INVARIANT CIRCLES AND THE ORDER STRUCTURE OF PERIODIC ORBITS IN MONOTONE TWIST MAPS

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This paper is dedicated to the memory of Charles Conley

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Let \( f \) be an area preserving monotone twist map of the annulus and \( \omega \) an irrational number which is between the rotation numbers of \( f \) restricted to the boundaries. A non-Birkhoff periodic orbit is one whose iterates are ordered around the annulus differently from those of rigid rotation. We show that \( f \) has no invariant circle with rotation number \( \omega \) if and only if \( f \) has a non-Birkhoff periodic orbit whose rotation number is a convergent of \( \omega \). The proof that the existence of a non-Birkhoff orbit excludes the existence of certain invariant circles utilizes a comparison between the map of the annulus and a non-injective circle map constructed from the non-Birkhoff orbit. The converse is proven using a theorem of Birkhoff on the dynamics in zones of instability coupled with elementary index arguments.

§1

The relation between the dynamics of a diffeomorphism of a surface and the periodic orbits of that diffeomorphism has been studied by many authors. In this paper we consider the relation between periodic orbits and the existence of invariant circles for area preserving monotone twist maps. Periodic orbits on which the twist map does not preserve the ordering of the angular coordinate are called non-Birkhoff orbits.

THEOREM A. An area preserving monotone twist map \( f \) of an annulus \( A \) has no invariant circle of irrational rotation number \( \omega \) if and only if \( f \) has a non-Birkhoff periodic orbit with rotation number equal to a convergent of \( \omega \).

Invariant circles of area preserving monotone twist maps have been intensely studied by mathematicians and physicists. In particular, a number of other necessary and sufficient conditions for the non-existence of invariant circles have appeared. In [19] Mather and Katok give a condition based on variational techniques. Related work has been done by Aubry, Le Daeron and Andre in [2], Newman and Percival [21], MacKay and Percival [15] and Stark [22]. The relation of these conditions to the more topological one of Theorem A is unclear.

Theorem A is an immediate consequence of the following two stronger theorems.

THEOREM B. If \( f \) is a monotone twist map of the annulus which has a non-Birkhoff periodic orbit with rotation number \( p/q \), period \( q \) and \((p, q) = 1\) and \( p/q \) is a convergent of \( \omega \), then \( f \) has no invariant circles with rotation number \( \omega \).
Note that this theorem does not require that \( f \) be area preserving. If \( f \) is area preserving, the homotopically non-trivial invariant circles of \( f \) are graphs of uniformly Lipschitz functions (see [6], [11] or [13]) and thus form a closed set. Since a non-Birkhoff orbit cannot lie on such an invariant circle, one can show that the existence of a non-Birkhoff orbit as in Theorem B, excludes the existence of an invariant circle with rotation number \( p/q \). But the set of rotation numbers of invariant circles is also closed, so the existence of such a non-Birkhoff orbit implies the existence of an interval around \( p/q \) which does not contain the rotation number of any invariant circle of \( f \). Thus if \( f \) is area preserving, the main content of Theorem B lies in establishing a lower bound on the size of this neighborhood. In particular, using a well known fact about the convergents of a continued fraction (see, for example theorem 184, [10]) we have the following corollary.

**Corollary.** If \( f \) and \( p/q \) are as in Theorem B, then \( f \) has no invariant circle with rotation number \( \omega \) where \( |p/q - \omega| < 1/(2q^2) \).

The main idea of the proof of Theorem B is to use the monotone twist hypothesis to compare the dynamics on the invariant circle with the dynamics of a particular piecewise linear, non-injective circle map constructed from the non-Birkhoff orbit. This map, called the “tight circle map” on the non-Birkhoff orbit, is the simplest map of the circle which has a periodic orbit with the same order structure as the non-Birkhoff orbit.

The other half of Theorem A follows from

**Theorem C.** If \( f \) is an area preserving monotone twist map and \( f \) has no invariant circle with rotation number \( \omega \) then for every rational \( r/s \) sufficiently close to \( \omega \), \( f \) has a non-Birkhoff periodic orbit with rotation number \( r/s \).

Theorem C may be derived by the appropriate alterations of the methods of Mather in [20]. Using variational techniques, Mather shows that whenever an area preserving monotone twist map has no invariant circle with rotation number equal to the irrational number \( \omega \), then there exist uncountably many Denjoy minimal sets with rotation number \( \omega \) on which \( f \) does not preserve the radial order. These techniques may also be applied to the (easier) case of showing the existence of periodic orbits whose rotation numbers are close to \( \omega \) on which \( f \) does not preserve the radial order, i.e. the existence of non-Birkhoff periodic orbits whose rotation numbers are near \( \omega \).

There are also simple topological reasons why these periodic orbits should exist. For example, when no invariant circle with rotation number \( \omega \) is present one would (generically) expect there to be transverse heteroclinic crossing between stable and unstable manifolds of periodic orbits with rotation numbers smaller and larger than \( \omega \). One could then construct the required non-Birkhoff orbits using shadowing as in [1]. The crossings of the stable and unstable manifolds are expected because Birkhoff [3] and [5] (see also [11] and [17]) showed that in a zone of instability (i.e. in a region between two invariant circles which contains no other invariant circles) there always exist orbits which transit between neighborhoods of the boundary circles. The proof we give of Theorem C utilizes these orbits to show that the orbit of a strip connecting the inner and outer boundary of the zone of instability is complicated and eventually crosses itself in a non-trivial manner which implies the existence of the required non-Birkhoff periodic orbits. Although the proof is somewhat complicated the techniques are completely elementary. We hope the proof provides further insight into the topological behavior of monotone twist maps.
We begin with some standard definitions. Let \( S' = \mathbb{R}/\mathbb{Z}, A = S' \times [0, 1] \) and \( \tilde{A} = \mathbb{R} \times [0, 1] \). Given \( f: A \to A \), we shall say that \( \tilde{f}: \tilde{A} \to \tilde{A} \) is a lift of \( f \) if it is a lift in the usual sense and, in addition, \( F(0, 0) \in [0, 1] \times [0, 1] \). A map \( f: A \to A \) is called a monotone twist map if it is a \( C^2 \)-orientation preserving diffeomorphism that preserves the boundary of \( A \) and \( \partial(\tilde{f} \circ \tilde{F})/\partial y > 0 \) where \( \tilde{\pi}_1: \tilde{A} \to \mathbb{R} \) is projection on the first factor.

