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A linearized inverse boundary value problem for Maxwell's equations

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Abstract

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In this paper, we consider the problem of determining the electromagnetic state of a body from measurements made on the surface of the body. We study the full set of Maxwell's equations that govern time-harmonic electric and magnetic fields. We seek to reconstruct the magnetic permeability μ, the electric permittivity ϵ and the electric conductivity σ in the interior of the body from measurements made on the surface. We exhibit appropriate boundary measurements in the form of a boundary mapping, specifically the mapping from the tangential components of the electric field to the tangential components of the magnetic field. This data can be used to reconstruct μ, ϵ and σ approximately, provided they deviate only slightly from known constants. We also estimate the reconstruction errors.

Keywords: Impedance imaging, Maxwell's equations, distorted plane waves.

1. Introduction

In his paper [1], Calderón considered the problem of determining an approximation to the electrical conductivity in a bounded region Ω of $\mathbb{R}^3$ from electrical measurements made on the surface $\partial\Omega$. This *impedance imaging* problem has recently been studied by a number of authors, see, e.g., [4,6,7,9]. In these works, the time variation of the electromagnetic fields is neglected, so that the electric state of Ω is governed by a single second-order elliptic partial differential

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equation of electrostatics, where only one unknown function (electric conductivity) appears. In this article, we study the full set of Maxwell's equations that govern time-harmonic electric and magnetic fields. The inverse problem therefore involves three unknown coefficients.

Here we consider the case in which Ω is a bounded domain in $\mathbb{R}^3$. For simplicity, we assume that Ω has a C^2 boundary, i.e., each point at the boundary $\partial\Omega$ has a neighborhood U such that $U \cap \overline{\Omega}$ is C^2-isomorphic with the upper half space $\overline{\mathbb{R}^3_+} = \{x = (x_1, x_2, x_3) \mid x_3 \geqslant 0\}$. We shall consider the electromagnetic fields (E, H) in Ω satisfying frequency domain Maxwell's equations

$$\nabla \wedge E = \mathrm{i}\omega\mu H, \qquad \nabla \wedge H = -\mathrm{i}\omega\epsilon E + \sigma E, \tag{1.1}$$

in Ω, $\omega > 0$ being the time-harmonic frequency of the fields. The magnetic permeability μ, the electric permittivity ϵ and the conductivity σ are assumed to satisfy the following regularity conditions in Ω:

$$\mu, \epsilon, \sigma \in C^1(\overline{\Omega}) \cap C^2(\Omega), \tag{1.2}$$

$$\mu, \epsilon > 0, \quad \sigma \geqslant 0. \tag{1.3}$$

To state the inverse problem, the first question is what boundary data we should consider. The classical impedance imaging problem assumes the knowledge of the map that associates currents to voltages, namely the Dirichlet-to-Neumann map or its inverse, the resistivity map. In fact, this mapping is the kernel of the quadratic form that gives the total energy needed to maintain a given voltage or current pattern on the surface of the unknown conductor Ω. In the time-dependent case the total energy flux Φ through the boundary $\partial\Omega$ is

$$\Phi = \mathrm{Re}\int_{\partial\Omega} \nu \cdot (E \wedge \overline{H})\, \mathrm{d}S = \mathrm{Re}\int_{\partial\Omega} (\nu \wedge E) \cdot \overline{H}\, \mathrm{d}S. \tag{1.4}$$

Here, ν is the unit normal vector of $\partial\Omega$ pointing outwards from Ω. Formula (1.4) suggests that in the case of nonstatic theory we take the data to be the boundary mapping

$$\Lambda : \nu \wedge E\,|_{\partial\Omega} \mapsto \nu \wedge H\,|_{\partial\Omega}. \tag{1.5}$$

This map will be discussed further in Section 3.

In this work we suggest an approximate constructive procedure to recover the electromagnetic parameters ϵ, μ and σ from the knowledge of Λ. The method is similar to the distorted plane wave approximation given by Calderón. As in his work, this approximate linearization scheme is expected to work well in the case where the electromagnetic parameters deviate only slightly from constant values. In Section 2 we give the outline of the method, postponing the detailed discussion and proofs to Section 3.

2. The distorted plane wave approximation: an outline

In this section we explain the basic idea of the linearizing approximation scheme.

Assume that the coefficient functions μ, ϵ and σ can be written as

$$\mu(x) = \mu_0 + \delta\mu(x), \qquad \epsilon(x) = \epsilon_0 + \delta\epsilon(x), \qquad \sigma(x) = \sigma_0 + \delta\sigma(x), \tag{2.1}$$

where ϵ_0, μ_0 and σ_0 are constants, $\epsilon_0, \mu_0 > 0$ and $\sigma_0 \geqslant 0$, and the perturbations $\delta\mu$, $\delta\epsilon$ and $\delta\sigma$

are small to an order that will be specified later. Consider first the constant medium equations

$$\nabla \wedge E_0 = \mathrm{i}\omega\mu_0 H_0, \qquad \nabla \wedge H_0 = -\mathrm{i}\omega\epsilon_0 E_0 + \sigma_0 E_0. \tag{2.2}$$

