. By allowing zero maps among the $f_{l_1 l_2}$, we can make the same index set L_2 work for all $l_1 \in L_1$ simultaneously. We thus obtain

$$M = \bigoplus_{k \in K_1} M_k + \sum_{l_1 \in L_1} S_1 \mathrm{pr}_{I \setminus K_1}(S x_{l_1})$$
$$= \bigoplus_{k \in K_1} M_k + \sum_{l_1 \in L_1,\, l_2 \in L_2} S_1 f_{l_2 l_1}(x_{l_1}).$$

Next we let $K_2 \subseteq I$ be a finite set containing K_1 and the supports of the elements $f_{l_2 l_1}(x_{l_1})$ for all choices of l_1, l_2. Now we set $S_2 = \mathrm{End}_R(\bigoplus_{i \in I \setminus K_2} M_i)$ and, using the same guidelines as for the choice of L_1 and the $f_{l_2 l_1}$'s, we choose a finite index set L_3, disjoint from $L_1 \cup L_2$, and homomorphisms $f_{l_3 l_2}$, each from a suitable summand M_i of $\bigoplus_{k \in K_2} M_k$ to a summand M_j of $\bigoplus_{k \in I \setminus K_2} M_k$, such that

$$S_2 \mathrm{pr}_{I \setminus K_2}(S_1 f_{l_2 l_1}(x_{l_1})) = \sum_{l_3 \in L_3} S_2 f_{l_3 l_2} f_{l_2 l_1}(x_{l_1})$$

for all $l_1 \in L_1$ and $l_2 \in L_2$. This yields the equality

$$M = \bigoplus_{k \in K_2} M_k + \sum_{l_1 \in L_1,\, l_2 \in L_2,\, l_3 \in L_3} S_2 f_{l_3 l_2} f_{l_2 l_1}(x_{l_1}).$$

We repeat the above procedure to inductively obtain pairwise disjoint finite index sets $L_1, L_2, L_3, \ldots$, finite subsets $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ of I, and homomorphisms $f_{l_{n+1} l_n}$—for $l_n \in L_n$ and $l_{n+1} \in L_{n+1}$—from summands M_i of $\bigoplus_{k \in K_n} M_k$ to summands M_j of $\bigoplus_{i \in I \setminus K_n} M_i$, respectively, such that, for each $n \in \mathbb{N}$,

$$M = \bigoplus_{k \in K_n} M_k + \sum_{l_m \in L_m,\, 1 \le m \le n+1} S_n f_{l_{n+1} l_n} \cdots f_{l_2 l_1}(x_{l_1}),$$

where $S_n = \text{End}_R(\bigoplus_{i\in I\setminus K_n} M_i)$.

Let $\mathcal{G}$ be the graph having as vertices the indices in $\bigcup_{i\in\mathbb{N}} K_i$ and as edge paths those concatenations $f_{l_{n+1}l_n}\cdots f_{l_2l_1}$ which have the property that $f_{l_{n+1}l_n}\cdots f_{l_2l_1}(x_{l_1}) \neq 0$. Since all of the maps $f_{l_{i+1}l_i}$ are non-isomorphisms by construction, our T-nilpotence hypothesis excludes the occurrence of an infinite path. Therefore, König's graph theorem guarantees the existence of an upper bound on the lengths of paths in $\mathcal{G}$—N say—which shows that $M = \bigoplus_{k\in K_N} M_k$. But this says that $I = K_N$ is finite as required. ∎

We now come to our key lemma. In spite of the dual formats of Lemmas F and G, the latter cannot be obtained by dualizing the preceding argument.