For \( f: A \to A \), the rotation number of \( \beta \in A \) is defined as

\[
\rho(\beta, f) = \lim_{n \to \infty} \sup_{z} \frac{\pi_1(F^n(\tilde{\beta})) - \pi_1(\tilde{\beta})}{n}
\]

where \( \tilde{\beta} \) is a lift of \( \beta \). The orbit of \( \beta \) is denoted \( o(\beta, f) = \{ \ldots, f^{-1}(\beta), \beta, f(\beta), \ldots \} \) and the orbit is called \( p/q \)-periodic if it is periodic of least period \( q \) and \( \rho(\beta, f) = p/q \). Note that this implies that \( F^q(\tilde{\beta}) - (p, 0) = \tilde{\beta} \) and that we are not assuming \( p \) and \( q \) are relatively prime.

Next, for \( z \in A \), define the extended orbit of \( z, eo(z, F) \subset \tilde{A} \), as the union of all the orbits of \( F: \tilde{A} \to \tilde{A} \) that cover \( o(z, f) \), i.e. \( eo(z, F) = \{ F^i(\tilde{z}) + (j, 0) : \tilde{z} \text{ is a lift of } z \text{ and } i, j \in \mathbb{Z} \} \). For brevity of notation, we often let \( o(z) = o(z, f) \) and \( eo(z) = eo(z, F) \).

Standing assumptions. We shall always assume that a \( p/q \)-periodic orbit \( o(\beta, f) \) is such that \( \pi_1(eo(\beta)) \) is injective. This may always be arranged by a local change of variables.

A \( p/q \)-periodic orbit is called Birkhoff if \( F|_{eo(\beta)} \) is order preserving, i.e. if \( \beta_1, \beta_2 \in eo(\beta) \) and \( \pi_1(\beta_1) < \pi_1(\beta_2) \), then \( \pi_1(F(\beta_1)) < \pi_1(F(\beta_2)) \). Aubry and Mather [2] were the first to note the importance of the order structure of orbits in monotone twist maps. Note that any \( p/q \)-periodic orbit whose iterates are permuted around the annulus in a different order than those of rigid rotation is a non-Birkhoff orbit.

In what follows, we will need to associate a map of the circle with a given \( p/q \)-periodic orbit, \( o(\beta, f) \). This circle map will be (in some sense) the simplest map of the circle which has a periodic orbit with the same order structure around \( S' \) as \( o(\beta, f) \). To begin, define \( G: \pi_1(eo(\beta)) \to \pi_1(eo(\beta)) \) so that \( G(\pi_1(b)) = \pi_1(F(b)) \) for all \( b \in eo(\beta) \). Now extend \( G \) to \( G: \mathbb{R} \to \mathbb{R} \) in the simplest way, i.e. \( G \) is continuous and piecewise linear with discontinuous derivative only at points of \( \pi_1(eo(\beta)) \). Since \( G(x + 1) = G(x) + 1 \), \( G \) projects to a map \( g: S' \to S' \). We will call \( g \) (and sometimes \( G \)) the tight circle map associated with the \( p/q \)-periodic orbit \( o(\beta) \). Note that \( g \) is usually not a homeomorphism. (In fact, \( o(\beta) \) is non-Birkhoff if and only if \( g \) is not injective.)

Next, we recall some information about degree one maps of the circle. For such a \( g: S' \to S' \) and \( x \in S' \), define the rotation number of \( x \) as \( \rho(x, g) = \lim \sup_{n \to \infty} \frac{G^n(x) - x}{n} \). The rotation set of \( g \) is \( \rho(g) = \{ \rho(x, g) : x \in S' \} \). Ito [12] showed that \( \rho(g) \) is a closed interval which we denote as \( \rho(g) = [\rho_-, \rho_+] \). (For this reason, \( \rho(g) \) is usually called the rotation interval).

The study of the dynamics of degree one circle maps is greatly facilitated by defining auxiliary non-decreasing circle maps as follows: If \( G: \mathbb{R} \to \mathbb{R} \) is the lift of a non-injective degree one circle map, let

\[
G_+ = \inf \{ H : H \text{ is the lift of a circle homeomorphism and } H \geq G \}
\]

and

\[
G_- = \sup \{ H : H \text{ is the lift of a circle homeomorphism and } H \leq G \}.
\]

If we restrict to \( g: S' \to S' \), where \( G \) has a finite number of turning points in a fundamental domain (e.g. \( G \) is a tight circle map), one sees that \( G_+ \) agrees with \( G \) except for a finite number of intervals where \( G_+ \equiv 0 \) (see Fig. 2). Also, since \( G_+ \) is non-decreasing, \( \rho(x, G_+) \) is
independent of the choice of $x \in S^1$ and so $\rho(G_-)$ is a single number. Similar comments hold for $G_-$ as well. In addition, using elementary techniques as in [7], one can show that $\rho(G_-) = \rho_-(g)$ and $\rho(G_-) = \rho_-(g)$.

**Notation.** If $g$ is the tight circle map associated with $o(\beta, f)$, note that $\rho(G_-)$ only depends on $o(\beta)$ so we define $R_+(\beta) = \rho(G_+)$. Similarly, $R_-(\beta) = \rho(G_-)$.

Our last remarks apply to the tight circle map $G : R \rightarrow R$ associated with a non-Birkhoff $p/q$-periodic orbit $o(\beta, f)$. If $h \in \pi_1(eo(\beta, F))$ satisfies $G_-(b) \neq G(b)$ then if $b'$ is the closest point to the left of $b$ in $\pi_1(eo(\beta, F))$ that satisfies $G_+(b') = G(b')$ then $G(b') = G_-(b') = G_+(b)$. This is an easy consequence of the construction of $G_+$. This implies that if $x \in S^1$ is covered by $b \in \pi_1(eo(\beta, F))$ and $g_+ : S^1 \rightarrow S^1$ is the projection of $G_+$ then $o(x, g_+) \subseteq o(x, g)$. We will use this remark in Lemma 4.