For any $\xi \in \mathbb{R}^3$, assume that we can always find a pair (E_0, H_0) and (E_0^*, H_0^*) of solutions to the system (2.2) with the following properties: (E_0, H_0) and (E_0^*, H_0^*) are of the form

$$E_0(x) = \alpha\, \mathrm{e}^{\mathrm{i}\zeta\cdot x}, \qquad H_0(x) = \beta\, \mathrm{e}^{\mathrm{i}\zeta\cdot x}, \tag{2.3}$$

and

$$E_0^*(x) = \alpha^* \mathrm{e}^{\mathrm{i}\zeta^*\cdot x}, \qquad H_0^*(x) = \beta^* \mathrm{e}^{\mathrm{i}\zeta^*\cdot x}, \tag{2.4}$$

for some α, β, ζ and α^*, β^*, ζ^* in $\mathbb{C}^3$, where the complex wave vectors satisfy the extra condition

$$\zeta + \zeta^* = \xi. \tag{2.5}$$

Next, suppose that the complete equations (1.1) have a solution (E, H) of the form

$$E(x) = E_0(x) + \delta E(x), \qquad H(x) = H_0(x) + \delta H(x), \tag{2.6}$$

where (E_0, H_0) is the constant medium solution (2.3), and δE satisfies the boundary condition

$$\nu \wedge \delta E(x)\,|_{\partial\Omega} = 0, \tag{2.7}$$

and what is more, the correction terms δE and δH are small to first order in the perturbations $\delta\mu$, $\delta\epsilon$ and $\delta\sigma$, denoted as

$$\delta E(x) = \mathcal{O}(\delta\mu, \delta\epsilon, \delta\sigma), \qquad \delta H(x) = \mathcal{O}(\delta\mu, \delta\epsilon, \delta\sigma). \tag{2.8}$$

Then a straightforward calculation gives

$$\begin{aligned} &\int_{\partial\Omega} \nu\cdot(E\wedge H_0^* - E_0^*\wedge H)\,\mathrm{d}S \\ &\quad= \int_\Omega \nabla\cdot(E\wedge H_0^* - E_0^*\wedge H)\,\mathrm{d}x \\ &\quad= \int_\Omega ((\nabla\wedge E)\cdot H_0^* - (\nabla\wedge H_0^*)\cdot E - (\nabla\wedge E_0^*)\cdot H + (\nabla\wedge H)\cdot E_0^*)\,\mathrm{d}x \\ &\quad= \int_\Omega (\mathrm{i}\omega\delta\mu H\cdot H_0^* - \mathrm{i}\omega\delta\epsilon E\cdot E_0^* + \delta\sigma E\cdot E_0^*)\,\mathrm{d}x \\ &\quad= \int_\Omega (\mathrm{i}\omega\delta\mu H_0\cdot H_0^* - \mathrm{i}\omega\delta\epsilon E_0\cdot E_0^* + \delta\sigma E_0\cdot E_0^*)\,\mathrm{d}x + \mathcal{O}\big((\delta\mu, \delta\epsilon, \delta\sigma)^2\big), \end{aligned} \tag{2.9}$$

the notation above meaning that the error term is second order in the perturbations $\delta\mu$, $\delta\epsilon$ and $\delta\sigma$. Using the definition of the fields (2.3), (2.4) together with the boundary condition (2.7) we obtain from (2.9) the formula

$$\begin{aligned} d(\xi) &= \int_{\partial\Omega} ((\nu\wedge\alpha)\cdot\beta^*\mathrm{e}^{\mathrm{i}\xi\cdot x} + \Lambda(\nu\wedge\alpha\,\mathrm{e}^{\mathrm{i}\zeta\cdot x})\cdot\alpha^*\mathrm{e}^{\mathrm{i}\zeta^*\cdot x})\,\mathrm{d}S \\ &= \int_\Omega (\mathrm{i}\omega\delta\mu\beta\cdot\beta^* - \mathrm{i}\omega\delta\epsilon\alpha\cdot\alpha^* + \delta\sigma\alpha\cdot\alpha^*)\,\mathrm{e}^{\mathrm{i}\xi\cdot x}\,\mathrm{d}x + \mathcal{O}\big((\delta\mu, \delta\epsilon, \delta\sigma)^2\big) \\ &= \mathrm{i}\omega\beta\cdot\beta^*\widehat{\delta\mu} - \mathrm{i}\omega\alpha\cdot\alpha^*\widehat{\delta\epsilon} + \alpha\cdot\alpha^*\widehat{\delta\sigma} + \mathcal{O}\big((\delta\mu, \delta\epsilon, \delta\sigma)^2\big), \end{aligned} \tag{2.10}$$

the hats denoting the Fourier transforms of the corresponding functions.

The approximate linearized solution of the inverse problem is obtained by the following steps.

(i) From the knowledge of Λ, compute the left-hand side of formula (2.10).

(ii) By using the fact that the choice of the complex vectors α, β, α^* and β^* is not unique for a given pair ζ, ζ^*, solve for the Fourier transforms of the functions $\delta\mu$ and $\delta\epsilon + \mathrm{i}\delta\sigma/\omega$ in terms of the data and the residual term.

(iii) Apply the inverse Fourier transform to these equations to get an approximate solution for $\delta\mu$ and $\delta\epsilon + \mathrm{i}\delta\sigma/\omega$.

The crucial point that makes the above linearization possible is that one can prove estimates for the error term that is due to the second-order residual term $\mathcal{O}((\delta\mu, \delta\epsilon, \delta\sigma)^2)$ in (2.10). The main results of the paper are summarized in the theorem below, which we make more precise in Section 3. In the following, ψ denotes a smooth mollifier, i.e., $\psi \in C^\infty(\mathbb{R}^3)$, supp $\hat{\psi} \subset \{\xi \in \mathbb{R}^3 \mid |\xi| \leqslant 1\}$ and $\int \psi \, \mathrm{d}x = 1$. If $N > 0$, we write $\psi_N(x) = N^3\psi(Nx)$.

Theorem 2.1. *Assume that*

$$\|Q\|_\infty = \sup|Q(x)| = \sup\left|\begin{pmatrix} \delta\epsilon + \mathrm{i}\dfrac{\delta\sigma}{\omega} & 0 \\ 0 & \delta\mu \end{pmatrix}\right| \tag{2.11}$$

is small, and for some η, $0 < \eta < 1$, N is a large number of the order $(1-\eta)\log(1/\|Q\|_\infty)$. *Then the above linearization yields an approximation of the functions $\psi_N * \delta\mu$, $\psi_N * \delta\epsilon$ and $\psi_N * \delta\sigma$ where the error is of the order of magnitude* $\|Q\|_\infty^{1+\eta}$.

A more precise meaning for the smallness of the norm of Q as well as for the constants of proportionality are given in the following section.

