Topology and its Applications 159 (2012) 49–56

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On coincidence point and fixed point theorems for nonlinear multivalued maps

Wei-Shih Du

Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 802, Taiwan

A R T I C L E   I N F O

Article history:
Received 17 July 2010
Received in revised form 21 July 2011
Accepted 21 July 2011

MSC:
47H10
54H25

Keywords:
Coincidence point
$\mathcal{MT}$-function
Function of contractive factor
Berinde–Berinde fixed point theorem
Mizoguchi–Takahashi's fixed point theorem
Nadler's fixed point theorem
Banach contraction principle

A B S T R A C T

Several characterizations of $\mathcal{MT}$-functions are first given in this paper. Applying the characterizations of $\mathcal{MT}$-functions, we establish some existence theorems for coincidence point and fixed point in complete metric spaces. From these results, we can obtain new generalizations of Berinde–Berinde's fixed point theorem and Mizoguchi–Takahashi's fixed point theorem for nonlinear multivalued contractive maps. Our results generalize and improve some main results in the literature.

1. Introduction and preliminaries

Let us begin with some basic definitions and notation that will be needed in this paper. Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$, the set of positive integers and real numbers, respectively. Let (X,d) be a metric space. For each $x \in X$ and $A \subseteq X$, let $d(x, A) = \inf_{y \in A} d(x, y)$. Denote by $\mathcal{N}(X)$ the class of all nonempty subsets of X and $\mathcal{CB}(X)$ the family of all nonempty closed and bounded subsets of X. A function $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ defined by

$$\mathcal{H}(A, B) = \max\Big\{\sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B)\Big\}$$

is said to be the Hausdorff metric on $\mathcal{CB}(X)$ induced by the metric d on X.

Let $g : X \to X$ be a self-map and $T : X \to \mathcal{N}(X)$ be a multivalued map. A point x in X is a *coincidence point* of g and T if $gx \in Tx$. If $g = id$ is the identity map, then $x = gx \in Tx$ and call x a *fixed point* of T. The set of fixed points of T and the set of coincidence point of g and T are denoted by $\mathcal{F}(T)$ and $\mathcal{COP}(g, T)$, respectively. Recall that g is said to be *nonexpansive* if $d(gx, gy) \leqslant d(x, y)$ for all $x, y \in X$.

It is known that many metric fixed point theorems were motivated from the Banach contraction principle (see, e.g., [1]) which plays an important role in various fields of nonlinear analysis.

E-mail address: wsdu@nknucc.nknu.edu.tw.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.07.021

Theorem BCP *(Banach contraction principle). Let (X,d) be a complete metric space and $T: X \to X$ be a self-map. Assume that there exists a nonnegative number $\gamma < 1$ such that*

$$d\big(T(x), T(y)\big) \leqslant \gamma d(x, y) \quad \text{for all } x, y \in X.$$

Then T has a unique fixed point in X.

In 1969, Nadler [2] first gave a famous generalization of the Banach contraction principle for multivalued map. Since then a number of generalizations in various different directions of the Banach contraction principle and Nadler's fixed point theorem have been investigated by several authors; see [1,3–15] and references therein.

Theorem NA *(Nadler). Let (X,d) be a complete metric space and $T: X \to \mathcal{CB}(X)$ be a k-contraction; that is, there exists a nonnegative number $k < 1$ such that*

$$\mathcal{H}(Tx, Ty) \leqslant kd(x, y) \quad \text{for all } x, y \in X.$$

Then $\mathcal{F}(T) \neq \emptyset$.

Let f be a real-valued function defined on $\mathbb{R}$. For $c \in \mathbb{R}$, we recall that

$$\limsup_{x \to c} f(x) = \inf_{\varepsilon > 0} \sup_{0 < |x-c| < \varepsilon} f(x)$$

and

$$\limsup_{x \to c^+} f(x) = \inf_{\varepsilon > 0} \sup_{0 < x-c < \varepsilon} f(x).$$

Definition 1.1. ([3–5]) A function $\varphi : [0,\infty) \to [0,1)$ is said to be an *$\mathcal{MT}$-function* if it satisfies Mizoguchi–Takahashi's condition (i.e. $\limsup_{s \to t^+} \varphi(s) < 1$ for all $t \in [0,\infty)$).