LEMMA G. *Let $(M_i)_{i\in I}$ be a left semi-T-nilpotent family of modules with local endomorphism rings. If, for each cofinite subset I' of I, the direct sum $\bigoplus_{i\in I'} M_i$ is finitely cogenerated over its endomorphism ring, then I is finite.*

Proof. Again suppose that the endo-condition of our claim is satisfied and set $M = \bigoplus_{i\in I} M_i$, $S = \text{End}_R(M)$. Assume that I is infinite. In order to deduce that the family $(M_i)_{i\in I}$ then fails to be left semi-T-nilpotent, we consider the relative endosocle series $(\text{endosoc}'_\alpha M)_\alpha$, as introduced at the end of Section 2. As before, we denote the support of $\bigoplus_{\beta\leq\alpha} \text{endosoc}'_\beta M$ by I_α. By hypothesis, $\text{endosoc}\, M$ is finitely generated, whence the support I_1 is finite by Lemma B(2). An obvious induction, based on the hypothesis that the endosocle of any cofinite subsum $\bigoplus_{i\in I'} M_i$ has finite support in I', thus yields finiteness of I_k for all $k \in \mathbb{N}$.

Our hypothesis moreover guarantees that each of the relative endosocles $\text{endosoc}'_{k+1}M$ is essential in the corresponding direct sum $\bigoplus_{i\in I\setminus I_k} M_i$ viewed as a module over its endomorphism ring S_k. Since, under the canonical embedding of S_k into S, we have $J(S_k) \subseteq J(S)$, we deduce from Lemma D(2) that, for any nonzero S_k-submodule U of $\bigoplus_{i\in I\setminus I_k} M_i$, the intersection $(J(S)U) \cap \text{endosoc}'_k M$ is nonzero. For each $k \in \mathbb{N}$, we now consider a descending chain $A_{k1} \supseteq A_{k2} \supseteq A_{k3} \supseteq \cdots$ of nonzero S_k-submodules of $\text{endosoc}'_k M$, where the A_{ki} are inductively defined as follows: $A_{k1} = (J(S)\,\text{endosoc}'_{k+1}M) \cap \text{endosoc}'_k M$ for $k \geq 1$ and

$$A_{k,\,i+1} = \big(J(S)A_{k+1,\,i}\big) \cap \text{endosoc}'_k M.$$

Being contained in the finitely generated semisimple S_{k-1}-module $\text{endosoc}'_k M$, the chain $(A_{ki})_{i\geq 1}$ becomes stationary. Furthermore, by Lemma D(2), it consists of nonzero terms. Therefore, it converges to a nonzero S_{k-1}-submodule $C_k \subseteq \text{endosoc}'_k M$. From the definition of the A_{ki}, we moreover derive $C_k = (J(S)C_{k+1}) \cap \text{endosoc}'_k M$ for all $k \in \mathbb{N}$, which yields a sequence of inclusions

$$C_1 \subseteq J(S)C_2,\; C_2 \subseteq J(S)C_3,\; \ldots .$$

Since C_1 is an S-module, we can pick $i_1 \in I_1$ so that $M_{i_1} \cap C_1$ contains a nonzero element x. The above string of inclusions then yields disjoint finite sets L_n for $n \in \mathbb{N}$, together with families of homomorphisms $(f_{l_n})_{l_n \in L_n}$ in $J(S)$, each of the form $f_{l_n} = \mathrm{pr}_{v(l_n)} \cdot f_{l_n} \cdot \mathrm{in}_{u(l_n)} \in \mathrm{Hom}_R(M_{u(l_n)}, M_{v(l_n)})$ for suitable indices $u(l_n) \in I_{n+1} \backslash I_n$, $v(l_n) \in I_n$, and $v(l_1) = i_1$ for all $l_1 \in L_1$, such that

$$x \in \mathrm{Im}\left(\sum_{l_1 \in L_1, \ldots, l_n \in L_n} f_{l_1} f_{l_2} \cdots f_{l_n} \right)$$