3. Detailed discussion and proofs

Let us start this section by defining the basic function spaces. This will enable us to describe the mapping properties of the operator Λ.

The basic function space that we shall consider is

$$X = \epsilon^{-1/2}L^2(\Omega)^3 \times \mu^{-1/2}L^2(\Omega)^3, \tag{3.1}$$

i.e., the direct product of vector-valued weighted L^2-spaces equipped with the norm

$$\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\| = \left(\int_\Omega \epsilon(x)|u(x)|^2 \, \mathrm{d}x + \int_\Omega \mu(x)|v(x)|^2 \, \mathrm{d}x\right)^{1/2}, \tag{3.2}$$

and the corresponding inner product. Furthermore, define

$$H(\nabla\wedge) = \{u \in L^2(\Omega)^3 \mid \nabla \wedge u \in L^2(\Omega)^3\} \tag{3.3}$$

and

$$\mathring{H}(\nabla\wedge) = \left\{u \in H(\nabla\wedge) \mid \int_\Omega \nabla \wedge u \cdot v \, \mathrm{d}x = \int_\Omega u \cdot \nabla \wedge v \, \mathrm{d}x \text{ for all } v \in H(\nabla\wedge)\right\}. \tag{3.4}$$

Equivalently, $\overset{\circ}{H}(\nabla\wedge)$ is the completion of $C_0^\infty(\Omega)^3$ in the graph norm of $\nabla\wedge$. Since in general

$$\int_\Omega \nabla\wedge u\cdot v\,\mathrm{d}x=\int_\Omega u\cdot\nabla\wedge v\,\mathrm{d}x+\int_{\partial\Omega}(\nu\wedge u)\cdot v\,\mathrm{d}S, \tag{3.5}$$

u being in $\overset{\circ}{H}(\nabla\wedge)$ says that the tangential component of u vanishes on $\partial\Omega$ in the weak sense.

By $H^s(\Omega)^3$, $s\in\mathbb{R}$, we denote the L^2-based Sobolev spaces of vector-valued fields in Ω. Furthermore, the space of tangential vector fields on the boundary $\partial\Omega$ with components locally in the Sobolev space H^s is denoted by $TH^s(\partial\Omega)$.

Next, define the Maxwell operator L over Ω with the "electric" boundary condition (2.7) as

$$L=\begin{pmatrix}0 & \dfrac{\mathrm{i}}{\epsilon}\nabla\wedge\\ -\dfrac{\mathrm{i}}{\mu}\nabla\wedge & 0\end{pmatrix}:\mathscr{D}(L)\to X, \tag{3.6}$$

where the domain of L is

$$\mathscr{D}(L)=\overset{\circ}{H}(\nabla\wedge)\times H(\nabla\wedge). \tag{3.7}$$

With this domain of definition, L is easily proved to be self-adjoint [5].

Consider now the problem of finding a solution (E,H) for Maxwell's equation with a prescribed tangential electric field at the boundary. In order to formulate the problem in terms of L, we need to subtract off the boundary values. We do this as follows. Assume that we require

$$\nu\wedge E\,|_{\partial\Omega}=f\in TH^{1/2}(\partial\Omega). \tag{3.8}$$

Denote by R a right inverse of the tangential trace mapping

$$\gamma_t:H^1(\Omega)^3\to TH^{1/2}(\partial\Omega),\qquad \gamma_t:F\mapsto\nu\wedge F\,|_{\partial\Omega}. \tag{3.9}$$

For the proof of the existence of the mapping R, see [3]. With the notation $\tilde{E}=E-Rf$, Maxwell's equations can be written

$$(L-\omega-B)\begin{pmatrix}\tilde{E}\\ H\end{pmatrix}=\begin{pmatrix}J\\ K\end{pmatrix}, \tag{3.10}$$

where B is the pointwise multiplier given as

$$B=\begin{pmatrix}\mathrm{i}\dfrac{\sigma}{\epsilon}\mathbf{1} & 0\\ 0 & 0\end{pmatrix}, \tag{3.11}$$

and the inhomogeneity terms J and K are given as

$$J=\frac{\mathrm{i}}{\mu}\nabla\wedge Rf,\qquad K=\omega\frac{\gamma}{\epsilon}Rf. \tag{3.12}$$

In the above definitions, $\mathbf{1}$ is the 3×3 unit matrix, and $\gamma=\epsilon+\mathrm{i}\sigma/\omega$. We shall discuss now the solvability of (3.10).

The following result is well known at least in the nondissipative case $\sigma=0$, i.e., $B=0$, see, e.g., [5, Theorem 8.11]. We shall give here a modification of this argument to cover also the dissipative case.

Proposition 3.1. *The spectrum of the operator $L-B$ consists of a discrete set $\{\omega_n \mid n \in \mathbb{N}\}$ whose only cluster point is infinity. If B vanishes, the spectrum is real, while in the general case $\sigma \neq 0$, the spectrum lies in the strip* $-\max(\sigma(x)/\epsilon(x)) \leqslant \operatorname{Im}\,\omega \leqslant 0$.

Proof. If $B=0$, the reality of the spectrum follows by the self-adjointness of the operator L. Next, assume that $B \neq 0$ and $\operatorname{Im}\,\omega>0$. Since L is self-adjoint in X, we have for all $u=(u_1, u_2) \in \mathscr{D}(L)$ the estimate

$$|\langle (L-B-\omega)u, u\rangle| = \left|\langle Lu, u\rangle - \operatorname{Re}\,\omega \|u\|^2 - \mathrm{i}\left(\operatorname{Im}\,\omega \|u\|^2 + \int_\Omega \sigma |u_1|^2\,\mathrm{d}x\right)\right|$$
$$\geqslant \operatorname{Im}\,\omega \|u\|^2, \tag{3.13}$$

and consequently, by Schwarz's inequality,

$$\|u\| \leqslant \frac{1}{\operatorname{Im}\,\omega}\|(L-B-\omega)u\|. \tag{3.14}$$

In particular, $L-B-\omega$ is injective. To see that the range of $L-B-\omega$ is dense, assume that v is orthogonal to the range, i.e., for all $u \in \mathscr{D}(L)$,

$$\langle (L-B-\omega)u, v\rangle = 0. \tag{3.15}$$

Then, by the definition of the adjoint operator, $v \in \mathscr{D}(L^*) = \mathscr{D}(L)$, and we can choose $u=v$ above. But then the estimate (3.13) implies that $v=0$. Hence, $L-B-\omega$ is invertible and the the norm of the inverse operator does not exceed $1/\operatorname{Im}\,\omega$.

