Journal of the Egyptian Mathematical Society (2014) 22, 317–321

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Gap sequences of 1-Weierstrass points on non-hyperelliptic curves of genus 10 ☆

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Received 20 July 2013; revised 18 November 2013; accepted 11 December 2013
Available online 25 January 2014

KEYWORDS

1-Weierstrass points;
q-Gap sequence;
Flexes;
Sextactic points;
Tentactic points;
Canonical linear system;
Kuribayashi sextic curve

Abstract In this paper, we compute the 1-gap sequences of 1-Weierstrass points of non-hyperelliptic smooth projective curves of genus 10. Furthermore, the geometry of such points is classified as flexes, sextactic and tentactic points. Also, upper bounds for their numbers are estimated.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 14H55; 14Q05; 14Q99

0. Introduction

Weierstrass points on curves have been extensively studied in connection with many problems. For example, the moduli space M_g has been stratified with subvarieties whose points are isomorphism classes of curves with particular Weierstrass points. For more deatails, we refer for example to [1,2].

At first, the theory of the Weierstrass points was developed only for smooth curves and for their canonical divisors. In the last decades, starting from some papers by Lax and Widland [3–8], the theory has been reformulated for Gorenstein curves, where the invertible dualizing sheaf substitutes the canonical sheaf. In this context, the singular points of a Gorenstein curve are always Weierstrass points.

In [9], Notari developed a technique to compute the Weierstrass gap sequence at a given point, no matter it is simple or singular, on a plane curve, with respect to any linear system $V \subseteq H^0(C, O_C(n))$. This technique can be useful to construct examples of curves with Weierstrass points of a given weight or to look for conditions for a sequence to be a Weierstrass gap sequence. He used this technique to compute the Weierstrass gap sequence at a point of particular curves and of families of quintic curves.

The aim of this paper is to compute the 1-gap sequence of the 1-Weierstrass points on smooth non-hyperelliptic algebraic curves of degree 6, which are genus 10 curves, to investigate the geometry of such kind of points and to estimate an upper bound for the numbers of flexes, sextactic and tentactic points on such kind of algebraic curves.

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☆ arXiv: 1307.0078[math.AG].

Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

1110-256X
http://dx.doi.org/10.1016/j.joems.2013.12.005

1. Preliminaries

Throughout this section, we use the following notations:

$I(C_1, C_2; p)$	the intersection number of the curves C_1 and C_2 at the point p [10],
$G_p^{(q)}(Q)$	the q-gap sequence of the point p with respect to the linear system Q [11,12],
$\omega^{(q)}(p)$	the q-weight of the point p [13],
$N^{(q)}(C)$	the number of q-Weierstrass points on C [14],
$Q(-(\ell \cdot p))$	the set of divisors in the linear system Q with multiplicity at least ℓ at the point p [13].

Recall that a linear system Q is called a g_d^r if $dim\ Q = r$ and $deg\ Q = d$. We have the following result.

Lemma 1.1 [13]. *Let Q be a nonempty g_d^r linear system on an algebraic curve X, and fix a point $p \in X$. Then:*

- *The set of gap numbers $G_p(Q)$ is a finite set and $|G_p(Q)| = 1 + r$.*
- *$G_p(Q) \subset \{1, 2, \ldots, 1 + d\}$.*

Let X be a smooth projective plane curve of genus $g \geqslant 2$ and let D be a divisor on C with $dim|D| = r \geqslant 0$. We denote by $L(D)$ the $\mathbb{C}$-vector space of meromorphic functions f such that $div(f) + D \geqslant 0$ and by $l(D)$ the dimension of $L(D)$ over $\mathbb{C}$. Then, the notion of D-Weierstrass points [13] can be defined in the following way:

Definition 1.2. Let $p \in C$. If n is a positive integer such that

$$l(D - (n-1)\cdot p) > l(D - n \cdot p),$$

we call the integer n a D-gap number at p.

Lemma 1.3. *Let $p \in C$, then there are exactly $r+1$ D-gap numbers $\{n_1, n_2, \ldots, n_{r+1}\}$ such that $n_1 < n_2 < \cdots < n_{r+1}$. The sequence $\{n_1, n_2, \ldots, n_{r+1}\}$ is called the D-gap sequence at p.*

Definition 1.4. The integer $\omega_D(p) := \sum_{i=1}^{r+1}(n_i - i)$ is called D-weight at p. If $\omega_D(p) > 0$, we call the point p a D-Weierstrass point on C. In particular, for the canonical divisor K, the qK-Weierstrass points ($q \geqslant 1$) are called q-Weierstrass points and the qK-weight is called q-weight, denoted by $\omega^{(q)}(p)$.