for each n. Here all of the f_{l_n}'s are non-isomorphisms from $M_{u(l_n)}$ to $M_{v(l_n)}$, respectively, since they belong to $J(S)$ (as usual, we identify homomorphisms $M_i \to M_j$ with elements of S). Note that the sets $\{u(l_n) \mid l_n \in L_n\}$ for $n \in \mathbb{N}$ are pairwise disjoint by construction; on the other hand, we permit repetitions $u(l_k) = u(l'_k)$ and $v(l_k) = v(l'_k)$ for $l_k \neq l'_k$. Next, we let $Y \subset M_{i_1}$ be a maximal R-submodule of M_{i_1} with $x \notin Y$ and consider the tree having as root the index i_1 and as branches all those concatenations $f_{l_1} f_{l_2} \cdots f_{l_n}$ the image of which is not contained in Y. By construction, there is no upper bound on the lengths of the edge paths in this graph, and therefore König's graph theorem guarantees the existence of a path of infinite length. But, since M_{i_1}/Y is a finitely cogenerated factor module of M_{i_1}, this means that the family (M_i) fails to be left semi-T-nilpotent. Thus, our hypothesis ensures finiteness of I. ∎

This smooths the road to

THEOREM H. *Let $(M_i)_{i \in I}$ be a family of indecomposable R-modules and $(M_t)_{t \in T}$ a transversal of the isomorphism classes of the M_i.*

(I) *If all of the M_i have local endomorphism rings, the following statements are equivalent:*

(1) *$\bigoplus_{i \in I} M_i$ is endo-noetherian, and $(M_t)_{t \in T}$ is right semi-T-nilpotent.*

(2) *T is finite, and each M_t is endo-noetherian.*

(II) *The following statements are equivalent:*

(1) *$\bigoplus_{i \in I} M_i$ is endo-artinian, and $(M_t)_{t \in T}$ is left semi-T-nilpotent.*

(2) *T is finite, and each M_t is endo-artinian.*

(III) *Suppose that $(M_t)_{t \in T}$ is a left semi-T-nilpotent and right T-nilpotent family satisfying the following additional finiteness condition:*

(•) *For any cofinite subset $T' \subseteq T$, the endosocle of $\bigoplus_{t \in T'} M_t$ is finitely generated.*

Then T is finite.

Proof. Start by observing that, in each of the first two parts, the implication (2) $\Longrightarrow$ (1) is trivial. Indeed, it suffices to note that both endo-noetherianness and endo-artinianness are inherited by arbitrary powers and finite direct sums. Moreover, note that, in proving the remaining implications, it is innocuous to assume that $I = T$, since the conditions involved are stable under passage from $\bigoplus_{i\in I} M_i$ to $\bigoplus_{t\in T} M_t$. The implications (1) $\Longrightarrow$ (2) under (I) and (II) now follow from Lemmas F and G, respectively; note that (II.1) guarantees all of the M_i to have local endomorphism rings, while this is a blanket hypothesis for (I).

In order to prove (III), adopt all of the listed hypotheses. By part (1) of Proposition E, the M_t again have local endomorphism rings. Moreover, for each cofinite subset $T' \subseteq T$, the direct sum $\bigoplus_{t\in T'} M_t$ is finitely cogenerated over its endomorphism ring: Indeed, in view of Proposition E(2), right T-nilpotence of $(M_t)_{t\in T'}$ ensures that this direct sum has essential endosocle and, by condition ($\bullet$), this endosocle is finitely generated. Thus, once more, Lemma G yields finiteness of T. ∎

The first part of the following corollary results from a combination of Theorem H(I) and (II) with Proposition E; the second part is an immediate consequence of that proposition and Theorem H(III).

COROLLARY I. *Let $(M_i)_{i\in I}$ be a family of indecomposable R-modules and $(M_t)_{t\in T}$ a transversal of the isomorphism classes of the M_i.*

(I) *Then the following statements are equivalent*:

(1) *$\bigoplus_{i\in I} M_i$ is endo-noetherian, and $(M_t)_{t\in T}$ is right T-nilpotent.*

(2) *T is finite, and each M_t is endofinite.*

(II) *Suppose that $\bigoplus_{i\in I} M_i$ is Σ-algebraically compact, $(M_t)_{t\in T}$ left semi-T-nilpotent, and*

($\bullet$) *for each cofinite subset $T' \subseteq T$, the endosocle of $\bigoplus_{t\in T'} M_t$ is finitely generated.*

Then T is finite.