Assuming next that $\operatorname{Im}\,\omega < -\max(\sigma(x)/\epsilon(x))$, we get the estimate

$$\begin{aligned}|\langle (L-B-\omega)u, u\rangle| &\geqslant \left|\operatorname{Im}\,\omega \|u\|^2 + \int_\Omega \sigma |u_1|^2\,\mathrm{d}x\right| \\ &\geqslant |\operatorname{Im}\,\omega|\, \|u\|^2 - \int_\Omega \frac{\sigma}{\epsilon}\epsilon |u_1|^2\,\mathrm{d}x \\ &\geqslant \left(|\operatorname{Im}\,\omega| - \max\frac{\sigma(x)}{\epsilon(x)}\right)\int_\Omega \epsilon |u_1|^2\,\mathrm{d}x + |\operatorname{Im}\,\omega| \int_\Omega \mu |u_2|^2\,\mathrm{d}x \\ &\geqslant \left(|\operatorname{Im}\,\omega| - \max\frac{\sigma(x)}{\epsilon(x)}\right)\|u\|^2.\end{aligned} \tag{3.16}$$

Proceeding as in the previous case we get the invertibility of the mapping $L-B-\omega$.

To prove the discreteness of the spectrum we need the following fundamental result that is proved, e.g., in [5].

Lemma 3.2. *The range of the operator L is closed, and the mapping* $L^{-1}: \operatorname{Ran}(L) \to \operatorname{Ran}(L) \cap \mathscr{D}(L)$ *is compact.*

The above lemma implies in particular that X admits the decomposition

$$X = \operatorname{Ker}(L) \oplus \operatorname{Ran}(L), \tag{3.17}$$

the direct sum being orthogonal with respect to the inner product of X. We shall denote by P the orthogonal projection $P: X \to \mathrm{Ran}(L)$.

Now, let $\mathrm{Im}\ \omega < 0$ and $\mathrm{Re}\ \omega > 0$. We shall consider the equation

$$(L - B - \omega)u = v, \tag{3.18}$$

where $u \in \mathscr{D}(L)$ and $v \in X$ is arbitrary. The solution u of this equation is sought in the form

$$u = (B + \omega)^{-1}(v + q), \tag{3.19}$$

where $q \in \mathrm{Ran}(L)$ and $(B + \omega)^{-1}$ denotes simply the inverse of the matrix $B + \omega$. Plugging this expression into (3.18) gives

$$L(B + \omega)^{-1}(v + q) - q = 0, \tag{3.20}$$

or, since $L = LP$, we obtain by applying L^{-1} on this equation and rearranging terms,

$$P(B + \omega)^{-1}q - L^{-1}q = -P(B + \omega)^{-1}v. \tag{3.21}$$

In order to show that this equation is of the Fredholm type, we only have to verify that the operator

$$P(B + \omega)^{-1}|_{\mathrm{Ran}(L)}: \mathrm{Ran}(L) \to \mathrm{Ran}(L) \tag{3.22}$$

is an isomorphism. But for any $w \in \mathrm{Ran}(L)$, we have

$$|\langle w, P(B + \omega)^{-1}w\rangle| = |\langle w, (B + \omega)^{-1}w\rangle| \geqslant \frac{1}{\mathrm{Re}\ \omega}\|w\|^2, \tag{3.23}$$

and $P(B + \omega)^{-1}$ is bounded, so by the Lax–Milgram lemma [10] the mapping (3.19) is invertible. Now, by the analytic Fredholm theory (see [8]) we can deduce the discreteness of the spectrum. □ (Proposition 3.1)

We have now shown that Maxwell's equations (1.1) with boundary data given by (3.8) can be rewritten as (3.10), and this equation is uniquely solvable except for a discrete set of values of ω. In order to define Λ as in (1.5), we now need the tangential trace of the solution H, which is determined as follows. First, the tangential trace mapping γ_t is extended as

$$\gamma_t: H(\nabla \wedge) \to TH^{-1/2}(\partial\Omega) \tag{3.24}$$

by the weak form of Gauss' theorem (3.5),

$$\langle \Phi|_{\partial\Omega}, \gamma_t F\rangle_{\partial\Omega} = \langle \Phi, \nabla \wedge F\rangle_{\Omega} - \langle \nabla \wedge \Phi, F\rangle_{\Omega}, \tag{3.25}$$

where $F \in H(\nabla \wedge)$, $\Phi \in H^1(\Omega)^3$ is arbitrary and $\langle \cdot, \cdot\rangle_{\partial\Omega}$ and $\langle \cdot, \cdot\rangle_{\Omega}$ are the natural extensions of the L^2-dualities in the appropriate spaces (cf. [2]). Thus, we can define

$$\Lambda: TH^{1/2}(\partial\Omega) \to TH^{-1/2}(\partial\Omega) \tag{3.26}$$

through the decomposition

$$\Lambda: f \mapsto Rf \mapsto \begin{pmatrix} J \\ K \end{pmatrix} \mapsto \begin{pmatrix} \tilde{E} \\ H \end{pmatrix} \mapsto \gamma_t H. \tag{3.27}$$

Note that the extension mapping R is not unique. However, we have the following lemma.

Lemma 3.3. *The definition* (3.27) *of the mapping* Λ *is independent of the specific choice of the extension map* R.

