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On the Number of Bifurcation Branches of C^2-Maps

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1. INTRODUCTION

Let $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a C^2-map such that $f(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$. A standard question arising in this situation concerns the existence of bifurcation points of f (see Definition 1.1).

Up to now, a lot of theorems giving sufficient conditions for the existence of such points have been proved, among them Krasnosielski's theorem (see [5]) is the fundamental one.

In proofs of theorems of that type methods of algebraic topology are mainly used and the most popular tool is the topological degree.

Assume that the set of nontrivial zeroes of f is a 1-dimensional submanifold of $(\mathbb{R}^n \times \mathbb{R}) - (\{0\} \times \mathbb{R})$.

In this paper bifurcation branches of zeroes of f are investigated. More exactly, one can ask for the number of branches of zeroes of f emanating from an interval $\{0\} \times [\lambda_1, \lambda_2]$, where $(0, \lambda_1)$ and $(0, \lambda_2)$ are not bifurcation points.

Such a number is realized as the topological degree of a map which will be constructed below.

2. PRELIMINARIES

In this section we summarize without proofs the relevant material on bifurcation theory which will be used in our considerations.

Let $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a continuous map such that $f(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$. Define the set of nontrivial zeroes of f by

$$Z_f = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} \mid f(x, \lambda) = 0 \text{ and } x \neq 0\}.$$

DEFINITION 1.1. Any point $(0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}$ is said to be a bifurcation point of f provided $(0, \lambda_0) \in \mathrm{cl}(Z_f)$. The set of all bifurcation points of f we will denote by $B(f)$.

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For an arbitrary two points $(0, \lambda_1)$, $(0, \lambda_2)$ which are not bifurcation points one can always find a positive number α such that

$$(\|x\| \leqslant \alpha \wedge (x, \lambda) \in S^n(\lambda_1, \lambda_2) \wedge f(x, \lambda) = 0) \Rightarrow (x = 0),$$

where $S^n(\lambda_1, \lambda_2)$ is an n-dimensional sphere with the center at $(0, \frac{1}{2} \cdot (\lambda_1 + \lambda_2))$ and the radius which is equal to $\frac{1}{2} \cdot |\lambda_1 - \lambda_2|$.

Consider the $(n+1)$-dimensional disc $D^{n+1}(\lambda_1, \lambda_2)$ corresponding to the sphere $S^n(\lambda_1, \lambda_2)$. For any positive $\gamma \leqslant \alpha$ a map

$$(f, \theta_\gamma) : (D^{n+1}(\lambda_1, \lambda_2), S^n(\lambda_1, \lambda_2)) \to (\mathbb{R}^n, \mathbb{R}^n - \{0\})$$

can be defined as

$$(f, \theta_\gamma)(x, \lambda) = (f(x, \lambda), \|x\|^2 - \gamma^2).$$

The homotopy class of the map (f, θ_γ) does not depend on the choice of γ and that is why it will be simply denoted by (f, θ).

The above construction which is known as the complementing map construction is a standard trick often used in bifurcation theory.

Now, assume additionally that $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a C^1-map and if $f(x, \lambda) = 0$ and rank $[Df(x, \lambda)] < n$, then $x = 0$.

It follows that $f^{-1}(0) - \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} \mid x = 0\}$ is a 1-dimensional C^1-submanifold of $\mathbb{R}^n \times \mathbb{R} - (\{0\} \times \mathbb{R})$.

DEFINITION 1.2. A set S is called a branch of zeroes of f if it is a connected component of the manifold $f^{-1}(0) - (\{0\} \times \mathbb{R})$. The set of all branches we will denote by G.

In [7] the following theorem has been proved:

THEOREM 1.1. *Let* $F = (F_1, ..., F_{n-1})\colon U \to \mathbb{R}^{n-1}$, $G\colon U \to \mathbb{R}$ *be* C^2-*functions in an open set* $U \in \mathbb{R}^n$. *Assume that rank* $[DF(x_0)] = n - 1$, *where* $x_0 \in U$.

From the implicit function theorem $W = \{x \in U \mid F(x) = F(x_0)\}$ *is a* 1-*dimensional* C^2-*manifold in some neighbourhood of* x_0.

Let $\Delta = \partial(G, F_1, ..., F_{n-1})/\partial(x_1, ..., x_n)$ *be the Jacobian of a map* $(G, F_1, ..., F_{n-1})\colon U \to \mathbb{R}^n$, $H = (\Delta, F_1, ..., F_{n-1})\colon U \to \mathbb{R}^n$ *and let* $\Delta_1 = \partial(\Delta, F_1, ..., F_{n-1})/\partial(x_1, .., x_n) = \det[DH]$.