Before proceeding, we illustrate these definitions with a simple example. Let $f'$ be the monotone twist map associated with billiards on the table shown in Fig. 1 and let $o(\beta, f)$ be the $2/5$-periodic orbit associated with the trajectory shown (see [4] for the construction of a monotone twist map from billiards on a convex table). The tight circle map, $g$, associated with $o(\beta, f)$ is given in Fig. 2. One may easily check that $g_+$ has a $1/2$-periodic orbit and $g_-$ has a $1/3$-periodic orbit and so $\rho(g) = [1/3, 1/2]$. Note that $\rho(g)$ does not depend on the actual positions of the elements of $\pi_1(eo(z))$ but only on the permutation of these elements.

![Fig. 1. A billiards table with a non-Birkhoff 2/5 periodic orbit in the associated map of the annulus.](image1)

![Fig. 2. The tight circle map associated with the orbit of Fig. 1.](image2)
An invariant circle for $f: A \rightarrow A$ is a homotopically non-trivial Jordan curve $\Gamma \subseteq A$ with $f(\Gamma) = \Gamma$. Thus, $f$ restricted to $\Gamma$ is a circle homeomorphism which has a unique rotation number which we denote as $\rho(\Gamma)$. In a slight abuse of language, we speak of the rotation number of $\Gamma$ when we mean the rotation number of $f$ restricted to $\Gamma$. Our first theorem considers the possible rotation numbers of invariant circles when $f$ has a non-Birkhoff $p/q$-periodic orbit.

Intuitively speaking, a $p/q$-periodic orbit $o(\beta, f')$ is non-Birkhoff when it sometimes rotates faster than $p/q$ and sometimes slower. If an invariant circle is above the orbit, one would expect the monotone twist hypothesis to imply that $\rho(I)$ is larger than this faster rate. This faster rate may be quantified by the largest rotation number of the tight circle map $G$, associated with $o(\beta)$ and is thus equal to $R_+(\beta)$. Therefore, one would expect $\rho(\Gamma) \geq R_+(\beta)$ which we prove in Theorem 2. Similar comments hold for $R_-(\beta)$.

It will be convenient to work in a covering space that is particularly well suited to the situation at hand. We will use it in Lemma 1 to make precise the notion that the orbit of a point $x$ is "above" a periodic orbit, $o(\beta)$, and thus $\rho(x) \geq R_+(\beta)$. In particular, the covering space will allow us to use the monotone twist hypothesis without assuming that $f'$ is area preserving and thus that $\Gamma$ is the graph of a continuous function $S^1 \rightarrow [0, 1]$.

Fix an $f: A \rightarrow A$ with a $p/q$-periodic orbit, $o(\beta, f')$. Let $A_\beta = A \setminus \{o(\beta)\}$, and $A_\beta - \{e_0(\beta)\}$. Also, let $f: A_\beta \rightarrow A_\beta$ and $F: A_\beta \rightarrow A_\beta$ denote the restrictions as well as the original maps. Let $X$ be the $\mathbb{Z}$-cover of $A_\beta$ built by cutting $A_\beta$ on the vertical lines downward from the elements of $oe(\beta)$. We will describe $X$ embedded in $\mathbb{R}^3$ as it is pictured in Fig. 3. At each cut we bend the right flap down and the left flap up. $X$ is evidently a $\mathbb{Z}^2$-cover of $A_\beta$ and we denote the two deck transformations, $\sigma_1$ and $\sigma_2$, where $\sigma_1$ moves each point one domain to the right and $\sigma_2$ moves each point one domain upward. We have built $X$ by gluing together copies of an $\hat{A}_\beta$ that have been slit vertically downwards from each element of $e_0(\beta)$. Let $D$ denote one such copy. A level of $X$ is a set $\sigma_k^+ (D)$ for some fixed $k$. We have shaded a level of $X$ in Fig. 3. The rear boundary of a level is the lift of the top boundary of $A_\beta$ which is contained in that level. We indicate the level which contains $x \in X$ by letting $L(x)$ be the unique $k$ with $x \in \sigma_k^+(D)$. Let $\hat{f}: X \rightarrow X$ be the lift of $f: A_\beta \rightarrow A_\beta$ that leaves the rear boundary of each level invariant, and in addition, $\hat{f}$ satisfies $\rho(x, \hat{f}) = \rho(\hat{x}, F)$ where $x$ covers $\hat{x}$ and

$$\rho(x, \hat{f}) = \lim_{n \rightarrow \infty} \sup \frac{\pi(f^n(x)) - \pi(x)}{n}$$

Fig. 3.
with \( \pi: X \rightarrow \mathbb{R} \) the lift of \( \pi_1 \) and \( F: \tilde{A} \rightarrow \tilde{A} \) defined as in §2. Finally, let \( o^+ (x, \tilde{f}) \) 
\[ = \{ x, \tilde{f}(x), \tilde{f}^2(x), \ldots \} \] 

Our first lemma states that orbits of \( \tilde{f} \) never travel “up the levels” and that any orbit that does not travel “downwards” must have rotation number bigger than or equal to \( p/q \).

**Lemma 1.** If \( f: \mathcal{A} \rightarrow \mathcal{A} \) is a monotone twist map with periodic orbit \( o(\beta) \) and \( X \) and \( f \) are constructed as above then

(a) For all \( x \in X \), \( L(\tilde{f}(x)) \leq L(x) \)

(b) If \( L(o^+(x, \tilde{f})) \) is bounded then \( p(x) \geq R_+ (\beta) \).

**Proof.** We first note a few facts concerning \( F: \tilde{A} \rightarrow \tilde{A} \). Given \( \bar{x} \in \tilde{A} \), let \( I_{\bar{x}} \) be the vertical segment through \( \bar{x} \). By the monotone twist hypothesis, \( F(I_{\bar{x}}) \) is the graph of a continuous function from \( \pi_1(F(I_{\bar{x}})) \) to \([0, 1]\) and further, the image of the top point of \( I_{\bar{x}} \) under \( F \) is to the right of the image of the bottom point (see Fig. 4). Then, using the definition of \( X \) and the fact that \( \tilde{f} \) preserves the rear boundary one obtains the conclusion in (a).

