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Existence of directed triplewhist tournaments with the three person property $3PDTWh(v)$

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Abstract

A directed triplewhist tournament on v players, briefly $DTWh(v)$, is said to have the three person property if no two games in the tournament have three common players. We briefly denote such a design as a $3PDTWh(v)$. In this paper, we show that a $3PDTWh(v)$ exists whenever $v > 17$ and $v \equiv 1 \pmod 4$ with few possible exceptions.

Keywords: Whist tournament; 3PDTWh; 3PDTWh-frame

1. Introduction

A *whist tournament* $Wh(v)$ for $v = 4n$ (or $4n + 1$) is a schedule of games (a, b, c, d) where the unordered pairs $\{a, c\}$, $\{b, d\}$ are called *partners*, the pairs $\{a, b\}$, $\{c, d\}$, $\{a, d\}$, $\{b, c\}$ are called *opponents*, such that

(1) the games are arranged into $4n - 1$ (or $4n + 1$) rounds, each of n games;
(2) each player plays in exactly one game in each round (or all rounds but one);
(3) each player partners every other player exactly once;
(4) each player opposes every other player exactly twice.

The whist tournament problem was introduced by Moore [32]. Its existence attracted a lot of researchers such as Wilson, Baker, Hartman et al. A complete solution is given in [7] and [9]. Ever since the existence of whist tournaments was completely settled, the focus has turned to whist tournaments with additional properties. Such special whist tournaments include at least directedwhist tournaments, triplewhist tournaments, whist tournaments with the three-person property, and Z-cyclic whist tournaments. As more and more results have been obtained, the attention has turned to whist tournaments that satisfy more than one of the above-mentioned criteria simultaneously (see, for example, [8,11]). In what follows, for convenience, we shall provide a brief description of such types of tournaments and the known results associated with them.

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A whist tournament is said to have *three person property,* denoted by $3PWh(v)$ as in [18], if any two games do not have three common players. It was Hartman who first discussed this property in [28]. If we regard games in a $3PWh(v)$ as blocks, we obtain a super-simple $(v, 4, 3)$-BIBD (we call it a subdesign of the $3PWh(v)$). This kind of design was introduced and studied by Gronau and Mullin [27] and also studied by Chen [14,15]. Such designs with resolvable property were investigated by Ge and Lam [23] and Zhang and Ge [38]. For the existence of $3PWh(v)$s, Finizio et al. [18–20] obtained several infinite classes and some examples. In [8], Anderson and Finizio gave an asymptotic result. Subsequently, a complete solution was obtained in papers by Lu and Zhang [30] and Ge and Lam [24]. More formally, we state their results in the following theorem.

Theorem 1.1 (*[30,24]*). *Necessary conditions for the existence of a* $3PWh(v)$*, are* $v \equiv 0, 1 \pmod 4$ *and* $v \geq 8$. *These conditions are also sufficient with one definite exception for* $v = 12$*.*

We may think of (a, b, c, d) as the cyclic order of the four players sitting round a table. We refer to the pairs $\{a, b\}$ and $\{c, d\}$ as pairs of *opponents of the first kind*, and the pairs $\{a, d\}$ and $\{b, c\}$ as pairs of *opponents of the second kind*. We also refer to b as the *left-hand opponent* of a and as the *right-hand opponent* of c, and similar definitions apply to each of a, b, c, d. A *directedwhist tournament* $DWh(v)$ is a $Wh(v)$ in which each player is a left- (resp., right-) hand opponent of every other player exactly once. A $DWh(v)$ is associated with what has been referred to as a *resolvable* $(v, 4, 1)$*-perfect Mendelsohn design* or briefly a $(v, 4, 1)$-RPMD (see, for example, [31, 12]). A basic necessary condition for the existence of a $DWh(v)$ is $v \equiv 0, 1 \pmod 4$. It is fairly well known [12] that a $DWh(v)$ exists for all $v \geq 5$ whenever $v \equiv 1 \pmod 4$. On the other hand, the results for the existence of a $DWh(v)$ whenever $v \equiv 0 \pmod 4$ are still not conclusive. It is known [36,37] that a $DWh(v)$ exists for all $v \geq 4$ whenever $v \equiv 0 \pmod 4$, except for $v = 4, 8, 12$ and with at most 27 possible exceptions of which the largest is 188. More specifically, we have the following