Lemma 1.5. [13,15]. *Let X be a smooth projective plane curve of genus g. The number of q-Weierstrass points $N^{(q)}(C)$, counted with their q-weights, is given by*

$$N^{(q)}(C) = \begin{cases} g(g^2-1), & \text{if } q = 1 \\ (2q-1)^2(g-1)^2 g, & \text{if } q \geqslant 2. \end{cases}$$

In particular, for smooth projective plane sextic (i.e. $g = 10$), the number of 1-Weierstrass points is 990 counted with their weights.

Theorem 1.6 [13]. *Let X be a non-hyperelliptic curve of genus $\geqslant 3$. Write $G_p^{(1)}(Q) = \{n_1 < n_2 < \cdots < n_g\}$, then*

(1) $n_1 = 1$.
(2) $n_r \leqslant 2r - 2$ *for every* $r \geqslant 2$.
(3) $\omega^{(1)}(p) \leqslant \frac{(g-1)(g-2)}{2}$.
(4) *There are at least* $2g + 6$ *1-Weierstrass points on* X.

For more details on q-Weierstrass points on Riemann surfaces, we refer for example to [13,16].

2. Main results

Let X be a smooth projective plane curve of degree 6 and $Q := |K|$ the canonical linear system of X.

Proposition 2.1. *The linear system Q is g_{18}^9.*

Proof. The result follows directly from

$$dim\ Q := dim\ |K| = g - 1 = 9, \quad \text{and}$$
$$deg\ (Q) := deg\ (K) = 2(g-1) = 18. \qquad \square$$

Corollary 2.2. *Let $p \in X$, then $\sharp\left(G_p^{(1)}(Q)\right) = 10$ and $G_p^{(1)}(Q) \subset \{1, 2, 3, \ldots, 19\}$.*

Lemma 2.3. *The set of cubic divisors on X form a linear system which is g_{18}^9.*

2.1. Flexes

Definition 2.4. [11,17]. A point p on a smooth plane curve C is said to be a flex point if the tangent line L_p meets C at p with contact order $I_p(C, L_p; p)$ at least three. We say that p is i-flex, if $I_p(C, L_p) - 2 = i$. The positive integer i is called the flex order of p.

Our main results for this part are the following.

Theorem 2.5. *Let p be a flex point on a smooth projective non-hyperelliptic plane curve X of degree 6. Let L_p be the tangent lint to X at p such that $I(X, E_p) = \mu_f$. Then $G_p^{(1)}(Q) = \{1, 2, 3, 1+\mu_f, 2+\mu_f, 3+\mu_f, 2\mu_f+1, 2\mu_f+2, 3\mu_f+1, 3\mu f+2\}$. Moreover, the geometry of such points is given by Table 1:*

Proof. The dimension of $Q(-1\cdot p)$ and $Q(-2\cdot p)$ does not depend on whether p is 1-Weierstrass point or not, i.e., $1, 2 \in G_p^{(1)}(Q)$. The spaces $Q(-3\cdot p) = \ldots = Q(-\mu_f \cdot p)$ consist of divisors of cubic curves of the form L_pR, where R is an arbitrary conic. Hence, $dim\ Q(-\ell\cdot p) = 6$ for $\ell = 3, \ldots, \mu_f$. That is, $3 \in G_p^{(1)}(Q)$.

The space $Q(-(1+\mu_f)\cdot p)$ consists of divisors of cubic curves of the form L_pR, where R is a conic passing through p.

Table 1 Geometry of flexes.

$\omega_p^{(1)}(Q)$	$G_p^{(1)}(Q)$	Geometry
2	$\{1, 2, 3, \ldots, 8, 10, 11\}$	*1-Flex*
15	$\{1, 2, 3, 5, 6, 7, 9, 10, 13, 14\}$	*2-Flex*
28	$\{1, 2, 3, 6, 7, 8, 11, 12, 16, 17\}$	*3-Flex*