Then

(i) $G|_W$ *has a critical point at* x_0 *iff* $\Delta(x_0) = 0$ *and* $\Delta_1(x_0) = 0$,

(ii) $G|_W$ *has a nondegenerate critical point at* x_0 *iff* $\Delta(x_0) = 0$ *and* $\Delta_1(x_0) \neq 0$,

(iii) *if* $\Delta(x_0) = 0$ *and* $\Delta_1(x_0) > 0$ (<0), *then* $G|_W$ *has a minimum (maximum) at* x_0.

3. Main Results

Consider a C^2-map $T_1: (D^n, S^{n-1}) \to (\mathbb{R}^n, \mathbb{R}^n - \{0\})$ such that $0 \in \mathbb{R}^n$ is a regular value of T_1, where $D^n \subset \mathbb{R}^n$ is a unit disc and $S^{n-1} = \partial D^n$.

Then a map $T: D^{n+1} \to \mathbb{R}^n$ defined by the formula

$$T(x) = \begin{cases} \|x\|^7 \cdot T_1\left(\dfrac{x_1}{\|x\|}, ..., \dfrac{x_n}{\|x\|}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

has the following properties:

(1) T is a C^2-map,

(2)

$$T^{-1}(0) = \begin{cases} \{t \cdot x \mid t \in [0, 1] \wedge T(x) = 0 \wedge \|x\| = 1\} & \text{if } T_1^{-1}(0) \neq \varnothing \\ 0 & \text{if } T_1^{-1}(0) = \varnothing \end{cases}$$

(3) if $x \in T^{-1}(0)$ and $x \neq 0$, then rank $[DT(x)] = n$.

Now, for maps $T(x)$ and $\omega(x) = x_1^2 + \cdots + x_{n+1}^2$ the construction given in Theorem 1.1 can be applied.

As a result we obtain a map $H: (D^{n+1}, S^n) \to (\mathbb{R}^{n+1}, \mathbb{R}^{n+1} - \{0\})$, $H(x) = (\Delta(x), T(x))$ and using the same methods as in [7] we derive at once the equality

$$\deg(H, D^{n+1} - S^n, 0) = \# T_1^{-1}(0)$$

($\# A$ denotes the number of elements of a set A).

From now on, $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ denotes a map which satisfies the following conditions:

(1) f is a C^2-map,

(2) $f(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$,

(3) the set $B(f)$ of bifurcation points of f is discrete.

Lemma 2.1. *Let $(0, \lambda_1), (0, \lambda_2) \notin B(f)$. Assume that*

(1) *if $f(x, \lambda) = 0$ and* rank $[Df(x, \lambda)] < n$, *then $x = 0$,*

(2) *there exists a number $\beta > 0$ such that if $f(x, \lambda) = 0$ and $0 < \|x\| \leqslant \beta$, then $Df(x, \lambda): V_x \to \mathbb{R}^n$ is an isomorphism, where $V_x = \{(y, \lambda) \in \mathbb{R}^n \times \mathbb{R} \mid \langle x, y \rangle = 0\}$. Then there is only finite number of branches of zeroes of f bifurcating from the interval $\{0\} \times [\lambda_1, \lambda_2]$.*

Proof. Without loss of generality we can assume that the set $B(f) \cap (\{0\} \times [\lambda_1, \lambda_2]\})$ consists exactly of one point.

For any $\delta > 0$ put

$$C_\delta = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| = \delta \text{ and } \lambda_1 \leqslant \lambda \leqslant \lambda_2\}$$

and let $\hat{G}$ be the set of those branches $S \in G$ which bifurcate from the interval $\{0\} \times [\lambda_1, \lambda_2]$.

First, taking α as in the complementing map construction it will be shown that only a finite number of branches $S \in \hat{G}$ has a nonempty intersection with C_δ, for $\delta < \alpha$.

Suppose by contradiction that it is false. Then there is a sequence of points $\{(x_i, \lambda_i)\}_{i \in \mathbb{N}} \subset C_\delta$ such that different points belong to different branches.

By the compactness of C_δ one can choose a subsequence $\{(x_{i_k}, \lambda_{i_k})\}$ with $\lim_{k \to \infty}(x_{i_k}, \lambda_{i_k}) = (\bar{x}, \bar{\lambda}) \in C_\delta$. Of course, $f(\bar{x}, \bar{\lambda}) = 0$ and $\bar{x} \neq 0$.

Considering $f^{-1}(0)$ in any neighbourhood of $(\bar{x}, \bar{\lambda})$ we claim that it is not a submanifold of $\mathbb{R}^n \times \mathbb{R} - (\{0\} \times \mathbb{R})$, which contradicts Assumption 1.