Clearly, if $\varphi : [0,\infty) \to [0,1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class. An example which is not an $\mathcal{MT}$-function is given hereunder. Let $\varphi : [0,\infty) \to [0,1)$ be defined by

$$\varphi(t) := \begin{cases} \frac{\sin t}{t}, & \text{if } t \in (0, \frac{\pi}{2}], \\ 0, & \text{otherwise.} \end{cases}$$

Since $\limsup_{s \to 0^+} \varphi(s) = 1$, φ is not an $\mathcal{MT}$-function.

In 2007, M. Berinde and V. Berinde [6] proved the following interesting fixed point theorem.

Theorem BB *(M. Berinde and V. Berinde). Let (X,d) be a complete metric space, $T: X \to \mathcal{CB}(X)$ be a multivalued map, $\varphi : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function and $L \geqslant 0$. Assume that*

$$\mathcal{H}(Tx, Ty) \leqslant \varphi\big(d(x, y)\big)d(x, y) + Ld(y, Tx) \quad \text{for all } x, y \in X.$$

Then $\mathcal{F}(T) \neq \emptyset$.

It is quite obvious that if let $L = 0$ in Theorem BB, then we can obtain Mizoguchi–Takahashi's fixed point theorem [9] which is a partial answer of Problem 9 in Reich [10].

Theorem MT *(Mizoguchi and Takahashi). Let (X,d) be a complete metric space, $T: X \to \mathcal{CB}(X)$ be a multivalued map and $\varphi : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function. Assume that*

$$\mathcal{H}(Tx, Ty) \leqslant \varphi\big(d(x, y)\big)d(x, y) \quad \text{for all } x, y \in X.$$

Then $\mathcal{F}(T) \neq \emptyset$.

In fact, Mizoguchi–Takahashi's fixed point theorem is a generalization of Nadler's fixed point theorem, but its primitive proof in [9] is difficult. Another proof in [11] is not yet simple. Recently, Suzuki [10] gave a very simple proof of Theorem MT.

Several characterizations of $\mathcal{MT}$-functions are first given in this paper. Applying the characterizations of $\mathcal{MT}$-functions, we establish some existence theorems for coincidence point and fixed point in complete metric spaces. From these results, we can obtain new generalizations of Berinde–Berinde's fixed point theorem and Mizoguchi–Takahashi's fixed point theorem for nonlinear multivalued contractive maps. Our results generalize and improve some main results in [1–3,6–12].

2. Main results

In this section, we first give some characterizations of $\mathcal{MT}$-functions.

Theorem 2.1. *Let $\varphi : [0,\infty) \to [0,1)$ be a function. Then the following statements are equivalent.*

(a) *φ is an $\mathcal{MT}$-function.*
(b) *For each $t \in [0,\infty)$, there exist $r_t^{(1)} \in [0,1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \leqslant r_t^{(1)}$ for all $s \in (t, t+\varepsilon_t^{(1)})$.*
(c) *For each $t \in [0,\infty)$, there exist $r_t^{(2)} \in [0,1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \leqslant r_t^{(2)}$ for all $s \in [t, t+\varepsilon_t^{(2)}]$.*
(d) *For each $t \in [0,\infty)$, there exist $r_t^{(3)} \in [0,1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \leqslant r_t^{(3)}$ for all $s \in (t, t+\varepsilon_t^{(3)}]$.*
(e) *For each $t \in [0,\infty)$, there exist $r_t^{(4)} \in [0,1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \leqslant r_t^{(4)}$ for all $s \in [t, t+\varepsilon_t^{(4)})$.*
(f) *For any nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \leqslant \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*
(g) *φ is a* function of contractive factor [4]; *that is, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \leqslant \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*

Proof. (i) "(a) $\Leftrightarrow$ (b)".

We first show "(a) $\Rightarrow$ (b)". Suppose that φ is an $\mathcal{MT}$-function. Then for each $t \in [0,\infty)$, there exists $\varepsilon_t > 0$ such that

$$\sup_{t<s<t+\varepsilon_t} \varphi(s) < 1.$$

By the denseness of $\mathbb{R}$, there also exists $r_t \in [0,1)$ such that

$$\sup_{t<s<t+\varepsilon_t} \varphi(s) \leqslant r_t < 1,$$

which says that $\varphi(s) \leqslant r_t$ for all $s \in (t, t+\varepsilon_t)$. The converse part (i.e. (b) $\Rightarrow$ (a)) is obvious.

(ii) "(b) $\Leftrightarrow$ (c)".