Remarks. None of the semi-T-nilpotence conditions in the various parts of Theorem H and Corollary I is redundant, not even when all of the M_i are finitely generated over a finite-dimensional algebra—see Examples O below.

As for condition ($\bullet$) in Theorem H(III) and Corollary I(II): Dropping it means jeopardizing the conclusion in either case, as is illustrated by Example J(a) below. In case the individual modules M_i have endosocles of finite length, condition ($\bullet$) can clearly be weakened to the requirement that, for each cofinite subset $T' \subseteq T$, the endosocle of $\bigoplus_{t\in T'} M_t$ has finite support. We do not know whether, in the above statements, condition ($\bullet$) can always be relaxed in this fashion, but point out that finitely generated endosocles are not automatic in Σ-algebraically compact modules (see Example J(b)).

EXAMPLES J. (a) The direct sum $M = \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)$ is a Σ-algebraically compact $\mathbb{Z}$-module having as endomorphism ring the direct product of the rings of p-adic integers, where p runs through all primes. Moreover, the family $(\mathbb{Z}(p^\infty))_{p \text{ prime}}$ is clearly left semi-T-nilpotent and right T-nilpotent. However, the endosocle of M coincides with the $\mathbb{Z}$-socle and thus has infinite support. This shows that the endosocle condition in Theorem H(III) and Corollary I(II) is crucial.

(b) Set $R = K[X_n \mid n \in \mathbb{N}]/(X_n \mid n \in \mathbb{N})^k$, where K is a field and $k \geq 2$. Then the regular R-module is Σ-algebraically compact by [15]. On the other hand, its endosocle, $\operatorname{endosoc} R = \operatorname{soc}_R R = J^{k-1}$, fails to be finitely generated.

Prime targets for applications of the following consequence of Theorem H are families of generic or finitely generated modules over Artin algebras. In light of the subsequent proposition addressing families of finitely generated modules over Artin algebras, the picture can be sharpened in that situation.

COROLLARY K. *For any family $(M_i)_{i\in I}$ of indecomposable endofinite modules, the following statements are equivalent*:

(1) *The number of isomorphism types of M_i is finite.*

(2) *$\bigoplus_{i\in I} M_i$ is endo-noetherian, and the family (M_i) is right semi-T-nilpotent.*

(3) *$\bigoplus_{i\in I} M_i$ is endo-artinian, and the family (M_i) is left semi-T-nilpotent.*

(4) *If $(M_t)_{t\in T}$ is a transversal of the isomorphism types of the M_i, then (M_t) is left and right semi-T-nilpotent and, for any cofinite subset $T' \subseteq T$, the endosocle of $\bigoplus_{t\in T'} M_t$ has finite support in T.*

Proof. In view of Theorem H, only the implication (4) $\Longrightarrow$ (1) requires justification: Since the M_i are endofinite, each singleton family $\{M_i\}$ is right T-nilpotent by Proposition E(2), whence right T-nilpotence of the family (M_i) follows from right semi-T-nilpotence. ∎

The next result exhibits a simplification of the picture for Artin algebras. Namely, a direct sum $\bigoplus_{i\in I} M_i$ of finitely generated modules M_i is Σ-algebraically compact precisely when it is endo-artinian. In other words, for Artin algebras, Proposition L provides an "endo-artinian counterpart" to the final observation of Section 1: Namely, this observation remains true on replacing "endo-noetherian" by "endo-artinian" and substituting the ACC for (finite) matrix subgroups by the corresponding DCC.