To prove (b), first note that if \( k \) is such that \( L(\tilde{f}^k(x)) = \min \{ L(o^+(x, \tilde{f})) \} \) then using (a), \( L(o^+(\tilde{f}^k(x), \tilde{f})) \) is constant. Since \( p(f^k(x)) = p(x) \), we may assume without loss of generality that \( L(o^+(x, \tilde{f})) \) is constant.

Next, we show that if \( L(\tilde{f}(x)) = L(x) \), \( \bar{x} \in \tilde{A} \) is a projection of \( x \) and \( B \in \mathcal{G}(\beta, F^*) \) with \( \pi(B) < \pi(\bar{x}) \) then

\[ 1 \quad G(\pi(B)) = \pi(F(B)) \leq \pi(F(\bar{x})) \]

where \( G: \mathbb{R} \rightarrow \mathbb{R} \) is the tight circle map associated with \( o(\beta, f) \). The first equality follows from the definition of \( G \). To prove the inequality, assume to the contrary that \( \pi(F(\bar{x})) < \pi(F(B)) \). Using the monotone twist hypothesis this implies that \( F(I_{\bar{x}}) \) intersects \( I_{\pi(B)} \) below \( F(B) \) (see

![Fig. 4.](image-url)

![Fig. 5.](image-url)
By assumption, \( F(x) \) is to the left of this intersection implying that \( L(F(x)) < L(x) \), a contradiction.

Under the same assumptions as for (1), we now show that

\[
(2) \quad G_-(\pi_1(B)) \leq \pi_1(F(x)).
\]

Since \( G_+ (\pi_1(B)) \geq G(\pi_1(B)) \) and (1) implies (1') if \( G_+ (\pi_1(B)) = G(\pi_1(B)) \) we therefore assume that \( G_+ (\pi_1(B)) > G(\pi_1(B)) \). In this case, using the remarks at the end of §2, we may find a \( B' \in \mathcal{O}(\beta) \) with \( \pi_1(B') < \pi_1(B) < \pi_1(x) \) and \( G(\pi_1(B')) = G_+ (\pi_1(B')) = G_+ (\pi_1(B)) \) which also implies (2) using (1).

Now, define \( B(x) \) to be the element of \( \mathcal{E}(\beta, F) \) with \( \pi_1(B(x)) = \max\{\pi_1(B): \pi_1(B) < \pi_1(x)\} \). Since there exists a \( B'' \in \mathcal{E}(\beta, F) \) with \( G_+ (\pi_1(B''(x))) = \pi_1(B''(x)) \) we may use (2) to inductively prove that

\[
\pi_1(F^n(x)) \geq G_+. \pi_1(B(x))
\]

for \( n = 0, 1, \ldots \). This implies the desired conclusion. ■

**Theorem 2.** Let \( f: A \to A \) be a monotone twist map with a non-Birkhoff \( p/q \)-periodic orbit \( o(\beta, f) \). If \( f \) has an invariant circle \( \Gamma \) then \( \rho(\Gamma) \in (R_-(\beta), R_+(\beta)) \).

**Proof.** Since \( o(\beta, f) \) is non-Birkhoff, using the monotone twist hypothesis, it is not difficult to show that \( o(\beta, f) \notin \Gamma \). Since \( \Gamma \) goes around \( A \) once, \( A - \Gamma \) consists of two completely invariant sub annuli and so \( o(\beta, f) \) is contained in one of these sub annuli. We shall assume that \( \Gamma \) is homotopic to the outer boundary of \( A_\beta \) and show that \( \rho(\Gamma) \geq R_+(\beta) \). The proof that \( \rho(\Gamma) \leq R_-(\beta) \) when \( \Gamma \) is homotopic to the inner boundary of \( A_\beta \) is similar.

Construct \( X \) from \( o(\beta, f) \) as above. Since \( \Gamma \) is homotopically non-trivial in \( A \), we may lift it to a compact set in \( X/\sigma_1 \). This implies that if \( \tilde{\Gamma} \) is a lift of \( \Gamma \) to \( X \), \( L(\tilde{\Gamma}) \) is bounded and the theorem then follows from Lemma 1. ■

Since we are not assuming that \( f \) is area preserving, \( \Gamma \) may not be a graph of a continuous function \( S^1 \to [0, 1] \). However, if \( \rho(\Gamma) \) is irrational, then \( f|_\Gamma \) has a unique minimal set that is the \( \omega \)-limit set of all points in \( \Gamma \). One may use this fact in conjunction with Lemma 1 to show that \( L(\tilde{\Gamma}) \) is constant when \( \rho(\Gamma) \) is irrational. Thus \( \Gamma \) never crosses the cut lines in \( A_\beta \).

We also note that \( o(\beta, f) \) can be a non-Birkhoff \( p/q \)-periodic point and yet \( R_-(\beta) = R_+(\beta) \), in which case, Theorem 2 has no content. For example, billiards on an elliptical table defines an integrable monotone twist map with a 1/2-periodic elliptic point (see [B2]). Periodic orbits, \( o(\beta) \) around the elliptic points are non-Birkhoff \( n/2n \)-periodic orbits and yet \( R_-(\beta) = 1/2 = R_+(\beta) \). However, it usually happens that \( R_+(\beta) \) is strictly larger than \( R_-(\beta) \). In particular, this will be the case whenever \( p \) and \( q \) are relatively prime as will be shown in the next section.

In this section we prove two lemmas which give a lower bound for the width of the rotation interval of the tight circle map associated with a non-Birkhoff \( p/q \)-periodic orbit when \( p \) and \( q \) are relatively prime. The first lemma contains a fact from elementary number theory. We include a sketch of its proof for the convenience of the reader. Recall that if \( x \) has continued fraction expansion given in the usual notation as \( x = [a_0, a_1, \ldots, a_m, \ldots] \), then
the \(n\)th convergent of \(x\) is denoted \(p_n/q_n\) where \(p_n/q_n = [a_0, a_1, \ldots, a_n]\). Also recall that any rational may be written as a continued fraction of the form \([a_0, a_1, \ldots, a_m]\).