Clearly, "(c) $\Rightarrow$ (b)" is true for $r_t^{(1)} := r_t^{(2)}$ and $\varepsilon_t^{(1)} := \varepsilon_t^{(2)}$. Conversely, assume (b) holds. Let $t \in [0,\infty)$ be given. Then, by our hypothesis, there exist $r_t^{(1)} \in [0,1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \leqslant r_t^{(1)}$ for all $s \in (t, t+\varepsilon_t^{(1)})$. Put $\varepsilon_t^{(2)} = \varepsilon_t^{(1)}$ and

$$r_t^{(2)} := \max\bigl\{r_t^{(1)}, \varphi(t), \varphi\bigl(t+\varepsilon_t^{(1)}\bigr)\bigr\}.$$

Then $r_t^{(2)} \in [0,1)$ and $\varphi(s) \leqslant r_t^{(2)}$ for all $s \in [t, t+\varepsilon_t^{(2)}]$. So we prove "(b) $\Rightarrow$ (c)".

(iii) The implications "(c) $\Rightarrow$ (d) $\Rightarrow$ (b)" and "(c) $\Rightarrow$ (e) $\Rightarrow$ (b)" are obvious.

(iv) Let us prove "(e) $\Rightarrow$ (f)".

Suppose that (e) holds. Let $\{x_n\}_{n\in\mathbb{N}}$ be a nonincreasing sequence in $[0,\infty)$. Then $t_0 := \lim_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} x_n \geqslant 0$ exists. By our hypothesis, there exist $r_{t_0} \in [0,1)$ and $\varepsilon_{t_0} > 0$ such that $\varphi(s) \leqslant r_{t_0}$ for all $s \in [t_0, t_0+\varepsilon_{t_0})$. On the other hand, there exists $\ell \in \mathbb{N}$, such that

$$t_0 \leqslant x_n < t_0 + \varepsilon_{t_0}$$

for all $n \in \mathbb{N}$ with $n \geqslant \ell$. Hence $\varphi(x_n) \leqslant r_{t_0}$ for all $n \geqslant \ell$. Let

$$\eta := \max\bigl\{\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_{\ell-1}), r_{t_0}\bigr\} < 1.$$

Then $\varphi(x_n) \leqslant \eta$ for all $n \in \mathbb{N}$. Hence $0 \leqslant \sup_{n\in\mathbb{N}} \varphi(x_n) \leqslant \eta < 1$ and (f) holds.

(v) The implication "(f) $\Rightarrow$ (g)" is obvious.

(vi) Finally, we prove "(g) $\Rightarrow$ (e)".

Assume that φ is a function of contractive factor. On the contrary, suppose that there exists $\hat{t} \in [0,\infty)$ such that for each $r \in [0,1)$ and each $\epsilon > 0$ there is $s \in [\hat{t}, \hat{t}+\epsilon)$ with property $\varphi(s) > r$. So, for $r_1 := \varphi(\hat{t}) \in [0,1)$ and for $\epsilon_1 := 1 > 0$ it must exists $s_1 \in [\hat{t}, \hat{t}+\epsilon_1)$ with $\varphi(s_1) > r_1$. The last inequality also implies that $s_1 \neq \hat{t}$ and thus $\hat{t} < s_1$. Choose $\epsilon_2 > 0$ satisfying $\hat{t} + \epsilon_2 \leqslant s_1$, and set

$$r_2 := \max\left\{\varphi(s_1), 1 - \frac{1}{2}\right\}.$$

Then, for r_2 and for ϵ_2 as indicated, we can find $s_2 \in [\hat{t}, \hat{t}+\epsilon_2)$ with $\varphi(s_2) > r_2$. This also entails that $\hat{t} < s_2 < s_1$. Continuing this process, we can construct a strictly decreasing sequence $\{s_n\} \subset [\hat{t}, \infty) \subset [0,\infty)$ such that

$$\varphi(s_n) > r_n := \max\left\{\varphi(s_{n-1}), 1 - \frac{1}{n}\right\} \geqslant 1 - \frac{1}{n}$$

for all $n \in \mathbb{N}$. This yields $\sup_{n\in\mathbb{N}} \varphi(s_n) \geqslant 1$ which contradict that φ is a function of contractive factor. Therefore we show that "(g) $\Rightarrow$ (e)" is true.