Throughout our excursion to the narrowed setting, Λ will be an Artin algebra with center C, and D: Λ-mod $\to$ mod-Λ will denote the standard duality from the category of finitely generated left to the category of finitely generated right Λ-modules, i.e., $D = \operatorname{Hom}_C(-, E)$, where E is the C-injective envelope of $C/J(C)$. Note that D induces a Morita duality on C-mod as well; this obvious fact is useful, because endo-submodules of Λ-modules are, of course, C-submodules.

PROPOSITION L. *If Λ is an Artin algebra and $(M_i)_{i\in I}$ a family of finitely generated left Λ-modules, the following conditions are equivalent*:

(1) $\bigoplus_{i\in I} M_i$ *is Σ-algebraically compact.*

(2) $\bigoplus_{i\in I} M_i$ *is endo-artinian.*

(3) $\bigoplus_{i\in I} D(M_i)$ *is endo-noetherian.*

Proof. To verify (1) $\Longrightarrow$ (3), we adopt (1). This means that $M = \bigoplus_{i\in I} M_i$ satisfies the DCC for finite matrix subgroups, whence, by Proposition 3 of [17], the dual $D(M) = \prod_{i\in I} D(M_i)$ has the ACC for finite matrix subgroups. In view of the final observation of Section 1, this entails (3).

To prove (3) $\Longrightarrow$ (2), we assume that $\bigoplus_{i\in I} D(M_i)$ is endo-noetherian. We begin by noting that a C-submodule U of $\bigoplus_{i\in I} M_i$ is an endo-submodule of $\bigoplus_{i\in I} M_i$ if and only if $U = \bigoplus_{i\in I} U_i$, where the U_i are subgroups of the M_i with the property that, for all $i, j \in I$ and all $f \in \operatorname{Hom}_\Lambda(M_i, M_j)$, the image $f(U_i)$ is contained in U_j. Now suppose that U is an endo-submodule of $\bigoplus_{i\in I} M_i$ and, for $i \in I$, let ι_i: $U_i \hookrightarrow M_i$ be the canonical embedding. Moreover, let $V_i \subseteq D(M_i)$ denote the kernel of the dual map $D(\iota_i)$; i.e., $V_i = \{\rho \in \operatorname{Hom}_C(M_i, E) \mid \rho|_{U_i} = 0\}$. We claim that $V = \bigoplus_{i\in I} V_i$ is an endo-submodule of $\bigoplus_{i\in I} D(M_i)$. Clearly, V is a C-submodule, whence it suffices to check stability under homomorphisms $g \in \operatorname{Hom}_\Lambda(D(M_i), D(M_j))$. Suppose $g = D(f)$ with $f \in \operatorname{Hom}_\Lambda(M_j, M_i)$. Dualizing the fact that $f \cdot \iota_j = \iota_i \cdot (f|_{U_j})$, we obtain, for each $\rho \in V_i$, the equality $g(\rho)|_{U_j} = (\rho \cdot f)|_{U_j} = (\rho|_{U_i}) \cdot (f|_{U_j}) = 0$; in other words, $g(\rho)$ belongs to V_j as required.

Let $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ be a descending chain of endo-submodules of $\bigoplus_{i\in I} M_i$ and label the embedding $U_{k+1} \hookrightarrow U_k$ by ϕ_k. Extending the above notation, we moreover write the embedding $U_{ki} = M_i \cap U_k \hookrightarrow M_i$ as ι_{ki} and the kernel of the C-module epimorphism $D(\iota_{ki})$ as V_{ki}. The preceding paragraph shows that each of the direct sums $V_k = \bigoplus_{i\in I} V_{ki}$ is in fact an endo-submodule of $\bigoplus_{i\in I} D(M_i)$. Clearly, the V_k form an ascending chain which, by our hypothesis, becomes stationary. This means that the ascending chains $V_{1i} \subset V_{2i} \subset \cdots$ become uniformly stationary on I, and, in view of the exact sequences

$$0 \to V_{ki} \to D(M_i) \to D(U_{ki}) \to 0,$$

we see that it is again uniformly on I that the maps $D(\phi_k|_{U_{ki}})$—and hence also the $\phi_k|_{U_{ki}}$—become isomorphisms. This proves that our initial descending chain $(U_k)_{k\in\mathbb{N}}$ becomes stationary.