Now assume that for some $\delta < \min\{\alpha, \beta\}$ there exists a branch $S \in \hat{G}$ such that $S \cap C_\delta = \varnothing$. Let us define a map $g\colon \text{cl}(S) \to \mathbb{R}$ by the formula $g(x, \lambda) = x_1^2 + \cdots + x_n^2$. Clearly, g is bounded.

Denote by (x_M, λ_M) a point, g has a maximum at.

Since $f^{-1}(0)$ is a C^2-manifold in a neighborhood of (x_M, λ_M), a diffeomorphism $(x, \lambda)\colon (-\varepsilon, \varepsilon) \to S$ can be chosen in such a way that $(x(0), \lambda(0)) = (x_M, \lambda_M)$ and a map $h\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ defined by the equality $h(t) = g(x(t), \lambda(t))$ is a C^2-map, $\varepsilon > 0$.

It is easily seen that $h'(0) = 0$ and therefore

$$Df(x_M, \lambda_M)(x'(0), \lambda'(0)) = 0$$

for $(x'(0), \lambda'(0)) \in V_{x_M}$ and $(x'(0), \lambda'(0)) \neq 0$, which is impossible. ∎

COROLLARY 2.1. *For any $\delta < \min\{\alpha, \beta\}$ and any $S \in \hat{G}$ one has $S \cap C_\delta \neq \varnothing$.*

LEMMA 2.2. *Under the assumptions of Lemma 2.1 for any $\delta < \min\{\alpha, \beta\}$ and $S \in \hat{G}$ one has*

> $\#(S \cap C_\delta) = 2$ *if S is bounded and* cl$(S) - S$ *is contained in the interval* $\{0\} \times [\lambda_1, \lambda_2]$, *else* $\#(S \cap C_\delta) = 1$.

Proof. Fix $\delta < \min\{\alpha, \beta\}$ and assume that $\#(S \cap C_\delta) > 2$ for some $S \in \hat{G}$. Then for any three points $z_0, z_1, z_2 \in S \cap C_\delta$ there exists a compact connected subset $S_1 \subset S$ which is in fact a 1-dimensional submanifold of S with boundary, such that $z_0, z_1, z_2 \in S_1$.

Define a map $h: [-2, 2] \to \mathbb{R}$ by $h(t) = \|x(t)\|^2$, where $(x(t), \lambda(t))$: $[-2, 2] \to S_1$ is a C^2-diffeomorphism of manifolds with boundaries. For simplicity, put $(x, \lambda)(-1) = z_0$, $(x, \lambda)(0) = z_1$, $(x, \lambda)(1) = z_2$. Hence $h(-1) = h(0) = h(1)$, so there is a minimum of h in the interior of $[-1, 1]$, say t_1.

Of course, $0 < h(t_1) \leqslant \delta^2$, and for $0 \neq (x'(t_1), \lambda'(t_1))$ and $(x'(t_1), \lambda'(t_1)) \in V_{x(t_1)}$ we obtain $Df((x(t_1), \lambda(t_1))((x'(t_1), \lambda'(t_1)) = 0$, but in view of Assumption 2 we obtain a contradiction.

Now choose $S \in \hat{G}$ such that $\#(S \cap C_\delta) = 2$ and $\{z_0, z_1\} = S \cap C_\delta$. Suppose on the contrary, that S is unbounded or $\mathrm{cl}(S) - S$ is not contained in the interval $\{0\} \times [\lambda_1, \lambda_2]$.

Then there exists a point $z_2 \in S$ and a compact, connected submanifold $S_1 \subset S$ containing points z_0, z_1, z_2 such that for the map $h: [-2, 2] \to \mathbb{R}$ defined above one has $h(1) = \sum_{i=1}^{n} x_i^2(1) > \delta^2$.

As previously, we immediately obtain a minimum of h at a point $t_1 \in (-1, 1)$ which is less than or equal to δ^2. But it has already been shown that it is impossible.

Let $\#(S \cap C_\delta) = 1$ for some bounded $S \subset \hat{G}$ and assume that $(\mathrm{cl}(S) - S) \subset \{0\} \times [\lambda_1, \lambda_2]$. Defining a C^2-map $g: S \to \mathbb{R}$ by $g(x, \lambda) = \|x\|^2$ we claim that g is bounded and $g(x_0, \lambda_0) = \delta^2$ for some $(x_0, \lambda_0) \in S$. Choose a regular value of g, say y_0, such that $0 < y_0 < \delta^2$. Then $g^{-1}([y_0, \infty))$ is a 1-dimensional submanifold of S with boundary $g^{-1}(y_0)$ and $\# g^{-1}(y_0) \geqslant 2$.