By (i)–(vi), we complete the proof. □

Remark 2.1. In [5, Theorem 2.8], the author had proved that any $\mathcal{MT}$-function is a function of contractive factor.

The following existence theorem for coincidence point and fixed point is one of the main results of this paper.

Theorem 2.2. *Let (X, d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map, $g : X \to X$ be a continuous self-map and $\varphi : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function. Assume that*

(a) *Tx is g-invariant (i.e. $g(Tx) \subseteq Tx$) for each $x \in X$;*
(b_1) *there exists a function $h : X \to [0,\infty)$ such that*

$$\mathcal{H}(Tx, Ty) \leqslant \varphi\big(d(x, y)\big)d(x, y) + h(gy)d(gy, Tx) \quad \text{for all } x, y \in X.$$

Then $\mathcal{COP}(g, T) \cap \mathcal{F}(T) \neq \emptyset$.

Proof. Note first that for each $x \in X$, by (a), we have $d(gy, Tx) = 0$ for all $y \in Tx$. So, for each $x \in X$, by (b_1), we obtain

$$d(y, Ty) \leqslant \varphi\big(d(x, y)\big)d(x, y) \quad \text{for all } y \in Tx. \tag{1}$$

Let $x \in X$. Take $x_0 = x \in X$ and choose $x_1 \in Tx_0$. If $d(x_0, x_1) = 0$, then $x_0 = x_1 \in Tx_0$. Hence $x_0 \in \mathcal{F}(T)$ and we are done. Otherwise, if $d(x_0, x_1) > 0$ or $x_0 \neq x_1$, let $\kappa : [0,\infty) \to [0,1)$ be defined by $\kappa(t) = \frac{1+\varphi(t)}{2}$. Clearly, $0 \leqslant \varphi(t) < \kappa(t) < 1$ for all $t \in [0,\infty)$. By (1), it follows that

$$d(x_1, Tx_1) < \kappa\big(d(x_0, x_1)\big)d(x_0, x_1). \tag{2}$$

By (2), there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) < \kappa\big(d(x_0, x_1)\big)d(x_0, x_1).$$

If $d(x_1, x_2) = 0$, then $x_1 = x_2 \in Tx_1$ and hence $x_1 \in \mathcal{F}(T)$. Otherwise, there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) < \kappa\big(d(x_1, x_2)\big)d(x_1, x_2).$$

Let $\xi_n = d(x_{n-1}, x_n)$, $n \in \mathbb{N}$. By induction, we can obtain the following: for each $n \in \mathbb{N}$,

$$x_n \in Tx_{n-1}, \quad \text{and} \tag{3}$$

$$\xi_{n+1} < \kappa(\xi_n)\xi_n. \tag{4}$$

Since $\kappa(t) < 1$ for all $t \in [0,\infty)$, by (4), the sequence $\{\xi_n\}_{n=1}^{\infty}$ is strictly decreasing in $[0,\infty)$. Since φ is an $\mathcal{MT}$-function, by Theorem 2.1, $0 \leqslant \sup_{n\in\mathbb{N}} \varphi(\xi_n) < 1$. Then it follows that

$$0 < \sup_{n\in\mathbb{N}} \kappa(\xi_n) = \frac{1}{2}\Big[1 + \sup_{n\in\mathbb{N}} \varphi(\xi_n)\Big] < 1.$$

Let $\gamma := \sup_{n\in\mathbb{N}} \kappa(\xi_n)$. So $\gamma \in (0,1)$. By (4) again, we have

$$\xi_{n+1} < \kappa(\xi_n)\xi_n \leqslant \gamma\xi_n \quad \text{for each } n \in \mathbb{N}. \tag{5}$$

Thus it follows from (5) that

$$d(x_n, x_{n+1}) = \xi_{n+1} < \gamma\xi_n < \cdots < \gamma^n\xi_1 = \gamma^n d(x_0, x_1) \tag{6}$$

for each $n \in \mathbb{N}$. Let $\alpha_n = \frac{\gamma^n}{1-\gamma}d(x_0, x_1)$, $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $m > n$, we have form (6) that

$$d(x_n, x_m) \leqslant \sum_{j=n}^{m-1} d(x_j, x_{j+1}) < \alpha_n.$$

Since $0 < \gamma < 1$, $\lim_{n\to\infty} \alpha_n = 0$ and hence $\lim_{n\to\infty} \sup\{d(x_n, x_m)\colon\ m > n\} = 0$. This prove that $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. Since, by (3), $x_{n+1} \in Tx_n$, we have from (a) that