In view of the characterization of Σ-algebraic compactness quoted in Section 1, the implication (2) $\Longrightarrow$ (1) is clear. ∎

The next corollary results from a combination of Corollary K and Proposition L with the Harada–Sai Lemma [12].

COROLLARY M. *Suppose Λ is an Artin algebra and $(M_i)_{i\in I}$ a family of left Λ-modules having uniformly bounded (finite) composition lengths. Moreover, let $(M'_t)_{t\in T}$ be a transversal of the isomorphism types of the indecomposable direct summands of the M_i. Then the following conditions are equivalent:*

(1) *T is finite.*

(2) *$\bigoplus_{i\in I} M_i$ is endo-noetherian.*

(3) *$\bigoplus_{i\in I} M_i$ is endo-artinian.*

(4) *$\bigoplus_{i\in I} M_i$ is Σ-algebraically compact.*

(5) *For any cofinite subset $T' \subseteq T$, the endosocle of $\bigoplus_{t\in T'} M'_t$ has finite support.*

Proof. It suffices to observe that, due to the Harada–Sai Lemma, our blanket hypothesis forces the family (M'_t) to be both left and right T-nilpotent. ∎

The final corollary in this series highlights the role played by the endosocles of direct sums of indecomposable representations. It is an immediate consequence of Corollary M and the (confirmed) second Brauer–Thrall conjecture.

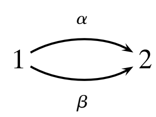
COROLLARY N. *For any finite-dimensional algebra Λ over an algebraically closed field, the following conditions are equivalent:*

(1) *Λ has finite representation type.*

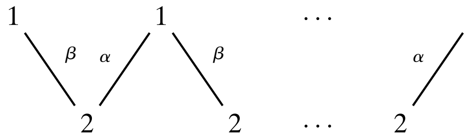
(2) *For every family $(M_i)_{i\in I}$ of pairwise non-isomorphic indecomposable modules of fixed finite dimension, the endosocle of $\bigoplus_{i\in I} M_i$ has finite support.*

The following examples attest to the fact that none of the T-nilpotence conditions in Theorem H and Corollaries I and K can be dropped without penalty.

EXAMPLES O. Let $\Lambda = K\Gamma$ be the Kronecker algebra over a field K, where Γ is the quiver



(1) Moreover, let $(M_n)_{n\in\mathbb{N}}$ be the family of all preinjective left Λ-modules, i.e., $M_1 = \Lambda e_1/J(\Lambda)e_1$, and, for $n \geq 2$, M_n is the string module with graph



having n summands S_1 in its top. Then the direct sum $\bigoplus_{n\in\mathbb{N}} M_n$ is Σ-algebraically compact by [19, Theorem 4.6], and Proposition L guarantees that $\bigoplus_{n\in\mathbb{N}} M_n$ is even endo-artinian.

(2) Let Λ still be the Kronecker algebra, but this time consider the family $(N_n)_{n\in\mathbb{N}}$ of all preprojective left Λ-modules, i.e., $N_n = D(M'_n)$, where M'_n is the preinjective right Λ-module with top dimension n. Since $\bigoplus_{n\in\mathbb{N}} M'_n$ is endo-artinian (see (1)), the direct sum $\bigoplus_{n\in\mathbb{N}} N_n$ of the duals of the M'_n is endo-noetherian, again by Proposition L.

At this point, we return to an arbitrary base ring. In light of our remarks concerning the exchange property of direct sums (in Section 3), we finally obtain the following consequence of Theorem H.