Theorem 1.2 (*[12,36,37]*). *Necessary conditions for the existence of a* $DWh(v)$ *are* $v \equiv 0, 1 \pmod 4$ *and* $v \geq 4$. *These conditions are also sufficient except for* $v = 4, 8, 12$ *and possibly for* $v \in \{16, 20, 24, 32, 36, 44, 48, 52, 56, 64, 68, 76, 84, 88, 92, 96, 104, 108, 116, 124, 132, 148, 152, 156, 172, 184, 188\}$.

For the existence of a $DWh(v)$ with the three person property, briefly denoted by $3PDWh(v)$, Finizio [18] was able to obtain several infinite classes and some examples where $v \equiv 1 \pmod 4$. Subsequently, for this case, a conclusive result was given by Bennett and Ge [11] and we now have the following theorem.

Theorem 1.3 (*[18,11]*). *There exists a* $3PDWh(v)$ *for all* $v > 5$*, where* $v \equiv 1 \pmod 4$.

A *triplewhist tournament* $TWh(v)$ is a $Wh(v)$ in which each player is an opponent of the first (resp., second) kind exactly once with every other player. The triplewhist tournament problem was first introduced by Moore [32] in 1896. For a long time there was no progress until Baker [9] proved in 1975 that a $TWh(v)$ exists for $v = 4, 8, 16, 24$ and for all large v, $v \equiv 1 (\mathrm{mod}\ 4)$ and $v \equiv 0, 4, 12 \pmod{16}$. In 1997, much progress was made by Lu and Zhu in [29]. They proved that the necessary condition for the existence of a $TWh(v)$, namely $v \equiv 0$ or $1 \pmod 4$, is also sufficient with 2 definite exceptions, namely $v = 5, 9$, as well as 15 possible exceptions, namely $v \in \{12, 56\} \cup \{13, 17, 45, 57, 65, 69, 77, 85, 93, 117, 129, 133, 153\}$. Subsequent improvements were made by Ge and Zhu in [26], Ge and Lam [25], and finally by Abel and Ge [5]. We summarize the known results for $TWh(v)$ in the following theorem.

Theorem 1.4 (*[5]*). *Necessary conditions for the existence of a* $TWh(v)$*, are* $v \equiv 0, 1 \pmod 4$ *and* $v \geq 4$. *These conditions are also sufficient except for* $v = 5, 9, 12, 13$ *and possibly for* $v = 17$.

The above theorem was recently extended to the case of $TWh(v)$s with the three person property (briefly denoted by $3PTWh(v)$) by Ge [22]. Concretely, we have the following theorem.

Theorem 1.5 (*[22]*). *The necessary conditions for the existence of a* $3PTWh(v)$*, namely,* $v \equiv 0, 1 \pmod 4$ *and* $v \geq 8$*, are also sufficient except for* $v = 9, 12, 13$ *and possibly for* $v = 17$.

Whist tournaments which are simultaneously both triplewhist and directedwhist are called *directed triplewhist tournaments* and denoted briefly by $DTWh(v)$. These were first investigated by Anderson and Finizio in [8]. In addition to the above, the following asymptotic result of Anderson and Finizio is contained in Theorem 4.1 of [8].

Theorem 1.6. *There exists a $3PDTWh(v)$ for all sufficiently large $v \equiv 1 \pmod 4$.*

In this paper, we shall investigate the problem of existence of $3PDTWh(v)$s for the case where $v \equiv 1$ (mod 4). From our earlier stated results, it is evident that a $3PDTWh(v)$ does not exist for $v = 4, 5, 8, 9, 12, 13$. In fact, to date, there are no known small examples of a $3PDTWh(v)$ where $v \equiv 0 \pmod 4$ and the general problem is far from being resolved. Our goal is to establish the existence of a $3PDTWh(v)$ for all $v > 17$, where $v \equiv 1 \pmod 4$ with just a few possible exceptions.