$$gx_{n+1} \in Tx_n \quad \text{for each } n \in \mathbb{N}. \tag{7}$$

Since g is continuous and $\lim_{n\to\infty} x_n = v$, we have

$$\lim_{n\to\infty} gx_n = gv. \tag{8}$$

Since the function $x \mapsto d(x, Tv)$ is continuous, we get from (b_1), (7), (8) and $\lim_{n\to\infty} x_n = v$ that

$$\begin{aligned} d(v, Tv) &= \lim_{n\to\infty} d(x_{n+1}, Tv) \\ &\leqslant \lim_{n\to\infty} \mathcal{H}(Tx_n, Tv) \\ &\leqslant \lim_{n\to\infty} \bigl\{\varphi\bigl(d(x_n, v)\bigr)d(x_n, v) + h(gv)d(gv, gx_{n+1})\bigr\} = 0. \end{aligned}$$

Hence $d(v, Tv) = 0$. Since Tv is closed in X, we have $v \in Tv$. By (a), $gv \in Tv$. Therefore, $v \in \mathcal{COP}(g, T) \cap \mathcal{F}(T)$ and the proof is complete. □

Here, we give an example illustrating Theorem 2.2.

Example A. Let ℓ^∞ be the Banach space consisting of all bounded real sequences with supremum norm d_∞ and let $\{e_n\}$ be the canonical basis of ℓ^∞. Let $\{\tau_n\}$ be a sequence of positive real numbers satisfying $\tau_1 = \tau_2$ and $\tau_{n+1} < \tau_n$ for $n \geqslant 2$ (for example, let $\tau_1 = \frac{1}{2}$ and $\tau_n = \frac{1}{n}$ for $n \in \mathbb{N}$ with $n \geqslant 2$). Thus $\{\tau_n\}$ is convergent. Put $v_n = \tau_n e_n$ for $n \in \mathbb{N}$ and let $X = \{v_n\}_{n\in\mathbb{N}}$ be a bounded and complete subset of ℓ^∞. Then (X, d_∞) be a complete metric space and $d_\infty(v_n, v_m) = \tau_n$ if $m > n$.

Let $T : X \to \mathcal{CB}(X)$ and $g : X \to X$ be defined by

$$Tv_n := \begin{cases} \{v_1, v_2\}, & \text{if } n \in \{1, 2\}, \\ X \setminus \{v_1, v_2, \ldots, v_n, v_{n+1}\}, & \text{if } n \geqslant 3, \end{cases}$$

and

$$gv_n := \begin{cases} v_2, & \text{if } n \in \{1, 2\}, \\ v_{n+1}, & \text{if } n \geqslant 3, \end{cases}$$

respectively. Then the following hold.

(a) Tx is g-invariant for each $x \in X$.
(b) $\mathcal{COP}(g, T) \cap \mathcal{F}(T) = \{v_1, v_2\}$.
(c) g is continuous on X.

Indeed, (a) and (b) are obviously true. To see (c), since

- $d_\infty(gv_1, gv_2) = 0 < \tau_1 = d_\infty(v_1, v_2)$,
- $d_\infty(gv_1, gv_m) = \tau_2 = \tau_1 = d_\infty(v_1, v_m)$ for any $m \geqslant 3$,
- $d_\infty(gv_2, gv_m) = \tau_2 = d_\infty(v_2, v_m)$ for any $m \geqslant 3$,
- $d_\infty(gv_n, gv_m) = \tau_{n+1} < \tau_n = d_\infty(v_n, v_m)$ for any $n \geqslant 3$ and $m > n$,

we prove that g is nonexpansive on X which implies that g is continuous on X.