Proof. Let R_1 and R_2 be two different right inverses of the trace mapping γ_t. Denote by $(\tilde{E}_j, H_j)$ the solutions of (3.10), when the inhomogeneities are chosen as

$$\begin{pmatrix} J_j \\ K_j \end{pmatrix} = \begin{pmatrix} \omega \dfrac{\gamma}{\epsilon} R_j f \\ -\dfrac{\mathrm{i}}{\mu} \nabla \wedge R_j f \end{pmatrix}, \quad j = 1, 2, \tag{3.28}$$

respectively. Denoting $E_j = \tilde{E}_j + R_j f$, $j = 1, 2$, and furthermore, $E = E_1 - E_2$ and $H = H_1 - H_2$, we note that $(E, H) \in \mathscr{D}(L)$ and

$$(L - B - \omega)\begin{pmatrix} E \\ H \end{pmatrix} = 0. \tag{3.29}$$

If ω is not in the spectrum of $L - B$, this implies in particular that $E = H = 0$ and thus the tangential boundary values of the fields H_1 and H_2 coincide. □

Now that we have discussed the boundary operator Λ, the next step is to investigate the constant medium solutions (E_0, H_0) and (E_0^*, H_0^*) of the form (2.3) and (2.4). The following simple lemma establishes the existence of these fields. In the following we shall use the notations $\gamma_0 = \epsilon_0 + \mathrm{i}\sigma_0/\omega$ and $k^2 = \omega^2 \mu_0 \gamma_0$.

Lemma 3.4. *Let* $\xi \in \mathbb{R}^3$ *be arbitrary. Assume that the coordinates are chosen so that* $\xi = (\xi_1, 0, 0)$. *Then the fields* (2.3), (2.4) *satisfy* (2.2) *and* (2.5) *if we choose*

$$\zeta = (\zeta_1, \zeta_2, \zeta_3) = \left(\tfrac{1}{2}\xi_1, \mathrm{i}\left(\tfrac{1}{4}\xi_1^2 + z_0^2\right)^{1/2}, \left(z_0^2 + k^2\right)^{1/2}\right), \quad z_0 \in \mathbb{C}, \ z_0^2 \neq k^2, \tag{3.30}$$

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) = \left(z_1, z_2, -\frac{\xi_1 z_1}{2\zeta_3} - \frac{\zeta_2 z_2}{\zeta_3}\right), \quad z_1, z_2 \in \mathbb{C}, \tag{3.31}$$

and

$$\beta = \frac{1}{\omega\mu_0} \zeta \wedge \alpha, \tag{3.32}$$

and the dual system of vectors as

$$\zeta^* = (\zeta_1, -\zeta_2, -\zeta_3), \tag{3.33}$$

$$\alpha^* = (\alpha_1, -\alpha_2, -\alpha_3), \tag{3.34}$$

and finally

$$\beta^* = (\beta_1, -\beta_2, -\beta_3). \tag{3.35}$$

Proof. Trivially,

$$\zeta + \zeta^* = \xi. \tag{3.36}$$

Also, by the definition of β,

$$\nabla \wedge (\alpha\, e^{i\zeta\cdot x}) = -i\alpha \wedge \zeta\, e^{i\zeta\cdot x} = i\omega\mu_0\beta\, e^{i\zeta\cdot x}, \tag{3.37}$$

so the first one of Maxwell's equations is satisfied. On the other hand, $\alpha\cdot\zeta = 0$ and $\zeta\cdot\zeta = k^2$, so

$$\begin{aligned}\nabla \wedge \beta\, e^{i\zeta\cdot x} &= -i\beta \wedge \zeta\, e^{i\zeta\cdot x} = -\frac{i}{\omega\mu_0}(\zeta\wedge\alpha)\wedge\zeta\, e^{i\zeta\cdot x}\\ &= -\frac{i}{\omega\mu_0}\zeta\cdot\zeta\alpha\, e^{i\zeta\cdot x} = -i\omega\gamma_0\alpha\, e^{i\zeta\cdot x},\end{aligned} \tag{3.38}$$

which is the second one of Maxwell's equations.

The same calculations apply for the dual system α^*, β^* and ζ^*. □

Having the constant medium solutions, we can consider the perturbations (2.1) of the medium and the corresponding solutions (2.6) with the boundary condition (2.7). Below we obtain the equations for the perturbations δE and δH. Combining the equations of E and H with those of E_0 and H_0 we get

$$\begin{aligned}\nabla \wedge \delta E &= i\omega\mu_0\delta H + i\omega\delta\mu(H_0+\delta H),\\ \nabla \wedge \delta H &= -i\omega\epsilon_0\delta E + \sigma_0\delta E + i\omega\left(\delta\epsilon + i\frac{\delta\sigma}{\omega}\right)(E_0+\delta E),\end{aligned} \tag{3.39}$$

with the boundary condition

$$\nu\wedge\delta E\,|_{\partial\Omega} = 0, \tag{3.40}$$

understood in the weak sense $\delta E \in \mathring{H}(\nabla\wedge)$. In operator notation, this equation can be written as

$$(L_0 - B_0 - \omega)\begin{pmatrix}\delta E\\ \delta H\end{pmatrix} = \omega Q\begin{pmatrix}\delta E\\ \delta H\end{pmatrix} + \omega Q\begin{pmatrix}E_0\\ H_0\end{pmatrix}, \tag{3.41}$$

where L_0 and B_0 are defined in the same way as L and B in (3.5) and (3.10), respectively, with ϵ, σ and μ replaced by ϵ_0, σ_0 and μ_0, and

$$Q = \begin{pmatrix}\dfrac{\delta\epsilon}{\epsilon_0} + i\dfrac{\delta\sigma}{\omega\epsilon_0} & 0\\ 0 & \dfrac{\delta\mu}{\mu_0}\end{pmatrix}. \tag{3.42}$$

Equation (3.41) is naturally understood in the space

$$X_0 = \epsilon_0^{-1/2}L^2(\Omega)^3 \times \mu_0^{-1/2}L^2(\Omega)^3, \tag{3.43}$$

and the domain of the operator L_0 coincides with that of the operator L.