Another problem of current interest mentioned earlier in [19] relates to the existence of Z-cyclic whist tournaments. A $Wh(v)$ is said to be Z-cyclic if the players are the elements in Z_v when $v \equiv 1 \pmod 4$ and in $Z_{v-1} \cup \{\infty\}$ when $v \equiv 0 \pmod 4$, and the rounds of the tournament are arranged so that each round is obtained from the previous round by adding 1(mod m) where $m = v - 1$ if $v \equiv 0 \pmod 4$ and $m = v$ if $v \equiv 1 \pmod 4$. An interesting feature of a Z-cyclic whist tournament is that the entire tournament can be described by what is usually referred to as the initial round of the tournament. In the process of establishing our main results, we shall also provide a plethora of examples of Z-cyclic $3PDTWh(v)$'s. In passing, it is also worth mentioning the fact that our results provide triplewhist tournaments that are also resolvable Mendelsohn designs, and which give rise to a pair of self-orthogonal Latin squares with a common symmetric orthogonal mate [8]. For general information on whist tournaments see the survey paper of Anderson [6].

2. Recursive constructions

To describe our recursive constructions, we need the following auxiliary designs. For the general background on design theory, the reader is referred to [13].

Suppose that S is a set of players, and $\mathbf{H} = \{S_1, S_2, \dots, S_n\}$ is a set of subsets (called *holes*), which form a partition of S. Let $s_i = |S_i|$ and $s = |S|$. A *holey round* with hole S_i is a set of games (a, b, c, d) which partition the set $S \setminus S_i$. A *whist tournament frame with three person property* (briefly $3PWh$-frame) of type $\{s_1, s_2, \dots, s_n\}$ is a schedule of games (a, b, c, d), where the unordered pairs $\{a, c\}, \{b, d\}$ are called *partners*, pairs $\{a, b\}, \{c, d\}, \{a, d\}, \{b, c\}$ are called *opponents*, such that

(1) the games are arranged into s holey rounds; for each i there are s_i holey rounds with hole S_i, each containing $(s - s_i)/4$ games;
(2) each player in hole S_i plays in exactly one game in each of $s - s_i$ holey rounds;
(3) each player partners every other player in distinct holes exactly once;
(4) each player opposes every other player in distinct holes exactly twice;
(5) any two games have at most two players in common.

A $3PWh$-frame of type $\{s_1, s_2, \dots, s_n\}$ will be called a *directed triple whist tournament frame* of the same type, briefly $3PDTWh$-frame, if each player is a left- (resp., right-) hand opponent of every other player exactly once and simultaneously an opponent of the first (resp., second) kind exactly once with every other player.

We shall use an "exponential" notation to describe types: so type $t_1^{u_1} \cdots t_m^{u_m}$ denotes u_i occurrences of t_i, $1 \le i \le m$ in the multiset $\{s_1, s_2, \dots, s_n\}$. It is easy to see that a $3PDTWh$-frame(1^v) with $v \equiv 1$ (mod 4) is just a $3PDTWh(v)$.

A *pairwise balanced design* (PBD) is a pair $(X, \mathbf{A})$ such that X is a set of elements (called *points*), and $\mathbf{A}$ is a set of subsets (called *blocks*) of X, each of cardinality at least two, such that every unordered pair of points is contained in a unique block in $\mathbf{A}$. If v is a positive integer and K is a set of positive integers, each of which is not less than 2, then we say that $(X, \mathbf{A})$ is a (v, K)-PBD if $|X| = v$, and $|A| \in K$ for every $A \in \mathbf{A}$. The integer v is called the *order* of the PBD. Using this notation, we can define a BIBD $B(k, 1; v)$ to be a $(v, \{k\})$-PBD. We shall denote by $B(K)$ the set of all integers v for which there exists a (v, K)-PBD. For convenience, we define $B(k_1, k_2, \dots, k_r)$ to be the set of all integers v such that there is a $(v, \{k_1, k_2, \dots, k_r\})$-PBD.