Define $\varphi : [0, \infty) \to [0, 1)$ by

$$\varphi(t) := \begin{cases} \frac{\tau_{n+2}}{\tau_n}, & \text{if } t = \tau_n \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases}$$

and $\hat{h} : X \to [0, \infty)$ by

$$\hat{h}(v_n) := \begin{cases} 0, & n \in \{1, 2\}, \\ n, & \text{if } n \geqslant 3. \end{cases}$$

Since $\limsup_{s\to t^+} \varphi(s) = 0 < 1$ for all $t \in [0, \infty)$, φ is an $\mathcal{MT}$-function. We claim that

$$\mathcal{H}_\infty(Tx, Ty) \leqslant \varphi\bigl(d_\infty(x, y)\bigr)d_\infty(x, y) + \hat{h}(gy)d_\infty(gy, Tx) \quad \text{for all } x, y \in X, \tag{$*$}$$

where $\mathcal{H}_\infty$ is the Hausdorff metric induced by d_∞. In order to verify that T satisfies ($*$), we consider the following four possible cases:

Case 1. $\varphi(d(v_1, v_2))d_\infty(v_1, v_2) + \hat{h}(gv_2)d_\infty(gv_2, Tv_1) = \tau_3 > 0 = \mathcal{H}_\infty(Tv_1, Tv_2)$.

Case 2. For any $m \geqslant 3$, we have

$$\varphi\big(d_\infty(v_1, v_m)\big)d_\infty(v_1, v_m) + \hat{h}(gv_m)d_\infty(gv_m, Tv_1) = \tau_3 + (m+1)\tau_2 > \tau_1 = \mathcal{H}_\infty(Tv_1, Tv_m).$$

Case 3. For any $m \geqslant 3$, we obtain

$$\varphi\big(d_\infty(v_2, v_m)\big)d_\infty(v_2, v_m) + \hat{h}(gv_m)d_\infty(gv_m, Tv_2) = \tau_4 + (m+1)\tau_2 > \tau_1 = \mathcal{H}_\infty(Tv_2, Tv_m).$$

Case 4. For any $n \geqslant 3$ and $m > n$, we get

$$\varphi\big(d_\infty(v_n, v_m)\big)d_\infty(v_n, v_m) + \hat{h}(gv_m)d_\infty(gv_m, Tv_n) = \tau_{n+2} = \mathcal{H}_\infty(Tv_n, Tv_m).$$

Hence, by Cases 1, 2, 3 and 4, we prove that T satisfies $(*)$. Therefore, all the assumptions of Theorem 2.2 are satisfied. Applying Theorem 2.2, we also prove $\mathcal{COP}(g,T) \cap \mathcal{F}(T) \neq \emptyset$. Notice that $\mathcal{H}_\infty(Tv_1, Tv_m) = \tau_1 > \tau_3 = \varphi(d_\infty(v_1, v_m))d_\infty(v_1, v_m)$ for all $m \geqslant 3$, so Mizoguchi–Takahashi's fixed point theorem is not applicable here.

As a direct consequence of Theorem 2.2, we obtain the following result.

Theorem 2.3. *Let (X, d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map, $g : X \to X$ be a continuous self-map and $\varphi : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function. Assume that*

(a) *Tx is g-invariant (i.e. $g(Tx) \subseteq Tx$) for each $x \in X$;*
(b_2) *there exist $L \geqslant 0$ and a function $\tau : X \to [0, L]$ such that*

$$\mathcal{H}(Tx, Ty) \leqslant \varphi\big(d(x,y)\big)d(x,y) + \tau(gy)d(gy, Tx) \quad \text{for all } x, y \in X.$$

Then $\mathcal{COP}(g,T) \cap \mathcal{F}(T) \neq \emptyset$.

Example B. Let ℓ^∞, d_∞, $\mathcal{H}_\infty$, $\{e_n\}$, $\{\tau_n\}$, X, T, g and φ be the same as in Example A. Define a function $\bar{h}$ by

$$\bar{h}(v_n) := \begin{cases} \frac{2}{\sqrt{5}}, & \text{if } n = 1, \\ \frac{1}{e}, & \text{if } n = 2, \\ 1 - \frac{\tau_n}{\tau_2}, & \text{if } n \geqslant 3. \end{cases}$$

Then $\bar{h}$ is a function from X into $[0,1]$. In order to verify that T satisfies

$$\mathcal{H}_\infty(Tx, Ty) \leqslant \varphi\big(d_\infty(x,y)\big)d_\infty(x,y) + \bar{h}(gy)d_\infty(gy, Tx) \quad \text{for all } x, y \in X, \tag{$**$}$$

we need to consider the following four possible cases:

Case 1. $\varphi(d(v_1, v_2))d_\infty(v_1, v_2) + \bar{h}(gv_2)d_\infty(gv_2, Tv_1) = \tau_3 > 0 = \mathcal{H}_\infty(Tv_1, Tv_2)$.