We show as follows that when the potential matrix Q is small, (3.41) admits a solution with the norm of the same order as the norm of Q. The spectrum $\sigma(L_0 - B_0)$ of the operator

$L_0 - B_0$ is known to be discrete. We can therefore choose $\omega > 0$ and $\omega \notin \sigma(L_0 - B_0)$. For this ω, let d denote the strictly positive distance of ω to the spectrum. Now, (3.41) is equivalent to

$$\begin{pmatrix} \delta E \\ \delta H \end{pmatrix} = \omega(L_0 - B_0 - \omega)^{-1} Q\left(\begin{pmatrix} \delta E \\ \delta H \end{pmatrix} + \begin{pmatrix} E_0 \\ H_0 \end{pmatrix}\right). \tag{3.44}$$

The operator on the right-hand side of this equation satisfies the norm estimate

$$\| \omega(L_0 - B_0 - \omega)^{-1} Q \| \leqslant \frac{\omega}{d} \| Q \|_\infty, \tag{3.45}$$

where

$$\| Q \|_\infty = \sup \left\| \begin{pmatrix} \delta\epsilon(x) + \mathrm{i}\dfrac{\delta\sigma(x)}{\omega} & 0 \\ 0 & \delta\mu(x) \end{pmatrix} \right\|. \tag{3.46}$$

In particular, the mapping is a contraction if $\| Q \|_\infty$ is small enough. We have therefore the following theorem.

Theorem 3.5. *Assuming that* $\omega \notin \sigma(L_0 - B_0)$, $\omega > 0$, *and*

$$\| Q \|_\infty < \frac{d}{\omega}, \tag{3.47}$$

(3.41) *admits a solution* $(\delta E, \delta H)$ *satisfying*

$$\left\| \begin{pmatrix} \delta E \\ \delta H \end{pmatrix} \right\| \leqslant \left(1 - \frac{\omega}{d} \| Q \|_\infty\right) \| Q \|_\infty \left\| \begin{pmatrix} E_0 \\ H_0 \end{pmatrix} \right\|. \tag{3.48}$$

It should be noted that in order to choose the frequency ω outside the spectrum of $L_0 - B_0$, one needs precise information about the geometry of the domain Ω as well as of the background material. However, if the background is conducting, i.e., $\sigma_0 \neq 0$, the choice of $\omega \in \mathbb{R}_+$ can be made arbitrarily. We have the following theorem.

Theorem 3.6. *The spectrum* $\sigma(L_0 - B_0)$ *of the operator* $L_0 - B_0$ *intersect the real axis at most at the origin.*

Proof. Since the operator $L_0 - B_0$ has a pure point spectrum, we only have to show that for each $\omega \in \mathbb{R}$, $\omega \neq 0$, the operator $A_\omega = L_0 - B_0 - \omega$ is injective. Therefore, assume that $A_\omega u = 0$ for some $u \in \mathscr{D}(A_\omega) = \mathscr{D}(L_0)$. Since $L_0 - \omega$ is self-adjoint, we have

$$0 = \operatorname{Im} \langle A_\omega u, u \rangle = -\operatorname{Im} \langle B_0 u, u \rangle = -\int_\Omega \sigma_0 | u_1(x) |^2 \, \mathrm{d}x, \tag{3.49}$$

implying that $u_1 = 0$. But then $A_\omega u = (L_0 - \omega)u$ and

$$0 = \langle (L_0 - \omega)u, u \rangle = -\omega \int_\Omega \mu_0 | u_2(x) |^2 \, \mathrm{d}x, \tag{3.50}$$

i.e., $u_2 = 0$. This completes the proof. □

Consider now (2.10) again. The above discussion of the perturbed solutions (2.6) has shown that the remainder term of (2.10) is indeed second order. For brevity, we use the notation $\delta\gamma = \delta\epsilon + \mathrm{i}\delta\sigma/\omega$ introduced in (3.12). Writing out explicitly the ξ-dependencies, we have

$$d(\xi) = \int_{\partial\Omega} \big((\nu \wedge \alpha(\xi)) \cdot \beta^*(\xi)\, \mathrm{e}^{\mathrm{i}\xi\cdot x} + \Lambda(\nu \wedge \alpha(\xi)\, \mathrm{e}^{\mathrm{i}\zeta\cdot x}) \cdot \alpha^*(\xi)\, \mathrm{e}^{\mathrm{i}\zeta^*\cdot x}\big)\, \mathrm{d}S$$
$$= \mathrm{i}\omega\big(-\alpha(\xi)\cdot\alpha^*(\xi)\widehat{\delta\gamma}(\xi) + \beta(\xi)\cdot\beta^*(\xi)\widehat{\delta\mu}(\xi)\big) + r(\xi), \tag{3.51}$$

where the residual term r is

$$r(\xi) = \mathrm{i}\omega\int_{\Omega}(-E_0^*(x, \xi)\cdot\delta E(x, \xi)\delta\gamma(x) + H_0^*(x, \xi)\cdot\delta H(x, \xi)\delta\mu(x))\, \mathrm{d}x. \tag{3.52}$$

From the proof of Lemma 3.4 it is evident that if we choose $\zeta \in \mathbb{C}^3$ as in (3.30) with z_0 fixed, the choice of the remaining parameters α, β, α^* and β^* is not unique. Assume that we pick two different vectors α_1 and α_2 according to (3.31) and following the guidelines of Lemma 3.4 form then the corresponding vectors β_j, α_j^* and β_j^*, $j = 1, 2$. Writing (3.51) first with the set $j = 1$ and then with $j = 2$, we get the matrix equation

$$\begin{pmatrix} d_1(\xi) \\ d_2(\xi) \end{pmatrix} = \mathrm{i}\omega \begin{pmatrix} -\alpha_1(\xi)\cdot\alpha_1^*(\xi) & \beta_1(\xi)\cdot\beta_1^*(\xi) \\ -\alpha_2(\xi)\cdot\alpha_2^*(\xi) & \beta_2(\xi)\cdot\beta_2^*(\xi) \end{pmatrix} \begin{pmatrix} \widehat{\delta\gamma}(\xi) \\ \widehat{\delta\mu}(\xi) \end{pmatrix} + \begin{pmatrix} r_1(\xi) \\ r_2(\xi) \end{pmatrix}, \tag{3.53}$$

or briefly, using the matrix notation,

$$D(\xi) = A(\xi)\begin{pmatrix} \widehat{\delta\gamma}(\xi) \\ \widehat{\delta\mu}(\xi) \end{pmatrix} + R(\xi). \tag{3.54}$$

To invert the matrix $A(\xi)$, we need the following result.