Hence, $dim\ Q(-(1+\mu_f)\cdot p)=5$. That is, $1+\mu_f \in G_p^{(1)}(Q)$. The space $Q(-(2+\mu_f)\cdot p)$ consists of divisors of cubic curves of the form L_pR, where R is a conic passing through p with contact order at least 2. Hence, $dim\ Q(-(2+\mu_f)\cdot p)=4$. That is, $2+\mu_f \in G_p^{(1)}(Q)$. The spaces $Q(-(3+\mu_f)\cdot p)=\cdots=Q(-2\mu_f\cdot p)$ consist of divisors of cubic curves of the form L_p^2H, where H is an arbitrary hyperplane. Hence, $dim\ Q(-\ell\cdot p)=3$ for $\ell=3+\mu_f,\ldots,2\mu_f$. That is, $3+\mu_f\in G_p^{(1)}(Q)$. The space $Q(-(2\mu_f+1)\cdot p)$ consists of divisors of cubic curves of the form L_p^2H, where H is a hyperplane through p. Hence, $dim\ Q(-(2\mu_f+1)\cdot p)=2$. That is, $2\mu_f+1\in G_p^{(1)}(Q)$. The spaces $Q(-(2\mu_f+2)\cdot p)=\cdots=Q\ (-3\mu_f\cdot p)$ consist of divisors of cubic curves of the form L_p^2H, where H is a hyperplane through p with contact order at least 2. Hence, $dim\ Q(-(2\mu_f+2)\cdot p)=1$. That is, $2\mu_f+2\in G_p^{(1)}(Q)$. The space $Q(-(3\mu_f+1)\cdot p)$ consists of the divisor of the cubed tangent line L_p^3. Hence, $dim\ Q(-(3\mu_f+1)\cdot p)=0$. That is, $3\mu_f+1\in G_p^{(1)}(Q)$. The spaces $Q(-\ell\cdot p)=\phi$ for $\ell\geqslant 3\mu_f+2$. That is, $3\mu_f+2\in G_p^{(1)}(Q)$. Consequently,

$$G_p^{(1)}(Q)=\{1,2,3,1+\mu_f,2+\mu_f,3+\mu_f,2\mu_f+1,2\mu_f+2,3\mu_f+1,3\mu_f+2\}.$$

Finally, by the famous Bezout's theorem, the tangent line meets X at the flex point p with $3\leqslant\mu_f\leqslant 6$. On the other hand, by Theorem 1.6, it follows that $n_{10}:=3\mu_f+2\leqslant 18$, hence $\mu_f\neq 6$. □

Corollary 2.6. *If a smooth projective curve X of degree 6 has 4-flex points, then X is hyperelliptic.*

Corollary 2.7. *On a smooth non-hyperelliptics projective plane curve X of degree 6, the 1-weight of a flex point is given by $\omega_p^{(1)}(Q)=13\mu_f-37$, where μ_f is the multiplicity of the tangent line L_p to X at p.*

Proof. Let p be a flex point on X, then, by Theorem 2.5,

$$G_p^{(1)}(Q)=\{1,2,3,1+\mu_f,2+\mu_f,3+\mu_f,2\mu_f+1,2\mu_f+2,3\mu_f+1,3\mu_f+2\}.$$

Consequently,

$$\begin{aligned}\omega_p^{(1)}(Q):=\Sigma_{r=0}^{g}(n_r-r)&=(1+\mu_f-4)+(2+\mu_f-5)\\&+(3+\mu_f-6)+(2\mu_f+1-7)+(2\mu_f+2-8)\\&+(3\mu_f+1-9)+(3\mu_f+2-10)=13\mu_f-37.\end{aligned}$$ □

Notation. $F_i^{(q)}(X)$ will denote the set of i-flex points which are q-Weierstrass points on X and $NF_i^{(q)}(X)$ will denote the cardinality of $F_i^{(q)}(X)$.

Corollary 2.8. *For a smooth non-hyperelliptic projective plane curve X of degree 6, the maximal cardinality of $F_i^{(1)}(X)$ is given by the inequality*

$$NF_i^{(1)}(X)\leqslant\left[\frac{990}{13(i+2)-37}\right],$$

where $i=1,2,3$ and $[v]$ is the greatest integer less than or equal to v.

2.2. Sextactic points

In analogy with tangent lines and flexes of projective plane curves, one can consider *osculating conics* and *sextactic points* in the following way:

Lemma 2.9 [12]. *Let p be a non-flex point on a smooth projective plane curve X of degree $d\geqslant 3$. Then there is an unique irreducible conic D_p with $I_p(X,D_p;p)\geqslant 5$. This unique irreducible conic D_p is called the osculating conic of X at p.*

Definition 2.10 [11]. A non-flex point p on a smooth projective plane curve X is said to be a *sextactic point* if the osculating conic D_p meets X at p with contact order at least six. A sextactic point p is said to be *i-sextactic*, if $I_p(X,D_p;p)-5=i$. The positive integer i is called the sextactic order.

Now, the main results for this part are the following.