A *group divisible design* (or GDD), is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:

1. $\mathcal{G}$ is a partition of a set X (of *points*) into subsets called *groups*;
2. $\mathcal{B}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point;
3. Every pair of points from distinct groups occurs in exactly λ blocks.

The *group type* (or *type*) of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. As with $3PWh$-frames, we shall use an "exponential" notation to describe group-type.

A GDD with block sizes from a positive integer set K is called a (K, λ)-GDD. When $K = \{k\}$, we simply write k for K. When $\lambda = 1$, we simply write K-GDD for a (K, λ)-GDD. A (k, λ)-GDD with group type 1^v is a balanced incomplete block design, denoted by (v, k, λ)-BIBD.

A GDD or a BIBD is said to be *resolvable* if its blocks can be partitioned into parallel classes each of which spans the set of points. We denote them by (K, λ)-RGDD or (v, k, λ)-RBIBD.

A *transversal design* (TD) TD(k, n) is a GDD of group type n^k and block size k. A resolvable TD(k, n) (denoted by RTD(k, n)) is equivalent to a TD$(k+1, n)$. It is well known that a TD(k, n) is equivalent to $k-2$ mutually orthogonal Latin squares (MOLS) of order n. In this paper, we mainly employ the following known results on TDs and PBDs.

Lemma 2.1 (*[4,2]*).

1. *An* RTD$(4, n)$ *exists for all* $n \geq 4$ *except for* $n = 6$ *and possibly for* $n = 10$.
2. *A* TD$(q + 1, q)$ *exists, where* q *is a prime power.*
3. *For all integers* v *where* $63 \leq v \leq 92$ *or* $v \geq 343$*, there exists a* $(v, \{7, 8, 9\})$*-PBD.*

Wilson's fundamental construction on $GDDs$ [34] can be adapted to obtain the following construction for $3PDTWh$-frames.

Construction 2.2 (*Weighting*). *Let* $(X, \mathcal{G}, \mathcal{B})$ *be a GDD with index unity, and let* $w : X \to Z^+ \cup \{0\}$ *be a weight function on* X*. Suppose that for each block* $B \in \mathcal{B}$*, there exists a* $3PDTWh$*-frame of type* $\{w(x) : x \in B\}$*. Then there is a* $3PDTWh$*-frame of type* $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.

To obtain our main results, we shall use the following basic recursive constructions, which are modifications of constructions for RGDDs and RBIBDs. Proofs for these can be found in [21]. Here, we just need to do the routine check for the three person property.

Construction 2.3 (*Inflating $3PDTWh$-frames by RTDs*). *If a* $3PDTWh$*-frame of type* h^u *and an* RTD$(4,\ m)$ *both exist, then there exists a* $3PDTWh$*-frame of type* $(mh)^u$.

Construction 2.4 (*Frame Constructions*). *Suppose that there is a* $3PDTWh$*-frame with type* $T = \{t_i : i = 1, 2, \ldots, n\}$*. Suppose also that there exists a* $3PDTWh(1 + t_i)$ *for* $i = 1, 2, \ldots, n$*. Then there exists a* $3PDTWh(u)$ *where* $u = 1 + \sum_{i=1}^{n} t_i$.