**Lemma 3.** Given \(p/q\) with \(0 < p/q < 1\) and \((p, q) = 1\), let \(h/k = \max\{r/s: r/s < p/q, (r, s) = 1\}\) and \(h'/k' = \min\{r/s: r/s > p/q, (r, s) = 1\}\). If \(p/q\) has continued fraction \(p/q = [a_0, a_1, \ldots, a_n]\) then \(h/k = \min\{p_n/q_n, p_{n-1}/q_{n-1}\}\) and \(h'/k' = \max\{p_n/q_n, p_{n-1}/q_{n-1}\}\).

**Proof.** Let \(\mathcal{F}_q\) denote the \(q\)th Farey series. Since \(h/k\) and \(h'/k'\) are consecutive terms in \(\mathcal{F}_{q-1}\), \(k \neq k'\) (theorem 31, p. 24, [10]). We also have \(kh' - hk' = 1\) (theorem 28, p. 23, [10]) and since \(h/k, p/q\) and \(h'/k'\) are consecutive terms of \(\mathcal{F}_q\), \(p/q = (h + h')/(k + k')\) (theorem 29, page 23 [10]). These are precisely the properties that characterize \(p_n/q_n\) when \(p/q\) is written with \([a_0, a_1, \ldots, a_n]\) (theorem 172, p. 140 [10]). Note that the hypothesis of this theorem requires \(q > 1\) but the proof also yields our desired conclusion for \(a_n = 1\).

If \(x, y \in \mathbb{R}\), we shall use that notation \(\langle x, y \rangle\) for the closed interval whose endpoints are \(x\) and \(y\). Results closely related to Lemma 4 are contained in [14] and [16].

**Lemma 4.** Let \(o(\beta, f)\) be a non-Birkhoff \(p/q\)-periodic orbit for \(f: A \to A\) with \((p, q) = 1\) and \(0 < p/q < 1\). If \(p/q\) has continued fraction \(p/q = [0, a_1, \ldots, a_n]\) (with \(n \geq 1\)) then

\[
\rho(g) = \min\{p_n/q_n, p_{n-1}/q_{n-1}\}
\]

Proof: Let \(G: \mathbb{R} \to \mathbb{R}\) be the tight circle map associated with \(o(\beta, f)\), \(G_+ : \mathbb{R} \to \mathbb{R}\) is the upper auxiliary map associated with \(G\) and \(g^+ : S^1 \to S^1\) is its projection. If \(x\) is a point of \(\pi_1(o(\beta, f))\), then by our last remark in \(\S 2\), \(o(x, g_+) \subseteq o(x, g)\). Now since \(o(\beta, f)\) is non-Birkhoff and \(G_+\) is non-decreasing, \(o(x, g_+) \neq o(x, g)\) is impossible. Therefore, \(o(x, g_+)\) is properly contained in the finite set \(o(x, g)\) and so \(o(x, g_+)\) contains a periodic orbit with period \(s < q\) and so \(\rho(g_+) = r/s\) for some \(r\) with \((r, s) = 1\).

Now if \(h'/k'\) is defined as in Lemma 3, then \(r/s \geq h'/k'\). Thus \(\rho(g_+) = r/s \geq h'/k' = \max\{p_n/q_n, p_{n-1}/q_{n-1}\}\). The proof that \(\rho(g_-) \leq \min\{p_n/q_n, p_{n-1}/q_{n-1}\}\) is similar. Since \(\rho(g)\) is a closed interval the conclusion follows.

We are now in a position to prove the following theorem which allows us to exclude invariant circles of certain rotation numbers whenever a monotone twist map has a non-Birkhoff \(p/q\)-periodic orbit with \((p, q) = 1\). Theorem 5 is a more detailed version of Theorem B of the introduction.

**Theorem 5.** Let \(f: A \to A\) be a monotone twist map with non-Birkhoff \(p/q\)-periodic orbit \(o(\beta, f)\) with \((p, q) = 1\) and \(0 < p/q < 1\). If \(f\) has an invariant circle \(\Gamma\) and \(p/q\) has continued fraction expansion \(p/q = [0, a_1, \ldots, a_n, 1]\) then \(\rho(\Gamma) \notin \text{Int} \left(\langle p_n/q_n, p_{n-1}/q_{n-1}\rangle\right)\). In particular, if \(\omega\) is irrational and \(p/q\) is a convergent of \(\omega\), then \(\rho(\Gamma) \neq \omega\).

**Proof.** The first conclusion follows directly from Theorem 2 and Lemma 4. To prove the second, assume that \(p/q\) is the \(m\)th convergent to \(\omega\), i.e. \(\omega = [0, b_1, \ldots, b_m, \ldots]\) and \(p/q = [0, b_1, \ldots, b_m]\). We distinguish two cases. If \(b_m = 1\), then by Lemma 4, \(\rho(\Gamma) \notin \text{Int} \left(\langle [0, b_1, \ldots, b_{m-1}, [0, b_1, \ldots, b_{m-2}]\rangle\right)\) but it is a standard fact that \(\omega\) is in that set. Now if \(b_m \neq 1\), we may write \(p/q = [0, b_1, \ldots, b_{m-1}, 1]\). In this case, one has \(\rho(\Gamma) \notin \text{Int} \left(\langle [0, b_1, \ldots, b_{m-1}], [0, b_1, \ldots, b_{m-1}]\rangle\right)\) and one may check that this set contains \(\text{Int} \left(\langle [0, b_1, \ldots, b_{m-1}, [0, b_1, \ldots, b_{m-1}]\rangle\right)\) which contains \(\omega\).
In using Theorem 5, one need only know that $o(\beta)$ is \( p/q \)-periodic with \( (p, q) = 1 \) and that the order structure of $o(\beta)$ around the annulus is different than that induced by rigid rotation by $p/q$. If the details of this order structure are known, then Theorem 2 may yield more information as it is often the case that $(R_+(\beta), R_-(\beta))$ is strictly bigger than the interior of $\langle p_n/q_n, p_{n-1}/q_{n-1} \rangle$.