Lemma 3.7. *The vectors α_j, β_j, α_j^* and β_j^*, $j = 1, 2$, satisfying Lemma 3.4 can be chosen such that for any $\lambda \in \mathbb{C}$,*

$$\det A(\xi) = \lambda\,|\xi|^2. \tag{3.55}$$

Proof. Let ζ, α and β be any vectors chosen as in Lemma 3.4. Then

$$\beta\cdot\beta^* = \left(\frac{1}{\omega\mu_0}\right)^2 (\zeta\wedge\alpha)\cdot(\zeta^*\wedge\alpha^*) = \left(\frac{1}{\omega\mu_0}\right)^2 \big((\zeta\cdot\zeta^*)(\alpha\cdot\alpha^*) - (\zeta^*\cdot\alpha)^2\big), \tag{3.56}$$

where we used the identity $\zeta\cdot\alpha^* = \zeta^*\cdot\alpha$. Furthermore, since

$$\zeta^*\cdot\alpha = (\xi - \zeta)\cdot\alpha = \xi\cdot\alpha, \tag{3.57}$$

we get with any choice of the vectors $\alpha_j = \alpha_j(\xi)$, $j = 1, 2$, the identity

$$\det A(\xi) = -(\alpha_1\cdot\alpha_1^*)(\beta_2\cdot\beta_2^*) + (\alpha_2\cdot\alpha_2^*)(\beta_1\cdot\beta_1^*)$$
$$= \left(\frac{1}{\omega\mu_0}\right)^2 \big((\alpha_1\cdot\alpha_1^*)(\xi\cdot\alpha_2)^2 - (\alpha_2\cdot\alpha_2^*)(\xi\cdot\alpha_1)^2\big). \tag{3.58}$$

To evaluate this expression, we write the product $\alpha\cdot\alpha^*$ explicitly. From (3.31) and (3.34),

$$\alpha\cdot\alpha^* = z_1^2 - z_2^2 + \left(\frac{\xi_1}{2\zeta_3}z_1 + \frac{\zeta_2}{\zeta_3}z_2\right)^2$$

$$= \frac{1}{\zeta_3^2}\left((\zeta_3^2+\tfrac{1}{4}\xi_1^2)z_1^2 + \zeta_2\xi_1 z_1 z_2 - (\tfrac{1}{4}\xi_1^2 + k^2)z_2^2\right). \tag{3.59}$$

We now make particular choices of z_1 and z_2 to obtain (3.55). We simplify the expression (3.59) by requiring that for some $\lambda\in\mathbb{C}$, z_1 and z_2 satisfy the equation

$$(\zeta_3^2+\tfrac{1}{4}\xi_1^2)z_1^2 + \zeta_2\xi_1 z_1 z_2 - (\tfrac{1}{4}\xi_1^2 + k^2)z_2^2 = -\tfrac{1}{2}(\omega\mu_0)^2\lambda\zeta_3^2. \tag{3.60}$$

By the Fundamental Theorem of Algebra, we can now choose $z_2^{(j)}$, $j=1, 2$, such that $(1, z_2^{(1)})$ and $(\mathrm{i}, z_2^{(2)})$ each satisfy (3.60). These choices, which correspond to

$$\alpha_1 = \left(1,\ z_2^{(1)},\ -\frac{\xi_1}{2\zeta_3} - \frac{\zeta_2}{\zeta_3}z_2^{(1)}\right) \tag{3.61}$$

and

$$\alpha_2 = \left(\mathrm{i},\ z_2^{(2)},\ -\mathrm{i}\frac{\xi_1}{2\zeta_3} - \frac{\zeta_2}{\zeta_3}z_2^{(2)}\right), \tag{3.62}$$

imply (3.55). □

From (3.60) it follows that for $|\xi|$ large, $z_2^{(j)} = \mathcal{O}(1)$. Therefore, from (3.31) and (3.32) with z_0 fixed, we have the estimates

$$|\alpha_j(\xi)| \leqslant C(1+|\xi|^2)^{1/2}, \tag{3.63}$$

$$|\beta_j(\xi)| \leqslant C(1+|\xi|^2). \tag{3.64}$$

Similar estimates hold also for α_j^* and β_j^*.

Multiplying (3.54) by $A(\xi)^{-1}$, $\xi\neq 0$, we get

$$A(\xi)^{-1}D(\xi) = \begin{pmatrix}\widehat{\delta\gamma}(\xi)\\ \widehat{\delta\mu}(\xi)\end{pmatrix} + A(\xi)^{-1}R(\xi). \tag{3.65}$$

The left-hand side is now completely determined by the data Λ, so to complete the approximation scheme we have to find an appropriate estimate for the inverse Fourier transform of the residual term $A^{-1}R$.