Construction 2.5 (*Generalized Frame Constructions*). *Suppose that there is a* $3PDTWh$*-frame with type* $T = \{t_i : i = 1, 2, \ldots, n\}$*. Let* $b > 0$*. If there exists a* $3PDTWh$*-frame of type* $1^{t_i} b^1$ *for* $i = 1, 2, \ldots, n - 1$*, then there exists a* $3PDTWh$*-frame of type* $1^{u-t_n} (t_n + b)^1$ *where* $u = \sum_{i=1}^{n} t_i$*. Furthermore, if a* $3PDTWh(t_n + b)$ *exists, then a* $3PDTWh(u + b)$ *exists.*

3. Direct constructions

The constructions used in this paper will combine both direct and recursive methods. For most of our direct constructions, we adapt the familiar difference method, where a finite abelian group is used to generate the set of blocks for a given design. That is, instead of listing all the blocks of the design, we shall list a set of base blocks and generate the others by an additive group and perhaps some further automorphisms.

Lemma 3.1. *A* $3PDTWh(v)$ *exists for each* $v \in \{25, 125\}$.

Proof. For $v = 25$, the required design is over $Z_5 \times Z_5$. The initial round of games is given by the following base blocks:

$$((4,4),(0,1),(0,2),(4,2)),\quad ((2,1),(2,4),(3,2),(4,1)),\quad ((1,4),(3,3),(1,0),(3,0)),$$
$$((3,1),(2,3),(0,4),(4,3)),\quad ((2,0),(0,3),(1,3),(3,4)),\quad ((1,2),(1,1),(2,2),(4,0)).$$

For $v = 125$, the required design is over GF(125). Let x be a primitive element of GF(125) satisfying $x^3 = x^2 + 2$. The initial round of games is obtained by multiplying the block $(x, x+1, x^2+2x, 3x^2+3x)$ by x^{4t} for $0 \leq t \leq 30$. $\square$

Lemma 3.2. *A Z-cyclic* $3PDTWh(v)$ *exists for each* $v \in \{33, 45, 57, 65, 69, 77, 81, 85, 93, 105\}$.

Proof. The following table displays a suitable initial round of games for all the given values of v:

v	Initial round games				
33	(3, 14, 23, 20),	(27, 26, 29, 21),	(11, 31, 32, 13),	(28, 22, 4, 6),	(12, 8, 16, 7),
	(10, 5, 17, 24),	(9, 30, 1, 19),	(15, 25, 18, 2).		
45	(20, 15, 14, 11),	(18, 12, 35, 1),	(5, 38, 19, 2),	(44, 42, 26, 13),	(16, 34, 39, 4),
	(33, 29, 31, 8),	(40, 3, 30, 6),	(41, 32, 22, 37),	(36, 10, 23, 43),	(17, 24, 9, 25),
	(7, 21, 27, 28).				
57	(20, 3, 8, 6),	(10, 52, 26, 46),	(1, 34, 51, 47),	(13, 42, 41, 33),	(55, 7, 9, 28),
	(50, 11, 23, 16),	(2, 5, 21, 27),	(45, 32, 22, 17),	(18, 54, 36, 14),	(56, 44, 25, 48),
	(53, 39, 4, 29),	(35, 24, 37, 38),	(31, 15, 30, 40),	(43, 12, 19, 49).	
65	(1, 29, 50, 12),	(11, 23, 38, 47),	(54, 2, 25, 58),	(53, 46, 56, 16),	(61, 18, 6, 5),
	(48, 24, 63, 42),	(19, 55, 7, 49),	(26, 45, 4, 59),	(33, 27, 14, 64),	(60, 43, 21, 3),
	(28, 36, 20, 31),	(62, 13, 51, 17),	(37, 40, 35, 39),	(44, 9, 10, 30),	(32, 34, 52, 57),
	(15, 41, 22, 8).				
69	(1, 46, 44, 56),	(11, 13, 9, 4),	(29, 19, 49, 35),	(15, 40, 18, 53),	(37, 65, 45, 58),
	(48, 67, 8, 16),	(64, 38, 30, 23),	(36, 5, 32, 55),	(43, 52, 7, 10),	(60, 28, 3, 42),
	(68, 20, 31, 14),	(17, 59, 6, 12),	(27, 63, 50, 39),	(57, 61, 26, 66),	(47, 62, 22, 21),
	(34, 54, 51, 33),	(2, 24, 41, 25).			
77	(1, 31, 35, 32),	(2, 69, 75, 30),	(6, 27, 39, 67),	(23, 48, 43, 41),	(17, 26, 66, 15),
	(61, 3, 14, 22),	(7, 42, 76, 64),	(16, 33, 19, 24),	(20, 51, 34, 10),	(13, 50, 37, 38),
	(55, 70, 49, 8),	(28, 12, 63, 57),	(52, 18, 68, 45),	(71, 60, 25, 47),	(44, 5, 46, 59),
	(11, 40, 21, 65),	(56, 74, 73, 53),	(72, 9, 54, 4),	(36, 29, 62, 58).	
81	(1, 54, 65, 31),	(2, 45, 71, 55),	(3, 30, 36, 10),	(7, 27, 18, 12),	(44, 29, 17, 58),
	(50, 51, 26, 5),	(39, 78, 37, 40),	(49, 20, 74, 16),	(11, 33, 48, 73),	(38, 69, 56, 60),
	(75, 6, 22, 57),	(59, 72, 80, 13),	(4, 67, 23, 53),	(46, 35, 15, 77),	(76, 68, 8, 52),
	(79, 47, 24, 41),	(61, 28, 62, 64),	(9, 14, 43, 19),	(63, 70, 66, 21),	(25, 34, 32, 42).
85	(1, 5, 70, 67),	(28, 7, 42, 11),	(12, 74, 62, 54),	(38, 10, 35, 68),	(2, 17, 4, 84),
	(8, 52, 23, 9),	(27, 59, 75, 20),	(6, 53, 31, 77),	(50, 61, 14, 83),	(21, 41, 65, 48),
	(34, 79, 47, 49),	(78, 71, 22, 82),	(3, 32, 80, 37),	(36, 58, 81, 30),	(60, 72, 26, 39),
	(63, 64, 73, 55),	(18, 45, 19, 13),	(57, 33, 51, 16),	(24, 15, 43, 69),	(56, 46, 44, 25),
	(29, 66, 76, 40).				
93	(58, 21, 74, 65),	(70, 22, 39, 12),	(60, 78, 56, 31),	(63, 48, 91, 49),	(4, 17, 29, 10),
	(32, 20, 23, 86),	(66, 27, 14, 35),	(64, 50, 6, 38),	(7, 90, 30, 54),	(88, 42, 1, 68),
	(82, 84, 18, 52),	(11, 61, 80, 79),	(43, 40, 41, 45),	(8, 81, 47, 36),	(62, 57, 73, 24),
	(55, 83, 5, 46),	(77, 69, 37, 72),	(26, 33, 9, 71),	(16, 76, 87, 34),	(25, 2, 44, 15),
	(53, 59, 67, 89),	(51, 13, 85, 28),	(92, 75, 19, 3).		
105	(45, 74, 18, 104),	(35, 90, 20, 79),	(31, 78, 71, 28),	(16, 61, 52, 22),	(51, 40, 17, 41),
	(65, 2, 33, 86),	(34, 38, 24, 63),	(32, 100, 88, 87),	(42, 15, 37, 12),	(30, 101, 81, 4),
	(73, 25, 77, 72),	(66, 53, 14, 55),	(44, 67, 50, 48),	(89, 5, 56, 46),	(62, 27, 85, 47),
	(76, 11, 83, 68),	(57, 64, 102, 99),	(13, 69, 96, 8),	(97, 36, 54, 60),	(19, 93, 95, 6),
	(98, 7, 39, 21),	(91, 59, 23, 43),	(103, 10, 75, 84),	(92, 70, 29, 3),	(49, 82, 58, 94),
	(9, 1, 26, 80). □				

Lemma 3.3. *A Z-cyclic* $3PDTWh(v)$ *exists for each* $v \in \{49, 97, 121, 133, 169, 193\}$.