Case 2. For any $m \geqslant 3$, we have

$$\begin{aligned} \varphi\big(d_\infty(v_1, v_m)\big)d_\infty(v_1, v_m) + \bar{h}(gv_m)d_\infty(gv_m, Tv_1) &= \tau_3 + \left(\frac{\tau_2 - \tau_{m+1}}{\tau_2}\right)\tau_2 \\ &= \tau_2 + (\tau_3 - \tau_{m+1}) > \tau_1 \\ &= \mathcal{H}_\infty(Tv_1, Tv_m). \end{aligned}$$

Case 3. For any $m \geqslant 3$, we obtain

$$\varphi\big(d_\infty(v_2, v_m)\big)d_\infty(v_2, v_m) + \bar{h}(gv_m)d_\infty(gv_m, Tv_2) = \tau_2 + (\tau_4 - \tau_{m+1}) \geqslant \tau_1 = \mathcal{H}_\infty(Tv_2, Tv_m).$$

Case 4. For any $n \geqslant 3$ and $m > n$, we get

$$\varphi\big(d_\infty(v_n, v_m)\big)d_\infty(v_n, v_m) + \bar{h}(gv_m)d_\infty(gv_m, Tv_n) = \tau_{n+2} = \mathcal{H}_\infty(Tv_n, Tv_m).$$

Hence we know that T satisfies $(**)$. Applying Theorem 2.3, we have $\mathcal{COP}(g,T) \cap \mathcal{F}(T) \neq \emptyset$.

Theorem 2.4. *Let (X,d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map, $g : X \to X$ be a continuous self-map, $\varphi : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function and $L \geqslant 0$. Assume that*

(a) *Tx is g-invariant (i.e. $g(Tx) \subseteq Tx$) for each $x \in X$;*
(b_3) *there exists $L \geqslant 0$ such that*

$$\mathcal{H}(Tx,Ty) \leqslant \varphi\big(d(x,y)\big)d(x,y) + Ld(gy,Tx) \quad \text{for all } x, y \in X.$$

Then $\mathcal{COP}(g,T) \cap \mathcal{F}(T) \neq \emptyset$.

Proof. Define $\tau : X \to [0,L]$ by $\tau(x) = L$ for all $x \in X$. So, (b_3) implies (b_2) and hence the conclusion follows from Theorem 2.3. □

Example C. Let ℓ^∞, d_∞, $\mathcal{H}_\infty$, $\{e_n\}$, $\{\tau_n\}$, X, T, g and φ be the same as in Example A. It is not hard to verify that T satisfies

$$\mathcal{H}(Tx,Ty) \leqslant \varphi\big(d(x,y)\big)d(x,y) + d(gy,Tx) \quad \text{for all } x, y \in X.$$

Applying Theorem 2.4 with $L = 1$, we obtain $\mathcal{COP}(g,T) \cap \mathcal{F}(T) \neq \emptyset$.

Remark 2.2. It is quite obvious that (b_2) implies (b_3) (since $\tau(x) \leqslant L$ for all $x \in X$), so (b_2) and (b_3) are indeed equivalent. Hence Theorem 2.3 and Theorem 2.4 are real logical equivalent.

The following intersection theorem is also immediate from Theorem 2.2.

Theorem 2.5. *Let (X,d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map, $g : X \to X$ be a continuous self-map and $\varphi : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function. Assume that*

(a) *Tx is g-invariant (i.e. $g(Tx) \subseteq Tx$) for each $x \in X$;*
(b_4) *there exist $L \geqslant 0$ and a function $\mu : X \to [L,\infty)$ such that*

$$\mathcal{H}(Tx,Ty) \leqslant \varphi\big(d(x,y)\big)d(x,y) + \mu(gy)d(gy,Tx) \quad \text{for all } x, y \in X.$$

Then $\mathcal{COP}(g,T) \cap \mathcal{F}(T) \neq \emptyset$.

Example D. Let ℓ^∞, d_∞, $\mathcal{H}_\infty$, $\{e_n\}$, $\{\tau_n\}$, X, T, g and φ be the same as in Example A. Define a function $\tilde{h}$ by

$$\tilde{h}(v_n) := 1 + \tau_n \quad \text{for any } n \in \mathbb{N}.$$

Then $\bar{h}$ is a function from X into $[1,\infty)$. Following a similar argument as in Example A, we can show that T satisfies

$$\mathcal{H}_\infty(Tx,Ty) \leqslant \varphi\big(d_\infty(x,y)\big)d_\infty(x,y) + \tilde{h}(gy)d_\infty(gy,Tx) \quad \text{for all } x, y \in X,$$

and $\mathcal{COP}(g,T) \cap \mathcal{F}(T) \neq \emptyset$ follows from Theorem 2.5.