COROLLARY P. *An endo-noetherian direct sum $\bigoplus_{i\in I} M_i$ of indecomposable modules M_i (over any ring R) has the exchange property if and only if each M_i has local endomorphism ring and the M_i fall into finitely many isomorphism classes.*

Viewing the Kronecker examples in light of this fact, we see that the direct sum of the preprojective modules over the Kronecker algebra fails to have the exchange property. On the other hand, the direct sum ranging over the preinjective modules does enjoy this property, as do all algebraically compact modules (see [16]).

Variants of the ACC and DCC for endo-submodules have made numerous appearances in the representation theory of Artin algebras. Our concluding remarks, relating ascending endo-chain conditions of direct sums to preprojective partitions and left almost split morphisms, are merely the "tip of the iceberg." (Naturally, certain relaxed descending endo-chain conditions are linked with preinjective partitions and the existence of right almost split maps in a dual fashion.)

Remarks. Due to [3, 13], respectively, we have the following connections between weakened endo-noetherian conditions for direct sums $\bigoplus_{i\in I} M_i$ of finitely generated indecomposable modules, on the one hand, and the existence of preprojective partitions (resp. strong preprojective partitions) on the other: From now on, suppose that $(M_i)_{i\in I}$ is a family of finitely generated indecomposable modules and $(M_t)_{t\in T}$ a transversal of its isomorphism types. Recall from [3] that a *preprojective partition* of $(M_i)_{i\in I}$ is a partition

$$\{M_t \mid t \in T\} = \bigsqcup_{n\leq \omega} \mathscr{C}_n$$

such that, for all $n < \omega$, the set $\mathscr{C}_n$ is a minimal finite generating set for $\bigsqcup_{n\leq m\leq \omega} \mathscr{C}_m$. A *strong preprojective partition* extends this principle to arbitrary ordinals; i.e., it is a partition $\{M_t \mid t \in T\} = \bigsqcup_{\alpha\leq\gamma} \mathscr{C}_\alpha$, where γ is an ordinal number and each $\mathscr{C}_\alpha$ is a minimal finite generating set for $\bigsqcup_{\alpha\leq\beta\leq\gamma} \mathscr{C}_\beta$. If the M_i have perfect endomorphism rings, such a (strong) preprojective partition is unique in the case of existence. The existence of a preprojective partition of (M_i), on the other hand, is equivalent to the following endo-condition for $\bigoplus_{i\in I} M_i$: For each cofinite subset $T_1 \subseteq T$, there exists a cofinite subset $T_2 \subseteq T_1$ such that

$$\bigoplus_{t\in T_1} M_t \subseteq \left(\mathrm{End}_R\left(\bigoplus_{t\in T} M_t\right)\right) \cdot \left(\bigoplus_{t\in T_1\setminus T_2} M_t\right).$$

Moreover, the condition that $\bigoplus_{i\in I} M_i$ be endo-noetherian implies the existence of strong proprojective partitions for arbitrary subfamilies of $(M_i)_{i\in I}$.

The connection between endofiniteness conditions and strong preprojective partitions was recently refined by Dung [8, 3.9 and 3.11] for the case where the M_i have perfect endomorphism rings. Namely, he established the following bridges among the conditions labeled (a), (b), and (c) below:

(a) $\mathrm{Hom}_R(M_k, \bigoplus_{i\in I} M_i)$ is noetherian as a left module over $\mathrm{End}_R(\bigoplus_{i\in I} M_i)$ for all $k \in I$.

(b) For every subfamily $\mathscr{C}$ of (M_i), the full subcategory $\mathrm{add}\,\mathscr{C}$ of the category of all finitely generated modules has left almost split morphisms.

(c) Every subfamily of (M_i) has a strong preprojective partition.

Under the given hypotheses for the M_i, condition (a) implies (b). If, moreover, the direct sum $\bigoplus_{i\in I} M_i$ is finitely generated over its endomorphism ring, then (b) implies (c).

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