Proof. These are obtained like the designs in the previous lemma, except that a multiplier of order 3 or 5 is used. For $v = 121$, the initial round is obtained by multiplying the blocks below by 3^i for $0 \le i \le 4$. For the other values, we give a multiplier w of order 3; the initial round is then obtained by multiplying the given blocks by w^i for $0 \le i \le 2$. □

v	w	Initial games			
49	18	(10, 14, 1, 32),	(4, 12, 15, 35),	(31, 44, 38, 24),	(16, 41, 26, 36).
97	35	(19, 73, 43, 94),	(29, 60, 95, 9),	(67, 66, 70, 44),	(51, 18, 52, 62),
		(20, 47, 37, 5),	(1, 53, 55, 15),	(84, 90, 59, 38),	(75, 8, 41, 22).
121		(109, 59, 118, 12),	(1, 78, 107, 32),	(93, 25, 50, 58),	(7, 23, 11, 52),
		(103, 18, 62, 102),	(8, 45, 66, 10).		
133	11	(57, 103, 115, 66),	(12, 51, 86, 128),	(3, 96, 24, 77),	(102, 36, 93, 107),
		(47, 56, 79, 20),	(119, 129, 37, 42),	(10, 121, 70, 52),	(34, 67, 21, 117),
		(76, 120, 9, 80),	(27, 85, 55, 74),	(17, 2, 39, 50).	
169	22	(23, 63, 62, 59),	(82, 121, 96, 41),	(3, 36, 50, 45),	(11, 54, 8, 128),
		(130, 139, 21, 67),	(48, 55, 75, 140),	(46, 76, 91, 15),	(152, 4, 32, 69),
		(9, 105, 40, 30),	(102, 74, 150, 97),	(25, 117, 108, 6),	(119, 56, 163, 92),
		(141, 61, 26, 68),	(22, 44, 93, 19).		
193	84	(44, 73, 113, 110),	(128, 4, 142, 111),	(5, 68, 52, 47),	(12, 69, 9, 132),
		(2, 17, 41, 150),	(25, 39, 40, 27),	(70, 100, 108, 3),	(15, 36, 172, 80),
		(63, 124, 98, 114),	(82, 65, 180, 144),	(127, 38, 11, 179),	(95, 117, 103, 183),
		(109, 174, 135, 190),	(16, 20, 14, 92),	(165, 24, 57, 176),	(42, 139, 189, 106).

Lemma 3.4. *There exists a Z-cyclic* $3PDTWh(v)$ *for* v *prime,* $v \equiv 1 \pmod 4$ *and* $29 \le v \le 241$.

Proof. For $v = 97$ and 193, see Lemma 3.3. For the other values of v, let x be any primitive element in GF(v). For $v \equiv 5 \pmod 8$ the initial round is obtained by multiplying one initial block by x^{4t} for $0 \le t < (v-1)/4$:

v	Initial block	v	Initial block	v	Initial block	v	Initial block
29	(1, 3, 13, 8)	37	(1, 2, 4, 17)	53	(1, 2, 11, 34)	61	(1, 2, 4, 10)
101	(1, 2, 4, 98)	109	(1, 2, 8, 64)	149	(1, 2, 4, 18)	157	(1, 2, 4, 116)
173	(1, 2, 4, 11)	181	(1, 2, 12, 63)	197	(1, 2, 6, 18)	229	(1, 2, 4, 145)

For $v \equiv 9 \pmod{16}$ the initial round is obtained by multiplying two initial blocks by x^{8t} for $0 \le t < (v-1)/8$:

v	Initial blocks	v	Initial blocks	v	Initial blocks	v	Initial blocks
41	(1, 2, 4, 17)	73	(1, 3, 9, 14)	89	(1, 3, 9, 22)	137	(1, 2, 4, 17)
	(3, 12, 5, 22)		(11, 63, 31, 25)		(5, 54, 41, 13)		(3, 47, 89, 116)
233	(1, 3, 9, 14)						
	(5, 35, 84, 159)						