Remark 2.3. In fact, Theorem 2.4 can be proved by Theorem 2.5. Indeed, under the assumptions of Theorem 2.4, let $\mu : X \to [L,\infty)$ be defined by $\mu(x) = L$ for all $x \in X$. So (b_3) implies (b_4). Hence Theorem 2.5 implies Theorem 2.4. Notice that Theorem 2.5 also implies Theorem 2.3 since $\tau(x) \leqslant L \leqslant \mu(x)$ for all $x \in X$.

Applying Theorem 2.2, we get the following generalization of Berinde–Berinde's fixed point theorem.

Theorem 2.6 *(Generalized Berinde–Berinde's fixed point theorem). Let (X,d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map, $\varphi : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function and $h : X \to [0,\infty)$ be a function. Assume that*

$$\mathcal{H}(Tx,Ty) \leqslant \varphi\big(d(x,y)\big)d(x,y) + h(y)d(y,Tx) \quad \text{for all } x, y \in X. \tag{9}$$

Then $\mathcal{F}(T) \neq \emptyset$.

Proof. Let $g = id$ be the identity map. It is easy to verify that all the conditions of Theorem 2.2 are satisfied. Hence the conclusion follows from Theorem 2.2. □

Example E. Let ℓ^∞, d_∞, $\mathcal{H}_\infty$, $\{e_n\}$, $\{\tau_n\}$, X, T, φ and $\hat{h}$ be the same as in Example A. Then $\mathcal{F}(T) = \{v_1, v_2\}$. Following a similar argument as in Example A, we can prove that T satisfies

$$\mathcal{H}_\infty(Tx, Ty) \leqslant \varphi\big(d_\infty(x, y)\big)d_\infty(x, y) + \hat{h}(y)d_\infty(y, Tx) \quad \text{for all } x, y \in X.$$

Using Theorem 2.6, we get $\mathcal{F}(T) \neq \emptyset$ (in fact $\mathcal{F}(T) = \{v_1, v_2\}$).

Remark 2.4.

(1) In Example B, since

$$\begin{aligned}\varphi\big(d_\infty(v_2, v_3)\big)d_\infty(v_2, v_3) + \bar{h}(v_3)d_\infty(v_3, Tv_2) &= \tau_4 + \left(\frac{\tau_2 - \tau_3}{\tau_2}\right)\tau_2 \\ &= \tau_2 + (\tau_4 - \tau_3) < \tau_1 \\ &= \mathcal{H}_\infty(Tv_2, Tv_m),\end{aligned}$$

T does not satisfy (9). So Theorem 2.6 cannot be applicable to Example B. Therefore Theorem 2.2 is a proper extension of Theorem 2.6.

(2) Theorems 2.2, 2.3, 2.4, 2.5 and 2.6 all generalize and improve Berinde–Berinde's fixed point theorem, Mizoguchi–Takahashi's fixed point theorem, Nadler's fixed point theorem, Banach contraction principle and some main results in [1–3,6–14]. By applying Theorem 2.1, [3, Theorem 2.1] and [14, Theorem 3.1] are indeed real logical equivalent.

(3) Let (X, d) be a metric space. Recall that a single-valued map $T : X \to X$ is called a *generalized Berinde map* [15] if there exist $r \in [0, 1)$ and a function b from X into $[0, \infty)$ such that

$$d(Tx, Ty) \leqslant rd(x, y) + b(y)d(y, Tx) \quad \text{for all } x, y \in X.$$

In particular, if there exists $B \in [0, \infty)$ such that $b(x) = B$ for all $x \in X$, then T is called a *Berinde map* [13,15]. In [15], Suzuki proved some new fixed point theorems for generalized Berinde maps with constants. He also gave an example illustrating that there exists a generalized Berinde map which is not a Berinde map; for more details, see [15].

Acknowledgements

The author wishes to express his hearty thanks to Professor *Thai Doan Chuong*, Professor *Hsien-Chung Wu* and the anonymous referees for their valuable suggestions and comments. This research was supported by the National Science Council of the Republic of China.

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