§6

In this section we prove Theorem C, restated here for convenience.

**Theorem C.** Suppose $f: A \to A$ is an area preserving monotone twist map and $\omega$ is an irrational between the rotation numbers of the boundaries of $A$. If $f$ has no invariant circle with rotation number $\omega$ then for every rational $r/s$, $(r, s) = 1$, sufficiently close to $\omega$, there exists a non-Birkhoff $r/s$-periodic orbit.

The proof we give is entirely elementary, explicitly following a strip of points which connects the inner and outer boundary components of $A$ and showing that the image of this strip intersects itself in a manner which forces the existence of the required periodic points.

We begin with several lemmas. The first two, due to Birkhoff, give some properties of area preserving monotone twist maps without invariant circles (see [3], [5] and [6]; for a modern treatment with generalizations of Birkhoff's work [11], [8], [17], [18] and [13]). The third lemma exploits the monotone twist condition to show that certain intersection properties persist under iteration and the fourth is a fixed point lemma.

**Lemma 5.** (Birkhoff) Suppose $f$ and $\omega$ are as in the statement of the theorem. Then there exists an annular region $\hat{A} \subseteq A$ whose boundary components are invariant circles of $f$ with $\omega$ between the rotation numbers of $f$ on these boundary circles and $f$ has no invariant circles in $\hat{A}$.

**Notation.** We will be interested only in $f$ restricted to $\hat{A}$, and $f$ takes $\hat{A}$ diffeomorphically onto itself so for notational convenience we will assume $A = \hat{A}$.

**Lemma 6.** (Birkhoff) Suppose $f: A \to A$ is an area preserving monotone twist map with no invariant circles (other than the boundary curves). Then for any neighborhoods $U_\omega$, $U_1$ of the inner and outer boundaries of $A$ respectively there exists points $\tau_\omega \in U_\omega$, $\tau_1 \in U_1$ and $n_\omega, n_1 > 0$ such that $f^n(\tau_\omega) \in U_1$ and $f^{n_1}(\tau_1) \in U_\omega$.

**Notation.** The following will occur repeatedly in the proof below, so we will make some specialized notation and terminology. They are illustrated in Fig. 6.
Let $\pi_2 : \mathbb{R} \to [0, 1]$ be the projection on the vertical coordinate:

1. If $a \in A$ let $l^-(a) = \{ z \in \mathbb{R} : \pi_1(z) = \pi_2(a) \}$ and $l^+(a) = \{ z \in \mathbb{R} : \pi_1(z) > \pi_2(a) \}$.

2. If $a, b \in A$ let
   \[ V(a, b) = \{ z \in A : \pi_1(a) < \pi_1(z) < \pi_1(b) \} \]
   and let $\bar{V}(a, b) = \{ z \in A : \pi_1(a) \leq \pi_1(z) \leq \pi_1(b) \}$.

3. For $a, b \in A$ with $\pi_1(a) < \pi_1(b)$ we say a set $B \subseteq \bar{V}(a, b)$ crosses $\bar{V}(a, b)$ diagonally if $B$ satisfies the following conditions:
   
   a. $B$ is equal to the closure of its interior and the boundary of $B$ is a piecewise smooth Jordan curve,
   
   b. $B$ is connected, and simply connected,
   
   c. $B$ intersects both $l^-(a)$ and $l^+(b)$,

   d. $B \cap (l^+(a) \cup \{ a \} \cup l^-(b) \cup \{ b \}) = \emptyset$.

Since $B$ is connected, $\partial B \cap V(a, b)$ will contain precisely two connected arcs $J_a$ and $J_b$, whose closures will intersect both $l^+(b) \cup \{ z \in \bar{V}(a, b) : \pi_2(z) = 1 \}$ and $l^-(a) \cup \{ z \in \bar{V}(a, b) : \pi_2(z) = 0 \}$. We call these arcs the upper and lower boundaries of $B$.

**Lemma 7.** Suppose $f : A \to A$ is a monotone twist map with lift $F : \hat{A} \to \hat{A}$. Suppose $a, b \in A$ satisfy $\pi_1(F^n(a)) < \pi_1(F^n(b))$ for all $n \geq 0$. Finally, suppose $B$ crosses $\bar{V}(a, b)$ diagonally. Then for each $n > 0$ there is a subset $B_n \subseteq B$ such that $F^n(B_n)$ crosses $F^n(B_n)$ diagonally.

**Proof:** After proving the lemma for $n = 1$, the result follows easily by induction. By the monotone twist condition

\[
\forall z \in l^-(a), \pi_1(F(z)) < \pi_1(F(a)), \forall z \in l^+(b), \pi_1(F(z)) > \pi_1(F(b))
\]

and $F(\bar{V}(a, b)) \cap (l^+(F(a)) \cup l^-(F(b))) = \emptyset$. Hence $F(B) \cap \bar{V}(F(a), F(b))$ contains the required subset (see Fig. 7).

**Lemma 8.** Suppose $f : A \to A$ is an orientation preserving diffeomorphism which preserves the boundary components, with lift $F : \hat{A} \to \hat{A}$. Suppose $a, b \in \hat{A}$ are fixed points of $F$ with $\pi_1(a) < \pi_1(b)$. Suppose $F(\bar{V}(a, b)) \cap \bar{V}(a, b)$ contains a component $C$ which crosses $\bar{V}(a, b)$ diagonally, such that the upper or lower boundary of $C$ is an image of a segment in $l^+(a)$ while the other is the image of a segment in $l^-(b)$ and $\partial C \cap \text{Int} V(a, b)$ is contained in $F(l^+(a) \cup l^-(b))$.

Then there exists $\zeta \in C$ such that $F(\zeta) = \zeta$.

**Proof:** This is the same fixed point lemma as in Hall [9]. The proof follows easily by computing the index of the vector field $F(z) - z$ around $\partial(F^{-1}(C))$ (see Fig. 8).
Before giving the proof of Theorem C, we sketch its main ideas. We consider the behavior under iteration of points in a strip $\tilde{V}(z_0, z_1)$ where $z_0, z_1$ are adjacent points of the extended orbit of a $p_1/q_1$-Birkhoff periodic point with $p_1/q_1$ near $\omega$ (see Fig. 9). Since by Lemma 6 there is a point, say $c_0$, which moves from the inner to the outer boundary of $\tilde{A}$ we see that some iterate of $\tilde{V}(a, b)$ will look as shown in Fig. 10.