Lemma 3.8. *Assume that $\omega>0$ is not in the spectrum of the operator $L_0 - B_0$, and assume that Q satisfies the estimate* (3.47) *of Theorem* 3.5. *Further, let $M>0$ be large enough to satisfy $\Omega\subset\{x\in\mathbb{R}^3 \mid |x|<M\}$. Then, for some constant $C>0$ independent of $\xi\in\mathbb{R}^3$,*

$$|A(\xi)^{-1}R(\xi)| \leqslant \frac{C}{|\xi|^2}(1+|\xi|^2)^3\,\mathrm{e}^{2M|\xi|}\,\|Q\|_\infty^2. \tag{3.66}$$

Proof. Denoting by $(E^*_{0,j}, H^*_{0,j})$ and $(\delta E_j, \delta H_j)$ the fields corresponding to the choice α_j, $j=1$, 2, we get by Schwarz's inequality and definition (3.52),

$$|r_j(\xi)| = |\mathrm{i}\omega\int_\Omega \left(-E^*_{0,j}(x\ \xi)\cdot\delta E_j(x,\xi)\delta\gamma(x) + H^*_{0,j}(x,\xi)\cdot\delta H_j(x,\xi)\delta\mu(x)\right)\mathrm{d}x\,|$$

$$\leqslant \omega\,\|Q\|_\infty \left\|\begin{pmatrix}E^*_{0,j}\\ H^*_{0,j}\end{pmatrix}\right\|_2 \left\|\begin{pmatrix}\delta E_j\\ \delta H_j\end{pmatrix}\right\|_2, \tag{3.67}$$

where the norms with the subindex 2 denote the usual unweighted norms of the space $L^2(\Omega)^3 \times L^2(\Omega)^3$. From the remark after the previous lemma as well as from the definition of ζ^* it follows now that for some constant $C>0$,

$$\left\|\begin{pmatrix}E^*_{0,j}\\ H^*_{0,j}\end{pmatrix}\right\|_2 \leqslant C\left(1+|\xi|^2\right)\mathrm{e}^{M|\xi|}. \tag{3.68}$$

Furthermore, from (3.63), (3.64), (3.55) and Cramer's rule, we get

$$|A(\xi)^{-1}| \leqslant \frac{C}{|\xi|^2}\left(1+|\xi|^2\right), \tag{3.69}$$

so combining these estimates with (3.48) proved in Theorem 3.5, the claim follows. □

Having established the second-order smallness of the residual term, we can now show that taking the inverse Fourier transform of (3.65) yields a good approximation to $\delta\gamma$ and $\delta\mu$. This argument follows that of [1]. Let ψ be a smooth mollifier in $\mathbb{R}^3$, i.e. $\psi \in C^\infty(\mathbb{R}^3)$, $\hat{\psi}(\xi)=0$ as $|\xi|>1$ and $\int\psi(x)\,\mathrm{d}x = 1$. For $N>0$ we define ψ_N by setting $\hat{\psi}_N(\xi) = \psi(1/N\cdot\xi)$, or $\psi_N(x) = N^3\psi(Nx)$. Multiplying (3.65) by $\hat{\psi}_N(\xi)$ and applying the inverse Fourier transform $\mathscr{F}^{-1}$ on both sides, we get

$$\psi_N * \mathscr{F}^{-1}(A^{-1}D) = \begin{pmatrix}\psi_N * \delta\gamma\\ \psi_N * \delta\mu\end{pmatrix} + \psi_N * \mathscr{F}^{-1}(A^{-1}R), \tag{3.70}$$

where $*$ is the convolution. When N is large, the first term on the right gives a reasonable approximation of the unknown functions $\delta\gamma$ and $\delta\mu$. (Remember that we assumed that $\delta\gamma$ and $\delta\mu$ are twice continuously differentiable.) Therefore, we have to evaluate the size of the second term on the right. It turns out that the second term is indeed small for large values of N if the perturbations are small enough. The precise relation between the sizes of N, $\|Q\|_\infty$ and the norm of the residual term is given in the following theorem.

Theorem 3.9. *Assume that $N>0$ is chosen so that for some η, $0<\eta<1$,*

$$N^7\mathrm{e}^{2MN} < \frac{1}{\|Q\|_\infty^{1-\eta}}. \tag{3.71}$$

Then, under the assumptions of the previous lemma,

$$\|\psi_N * \mathscr{F}^{-1}(A^{-1}R)\|_\infty \leqslant C\,\|Q\|_\infty^{1+\eta}. \tag{3.72}$$

Proof. By Lemma 3.8, we get the estimate

$$\begin{aligned}\|\psi_N * \mathscr{F}^{-1}(A^{-1}R)\|_\infty &\leqslant \int |\hat{\psi}_N(\xi)|\,|(A^{-1}R)(\xi)|\ \mathrm{d}\xi \\ &\leqslant C(1+N^2)^3 \mathrm{e}^{2MN} \int \frac{1}{|\xi|^2} \hat{\psi}\left(\frac{1}{N}\xi\right) \mathrm{d}\xi \,\|Q\|_\infty^2 \\ &\leqslant CN^7 \mathrm{e}^{2MN} \|Q\|_\infty^2, \end{aligned} \tag{3.73}$$

which clearly implies the claim. □

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References

[1] A.P. Calderón, On an inverse boundary value problem, in: *Seminar on Numerical Analysis and its Applications to Continuum Physics* (Soc. Brasileira de Matematica, Rio de Janeiro, 1980) 65–73.
[2] V. Girault and P.-A. Raviart, *Finite Element Approximation of the Navier–Stokes Equations,* Lecture Notes in Math. **749** (Springer, Berlin, 1979).
[3] P. Grisvard, *Elliptic Problems in Nonsmooth Domains* (Pitman, Boston, 1985).
[4] R. Kohn and M. Vogelius, Determining conductivity by boundary measurements, *Comm. Pure Appl. Math.* **37** (1984) 113–123.
[5] R. Leis, *Initial Boundary Value Problems in Mathematical Physics* (Teubner, Stuttgart/Wiley, New York, 1986).
[6] A. Nachman, Reconstructions from boundary measurements, *Ann. of Math.* **128** (1988) 531–576.
[7] A.G. Ramm, Multidimensional inverse problems and completeness of products of solutions to PDE, *J. Math. Anal. Appl.* **134** (1988) 211–253.
[8] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I* (Academic Press, New York, 1972).
[9] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, *Ann. of Math.* **125** (1987) 153–169.
[10] K. Yoshida, *Functional Analysis* (Springer, New York, 1980).