Let g have a maximum at (x_M, λ_M) and let S_1 be a connected component of $g^{-1}([y_0, \infty))$ including (x_M, λ_M). From our assumptions it follows that $g((x_M, \lambda_M)) > \delta^2$.

Taking once more a C^2-diffeomorphism $(x, \lambda): [-1, 1] \to S_1$ such that $\{(x(-1), \lambda(-1)), (x(1), \lambda(1))\} = \partial S_1$ and considering a map $h: [-1, 1] \to \mathbb{R}$ we obtain this time two points $t_1 \in (-1, 0)$ and $t_2 \in (0, 1)$ such that $h(t_1) = h(t_2) = \delta^2$, but this contradicts Assumption 2. ∎

From the above lemmas it is easily seen that if only $\delta < \min\{\alpha, \beta\}$ a map (f, θ_δ) obtained by the complementing map construction has the following properties:

(1) $(f, \theta_\delta): (D^{n+1}(\lambda_1, \lambda_2), S^n(\lambda_1, \lambda_2)) \to (\mathbb{R}^{n+1}, \mathbb{R}^{n+1} - \{0\})$ is a C^2-map,

(2) $0 \in \mathbb{R}^{n+1}$ is a regular value of (f, θ_δ),

(3) $\#(f, \theta_\delta)^{-1}(0) = b_1 + 2 \cdot b_2$, where $b_i = \#\{S \in \hat{G} \mid \#(S \cap C_\delta) = i\}$, $i = 1, 2$.

Therefore, the construction described in the first part of this paragraph can be applied to this map and as a result we obtain a map $H: (D^{n+2}, S^{n+1}) \to (\mathbb{R}^{n+2}, \mathbb{R}^{n+2} - \{0\})$ satisfying the following condition

$$\deg(H, D^{n+1} - S^{n+1}, 0) = \#(f, \theta_\delta)^{-1}(0).$$

Set $d(f; \lambda_1, \lambda_2) = \deg(H, D^{n+2} - S^{n+1}, 0)$.

Note that we have actually proved:

THEOREM 2.1. *Let* $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *be a* C^2*-map such that:*

(1) $f(0, \lambda) = 0$ *for all* $\lambda \in \mathbb{R}$,

(2) *if* $f(x, \lambda) = 0$ *and* rank $[Df(x, \lambda)] < n$, *then* $x = 0$,

(3) *there exists a number* $\beta > 0$ *such that if* $f(x, \lambda) = 0$ *and* $0 < \|x\| < \beta$ *then* $Df(x, \lambda): V_x \to \mathbb{R}^n$ *is an isomorphism, where* $V_x = \{(y, \lambda) \in \mathbb{R}^n \times \mathbb{R} \mid \langle x, y \rangle = 0\}$,

(4) *the set* $B(f)$ *of bifurcation points is discrete. Then, for each* $(0, \lambda_1)$, $(0, \lambda_2) \notin B(f)$, *the number of branches of zeroes of f which bifurcate from the interval* $\{0\} \times [\lambda_1, \lambda_2]$ *is equal to* $d(f; \lambda_1, \lambda_2)$.

COROLLARY 2.2. *If* $d(f; \lambda_1, \lambda_2)$ *is an odd number then there exists a bifurcation point outside the interval* $\{0\} \times [\lambda_1, \lambda_2]$ *or at least one branch of zeroes bifurcating from* $\{0\} \times [\lambda_1, \lambda_2]$ *is unbounded.*

To simplify the notation we write $\deg(f, \theta)$ instead of $\deg((f, \theta_\delta), \operatorname{int} D^{n+1}((0, \frac{1}{2} \cdot (\lambda_1 + \lambda_2)), \frac{1}{2} \cdot |\lambda_1 - \lambda_2|), 0)$.

COROLLARY 2.3. *For any* $\delta < \min\{\alpha, \beta\}$ *the following inequalities hold:*

$$|\deg(f, \theta)| \leqslant \#\{S \in \hat{G} \mid \#(S \cap C_\delta) = 1\} \leqslant \#\hat{G} = d(f; \lambda_1, \lambda_2).$$

In particular, if $B(f) \subset (\{0\} \times [\lambda_1, \lambda_2])$ *then the number of unbounded components of zeroes of f is greater than or equal to* $|\deg(f, \theta)|$.

COROLLARY 2.4. *For* $\delta < \min\{\alpha, \beta\}$ *we obtain*

$$\deg(f, \theta) \equiv d(f; \lambda_1, \lambda_2) \bmod 2.$$

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