We remark that the true image of $\tilde{V}(a, b)$ will probably be much more complicated, however the figures show the features of the image which are important for the proof. We single out the strip $F^i(B_2) = F^i(\tilde{V}(z_0, z_1))$ and note that since we may choose a point $c_2$ whose orbit moves from the upper to the lower boundary we see that for some $j > i$ we will have the situation in Fig. 11.

Since $F^j(B_2)$ crosses a strip diagonally with upper right corner near the upper boundary and lower left corner near the lower boundary we may arrange so that under iteration this strip will stretch far to the left of the image of $z_0$, e.g., after $k$ iterates we have the situation shown in Fig. 12.

Again using the point $c_0$ which moves from lower to upper boundary, we see that the
The image of $\bar{V}(z_0 + (l, 0), z_1 + (l, 0)) \cap F^k(B_2)$ will cross $\bar{V}(F^{2q_1}(z_0) + (l, 0), F^{2q_1}(z_1) + (l, 0))$ diagonally as shown in Fig. 13.

If we translate this picture left by $l + 2p_1$, then the map $F^{k+2q_1}(-) - (l + 2p_1, 0)$ will have a fixed point by Lemma 8. Moreover, this fixed point will correspond to a non-Birkhoff periodic orbit since it first moves far to the right of the image of $z_0$, then to the left. The details of the proof involve checking that the above can be made to happen so that a non-Birkhoff periodic orbit of any rational rotation number sufficiently close to $\omega$ will occur.

**Proof of Theorem C.** Fix $f: A \to A$ and $\omega$ as in the statement of the theorem. Fix a lift $F: \tilde{A} \to \tilde{A}$ of $f$. We will show the existence of the required periodic points of $f$ by finding fixed points of powers of $F$ composed with appropriate deck transformations. In order to guarantee that these are non-Birkhoff points we are forced to consider other points on the orbit as outlined above. The proof has five steps. The first two are used to pick periodic orbits and orbits that move between the boundary components. The third step follows a portion of a strip connecting upper and lower boundary components which is forced by the orbits chosen in the earlier steps to move in such a way that any periodic point in this set will be non-
Birkhoff. In the fourth step we fix an arbitrary rational sufficiently close to \( \omega \) and in the fifth step we show that the set studied in step three will contain a periodic point with the rotation number of step four.

**Step 1** Fix rationals \( p_0/q_0, p_1/q_1, p_2/q_2 \) strictly between the rotation numbers of the boundaries of \( \mathcal{A} \) such that \( p_0/q_0 < \omega < p_1/q_1 < p_2/q_2 \). Let \( b_0, b_1, b_2 \in \mathcal{A} \) be Birkhoff periodic points with rotation numbers \( p_0/q_0, p_1/q_1, p_2/q_2 \) respectively (such orbits exist by the Aubry–Mather theorem). We could carry out the proof with non-Birkhoff periodic points but the technical details would be more complicated. Let

\[
\beta_0 = \min \{ \pi_2(z) : z \in \text{eo}(b_0) \cup \text{eo}(b_1) \cup \text{eo}(b_2) \}
\]

\[
\beta_1 = \max \{ \pi_2(z) : z \in \text{eo}(b_0) \cup \text{eo}(b_1) \cup \text{eo}(b_2) \}.
\]

**Step 2.** By lemma (6) we may fix \( c_0, c_2 \in \mathcal{A} \) and integers \( 0 < q_1 < N_1 < N_2 < N_3 \) so that

\[
\pi_2(F^j(c_0)) < \beta_0, \pi_2(F^j(c_2)) > \beta_2, j = 0, \ldots, N_1
\]

\[
\pi_1(F^{N_1}(c_0)) - \pi_1(F^{N_1}(c_1)) > 4 + \pi_1(F^{N_1}(c_0)) - \pi_1(F^{N_1}(c_2)) - 4
\]

and

\[
\pi_2(F^j(c_0)) > \beta_2, \pi_2(F^j(c_2)) < \beta_0, j = N_2, \ldots, N_3
\]

\[
\pi_1(F^{N_1}(c_0)) - \pi_1(b_2) < \pi_1(F^{N_1}(c_0)) - \pi_1(F^{N_1-N_1}(b_2)) - 4
\]

Informally, \( c_0 \) spends \( N_1 \) iterates very close to the lower boundary, moving more slowly to the right under iteration than \( b_0 \), then moves close to the upper boundary where it moves more rapidly to the right than \( b_2 \) while \( c_2 \) does the opposite.

**Step 3.** Fix \( z_0, z_1 \in \text{eo}(b_1) \) so that \( V(z_0, z_1) \cap \text{eo}(b_1) = \emptyset \). We may assume \( \tau_0, \tau_1 \) are chosen so that \( \pi_1(F^2(z_0)) < \pi_1(F^2(z_1)) \) for some \( j \) between \( N_1 \) and \( N_2 \). Hence we may fix \( w_2 \in \text{eo}(b_2) \) so that \( (F^{N_1}V(z_0, z_1)) \cap V(F^{N_1}(z_1)) \) contains a component, \( F^{N_1}(B_1) \), which crosses diagonally (except it may contain \( F^{N_2}(z_0) \)) and \( \pi_1(w_2) > \pi_1(F^{N_1}(z_0)) + 3 \). (See Fig. 10 with \( i = N_3 \).) Next we choose \( c_2 \in \text{eo}(c_2) \) so that \( \pi_1(c_2) < \pi_1(F^{N_1}(z_0)) \), \( \pi_1(F^i(c_2)) < \pi_1(F^{N_1-N_1}(z_0)) + 3 \) for some \( j \) between \( 0 \) and \( N_3 \). Hence we may choose \( w_1 \in \text{eo}(b_1) \) with \( \pi_1(w_1) < \pi_1(F^{N_1}(z_0)) \) such that \( F^{2N_1}(z_0) \cap V(w_1, F^{2N_1}(w_1)) \) contains a component, \( F^{2N_1}(B_2) \), of \( F^{2N_1}(B_1) \) which crosses diagonally (see Fig. 11 with \( j = 2N_3 \)) such that, if \( z \in B_2 \) then \( \pi_1(F^{N_1}(z)) > \pi_1(F^{N_1}(z_1)) + 2 \) (this will imply that periodic points in \( B_2 \) are non-Birkhoff) and the upper and lower boundaries of \( F^{2N_1}(B_2) \) are the images of segments in \( l^+(z_0) \) and \( l^-(z_1) \) (one from each).