Theorem 2.11. *Let p be a sextactic point on a smooth projective non-hyperelliptic curve X of degree 6. Let D_p be the osculating conic to X at p such that $I(X,D_p;p)=\mu_s$. Then, $G_p^{(1)}(Q)=\{1,2,3,\ldots,7,1+\mu_s,2+\mu_s,3+\mu_s\}$.*

Proof. The idea of the proof is to investigate the existence of a curve H through p with multiplicity ℓ so that its divisor is in $Q(-\ell\cdot p)-Q(-(\ell+1)\cdot p)$, consequently, the integer $\ell\in G_p^{(1)}(Q)$. Now, the dimension of $Q(-1\cdot p)$ and $Q(-2\cdot p)$ does not depend on whether p is a 1-Weierstrass point or not, i.e., $1,2\in G_p^{(1)}(Q)$. Moreover, let L_p be the tangent line to X at p, then the divisor $div(L_pR_0)\in Q(-2\cdot p)-Q(-3\cdot p)$, where R_0 is a conic not through p. That is, $3\in G_p^{(1)}(Q)$. Furthermore, $div(L_pR_1)\in Q(-3\cdot p)-Q(-4\cdot p)$, where R_1 is a conic passing through p with multiplicity 1. That is, $4\in G_p^{(1)}(Q)$. Also, $div\left(L_p^2H_0\right)\in Q(-4\cdot p)-Q(-5\cdot p)$, where H_0 is a hyperplane not through p. That is, $5\in G_p^{(1)}(Q)$. Similarly, $div\left(L_p^2H_1\right)\in Q(-5\cdot p)-Q(-6\cdot p)$, where H_1 is a hyperplane passing through p with multiplicity 1. That is, $6\in G_p^{(1)}(Q)$.

Now, the spaces $Q(-7\cdot p)=\ldots=Q(-\mu_s\cdot p)$ consist of divisors of cubic curves of the form D_PH, where H is an arbitrary line. Hence, $7\in G_p^{(1)}(Q)$. On the other hand, the space $Q(-(1+\mu_s)\cdot p)$ consists of divisors of cubic curves of the form D_PH, where H is a hyperplane through p. Consequently, $1+\mu_s\in G_p^{(1)}(Q)$. Also, the space $Q(-(2+\mu_s)\cdot p)$ contains only the cubic divisor D_PL_p. Then, $2+\mu_s\in G_p^{(1)}(Q)$. Finally, $Q(-\ell.p)=\phi$, for $\ell\geqslant 3+\mu_s$. That is, $3+\mu_s\in G_p^{(1)}(Q)$. □

Corollary 2.12. *Let p be a sextactic point on a smooth projective non-hyperelliptic curve X of degree 6. Then, the geometry of such points is given by Table 2:*

Table 2 Geometry of sextactic points.

$\omega_p^{(1)}(Q)$	$G_p^{(1)}(Q)$	Geometry
3	$\{1, 2, 3, \ldots, 7, 9, 10, 11\}$	*3-Sextactic*
6	$\{1, 2, 3, \ldots, 7, 10, 11, 12\}$	*4-Sextactic*
9	$\{1, 2, 3, \ldots, 7, 11, 12, 13\}$	*5-Sextactic*
12	$\{1, 2, 3, \ldots, 7, 12, 13, 14\}$	*6-Sextactic*
15	$\{1, 2, 3, \ldots, 7, 13, 14, 15\}$	*7-Sextactic*

Proof. It follows, by Theorem 2.11 and Bezout's theorem, that D_p meets X at p with $8 \leqslant \mu_s \leqslant 12$. Hence, varying μ_s produces the last table. □

Corollary 2.13. *If a smooth projective curve X of degree 6 has 1-sextactic or 2-sextactic points, then X is hyperelliptic.*

Corollary 2.14. *On a smooth non-hyperelliptic projective curve X of degree 6, the 1-weight of a sextactic point is given by* $\omega_p^{(1)}(Q) = 3\mu_s - 21$, *where* μ_s *is the multiplicity of the osculating conic* D_p *at p.*

Proof. If p is a sextactic point on C, then $G_p^{(1)}(Q) = \{1, 2, 3, \ldots, 7, 1 + \mu_s, 2 + \mu_s, 3 + \mu_s\}$. Consequently,

$$\begin{aligned}\omega_p^{(1)}(Q) &:= \Sigma_{r=0}^{g}(n_r - r)\\ &= (1 + \mu_s - 8) + (2 + \mu_s - 9) + (3 + \mu_s - 10)\\ &= 3\mu_s - 21. \quad \square\end{aligned}$$

Notation. $S_i^{(q)}(X)$ will denote the set of i-sextactic points which are q-Weierstrass points on X and $NS_i^{(q)}(X)$ will denote the cardinality of the set $S_i^{(q)}(X)$.