**Step 4.** Fix \( r/s \in \mathbb{Q}, (r, s) = 1, p_0/q_0 < r/s < p_1/q_1, s > 6N_3 \) and

\[
(s - 2q_1 + 4N_3) \left| \frac{p_0}{q_0} - \frac{P_1}{q_1} \right| > (s - 2q_1 + 2N_3) \left| \frac{r}{s} - \frac{P_1}{q_1} \right| + \pi_1(F^{N_1}(\omega_2)) - \pi_1(\omega_0) + 2.
\]

Since \( \left| \frac{p_0}{q_0} - \frac{P_1}{q_1} \right| < 1 \) we see that for \( r/s \) sufficiently close to \( \omega \) (hence \( s \) large) the above inequality will be satisfied. We let \( \tilde{z}_0 = F^{-(2q_1+2N_3)}(z_0) + (r, 0), \tilde{z}_1 = F^{-(2q_1+2N_3)}(z_1) \)
34 Philip L. Boyland and Glen Richard Hall

\[ (+, 0). \] Then \( F^{s-(2q_1 + 2N_1)}(B_2) \) crosses \( \tilde{\nu}(F^{s-(2q_1 + 4N_1)}(w_0), F^{s-(2q_1 + 3N_1)}(w_2)) \) diagonally and

\[
\begin{align*}
\pi_1(F^{s-(2q_1 + 4N_1)}(w_0)) &< \pi_1(\tilde{z}_0) \\
&< \pi_1(\tilde{z}_1) \\
&< \pi_1(F^{s-(2q_1 + 3N_1)}(w_2))
\end{align*}
\]

and \( F^{s-(2q_1 + 2N_1)}(B_2) \cap \tilde{\nu}(\tilde{z}_0, \tilde{z}_1) \) contains a component \( C \) which connects \( l^- (\tilde{z}_0) \) with \( l^- (\tilde{z}_1) \).

By the choice of \( r/s \) we also note that if \( w \in e_0(b_0) \cap \tilde{\nu}(\tilde{z}_0, F^{s-(2q_1 + 3N_1)}(w_2)) \)

then

\[
\pi_1(F^{-1s-(2q_1 + 4N_1)}(w)) > \pi_1(F^{N_1}(w_2))
\]

and hence for \( w \in e_0(b_0) \cap \tilde{\nu}(\tilde{z}_0, F^{s-(2q_1 + 3N_1)}(w_2)) \) we may assume that \( C \) contains a component which crosses \( \tilde{\nu}(\tilde{z}_0, w) \) diagonally. Moreover, the upper and lower boundaries of \( C \) are images of segments in \( l^+(z_0) \) and \( l^-(z_1) \), one from each. (See Fig. 12 with \( k = s - (2q_1 + 2N_3) \) and \( \tilde{z}_1 = \tilde{z}_1 + (l, 0), i = 0, 1 \).)

**Step 5.** We choose \( \tilde{c}_0 \in e_0(c_0) \) so that for some \( j, N_1 < j < N_3 + q_1 \)

\[
\begin{align*}
\pi_1(F^{i}(\tilde{c}_0)) &< \pi_1(F^{i}(\tilde{c}_0)), i = q_1, \ldots, j + 1, \pi_1(F^{j}(\tilde{z}_1)) > \pi_1(F^{j}(\tilde{c}_0)) \\
\pi_1(F^{j+1}(\tilde{z}_1)) &< \pi_1(F^{j+1}(\tilde{c}_0)) \\
\pi_2(F^{j}(\tilde{c}_0)) &< \beta_0 \text{ for } i = q_1, \ldots, N_1.
\end{align*}
\]

Hence \( F^{2q_1 + 2N_1}(C) \cap \tilde{\nu}(F^{2q_1 + 2N_1}(\tilde{z}_0), F^{2q_1 + 2N_1}(\tilde{z}_1)) \) contains a component which crosses diagonally, i.e. \( F^{j}(\tilde{\nu}(z_0, z_1)) \cap \tilde{\nu}(z_0 + (r, 0), z_1 + (r, 0)) \) contains a component which crosses diagonally with upper and lower boundaries images of segments from \( l^+(z_0) \) and \( l^-(z_1) \). Hence Lemma 8 implies that there exists, \( \zeta \in \tilde{\nu}(\tilde{z}_0, z_1) \) with \( F^{j}(\zeta) - (r, 0) = \zeta, \) i.e. \( \zeta \) is an \( r/s \) periodic point of \( F \) and we may choose \( \zeta \) so that \( \pi_1(F^{N_1}(\zeta)) > \pi_1(F^{N_1}(z_0) + 2. \) Since \( \pi_1(F^{j}(\zeta)) < \pi_1(F^{j}(z_0)) \) we see that \( \zeta \) is a non-Birkhoff periodic orbit, and the proof of the theorem is complete.

As a concluding remark we note that no effort was made in the above proof to maximize the set of rotation numbers for which non-Birkhoff orbits exist. The constraints on the set of rationals were the numbers \( \left[ \begin{array}{c} p_0 - p_1 \\ q_0 - q_1 \end{array} \right] \) and \( \left[ \begin{array}{c} p_1 - p_2 \\ q_1 - q_2 \end{array} \right] \), which measure the width of the zone of instability around \( \omega \) and \( N_3 \), the transit time for orbits moving through the zone of instability. If the transit time for orbits moving through the zone of instability is small then one expects non-Birkhoff orbits of relatively small period and hence the type of periodic orbits which appear in a twist map can be used to bound the rate of transit of orbits.

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