Corollary 2.15. *For a smooth non-hyperelliptic projective curve X of degree 6, the maximal cardinality of* $S_i^{(1)}(X)$ *is given by*

$$NS_i^{(1)}(X) \leqslant \left[\frac{990}{3(i+5) - 21}\right],$$

where $i = 3, 4, 5, 6, 7$ *and* $[v]$ *is the greatest integer less than or equal to v.*

2.3. Tentactic points

Definition 2.16 (*11,18*). *A point p on a smooth plane curve C of genus* $g \geqslant 2$, *which is neither flex nor sextactic point, is said to be a tentactic point, if there a cubic* E_p *which meets C at p with contact order at least 10. The positive integer t such that* $i := I(C, E_p; p) - 9$ *is called the tentactic order of p. Moreover, the point p is said to be i-tentactic.*

Theorem 2.17. *Let p be a tentactic point on a smooth projective non-hyperelliptic curve X of degree 6 and let* E_p *be its osculating cubic curve such that* $I(X, E_p; p) = \mu_t$. *Then* $G_p^{(1)}(Q) = \{1, 2, 3, \ldots, 9, 1 + \mu_t\}$. *Moreover, the geometry of such points is given by the table:*

$\omega_p^{(1)}(Q)$	$G_p^{(1)}(Q)$	Geometry
1	{1,2,3,...,9,11}	*1-tentactic*
2	{1,2,3,...,9,12}	*2-tentactic*
3	{1,2,3,...,9,13}	*3-tentactic*
4	{1,2,3,...,9,14}	*4-tentactic*
5	{1,2,3,...,9,15}	*5-tentactic*
6	{1,2,3,...,9,16}	*6-tentactic*
7	{1,2,3,...,9,17}	*7-tentactic*
8	{1,2,3,...,9,18}	*8-tentactic*

Proof. Since the point p is neither flex nor sextactic, then

$$dim\ Q(-\ell \cdot p) = 9 - \ell \quad \text{for} \quad \ell = 1, 2, 3, \ldots, 9.$$

Hence, $1, 2, 3, \ldots, 9 \in G_p^{(1)}(Q)$. Moreover, assuming that $I(X, E_p) = \mu_t$, then

$$div\ (E_p) \in Q(-\mu_t \cdot p) - Q(-(1 + \mu_t) \cdot p).$$

Therefore, $1 + \mu_t \in G_p^{(1)}(Q)$. Consequently, $G_p(Q) = \{1, 2, 3, \ldots, 9, 1 + \mu_t\}$. Finally, $19 \notin G_p^{(1)}(Q)$ as $n_{10} \leqslant 18$. □

Corollary 2.18. *If a smooth projective curve X of degree 6 has 9-tentactic points, then X is hyperelliptic.*

Notation. $T_i^{(q)}(C)$ will denote the set of i-tentactic points which are q-Weierstrass points on X and $NT_i^{(q)}(X)$ will denote the cardinality of the set $T_i^{(q)}(X)$.

Corollary 2.19. *For a smooth projective non-hyperelliptic curve C of degree 6, the maximal cardinality of* $T_i^{(1)}(X)$ *is given by the following inequality*

$$NT_i^{(1)} \leqslant \left[\frac{990}{i}\right],$$

where $i = 1, 2, 3, \ldots, 8$ *and* $[v]$ *is the greatest integer less than or equal to v.*

3. Concluding remarks

We conclude the paper by the following remarks and comments.

- In this article, the 1-gap sequence of the 1-Weierstrass points on smooth non-hyperelliptic algebraic curves of degree 6 is computed. Furthermore, the geometry of Weierstrass points is classified as flexes, sextactic and tentactic points. On the other hand, we show that a smooth non-hyperelliptic curve of degree 6 has no 4-flex, no 1-sextactic, no 2-sextactic and no 9-tentactic points. Also, an upper bound for the numbers of flexes, sextactic and tentactic points on such curves is estimated.
- The main theorems constitute a motivation to solve more general problems. One of these problems is the investigation of the geometry of higher order and multiple Weierstrass points on non-hyperelliptic degree 6 curves.Another problem is combining the main results together with the classification of degree 6 non-hyperelliptic curves in [19]

to study the orbits and fixed points of cyclic automorphisms and its interrelation with q-Weierstrass points. However, these problems will be the object of a forthcoming work.

- The idea of the proof of Theorems 2.5, 2.11 and 2.17 may be applied for general non-singular plane curves of degree d in order to obtain a general statement covering the plane curves of any degree.

Acknowledgment

The authors would like to thank the referee for his valuable comments and suggestions which improved the manuscript in its present form.

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