For $v \equiv 17 \pmod{32}$ the initial round is obtained by multiplying four initial blocks by x^{16t} for $0 \le t < (v-1)/16$:

v	Initial blocks		v	Initial blocks	
113	(1, 2, 4, 10)	(5, 58, 81, 94)	241	(1, 2, 4, 7)	(12, 86, 130, 185)
	(3, 6, 13, 23)	(9, 59, 100, 63)		(3, 8, 19, 25)	(10, 17, 139, 202)

□

Summarizing the results of Lemmas 3.1–3.4, we have the following lemma:

Lemma 3.5. *There exists a* $3PDTWh(v)$ *for all* $v \equiv 1 \pmod 4$ *where* $25 \le v \le 241$*, with the possible exceptions of* $v \in \{117, 129, 141, 145, 153, 161, 165, 177, 185, 189, 201, 205, 209, 213, 217, 221, 225, 237\}$.

Lemma 3.6. *A* $3PDTWh$*-frame of type* 4^n *exists for each* $n \in \{7, 8, 9, 10, 11\}$.

Proof. These designs are over $Z_{4n-4} \cup \{\infty_0, \infty_1, \infty_2, \infty_3\}$. All points in the first block are distinct (mod 4); therefore for any $i \in \{0, 1, 2, 3\}$, adding $i, i+4, i+8, \ldots, i+4n-8$ to this block produces a partial parallel class missing the infinite points. The other blocks form a partial parallel class missing the group $\{0, n-1, 2(n-1), 3(n-1)\}$. Develop all the given blocks (mod $4n-4$). □

n	Initial games				
7	(0, 1, 3, 10),	(1, 4, 17, 8),	(7, 5, 2, 22),	(3, 11, 16, ∞_0),	(∞_1, 21, 13, 23),
	(9, 20, 19, ∞_2),	(15, 10, 14, ∞_3).			
8	(0, 10, 13, 11),	(20, 26, 16, 15),	(27, 24, 8, 19),	(6, 22, 9, 4),	(25, 1, 23, ∞_0),
	(18, 3, 12, ∞_1),	(∞_2, 17, 13, 5),	(∞_3, 10, 11, 2).		
9	(0, 14, 13, 7),	(28, 17, 5, 6),	(25, 12, 23, 27),	(19, 2, 15, 22),	(10, 20, 11, 14),
	(29, 31, 26, ∞_0),	(21, 1, 7, ∞_1),	(9, 18, 4, ∞_2),	(∞_3, 13, 30, 3).	
10	(0, 10, 29, 31),	(22, 19, 23, 11),	(14, 34, 26, 21),	(15, 16, 29, 35),	(4, 12, 24, 10),
	(6, 2, 1, 8),	(7, 30, 33, ∞_0),	(20, 5, 31, ∞_1),	(∞_2, 17, 32, 13),	(∞_3, 25, 3, 28).
11	(0, 17, 31, 34),	(12, 39, 1, 5),	(17, 9, 21, 22),	(26, 15, 33, 27),	(3, 36, 11, 35),
	(37, 6, 2, 24),	(34, 32, 13, 18),	(29, 8, 31, ∞_0),	(∞_1, 7, 38, 23),	(∞_2, 19, 16, 4),
	(28, 14, 25, ∞_3).				

Lemma 3.7. *A 3PDTWh-frame of type* $4^n 8^1$ *exists for each* $n \in \{7, 8, 9, 10\}$.

Proof. These designs are over $Z_{4n} \cup \{\infty_0, \infty_1, \infty_2, \ldots, \infty_7\}$ and are obtained like those in the previous lemma. All points in each of the first two blocks are distinct (mod 4); therefore each of these blocks generates four partial parallel classes missing the infinite points. The other blocks form a partial parallel class missing the group $\{0, n, 2n, 3n\}$. □

n	Initial games				
7	(0, 22, 11, 9),	(0, 9, 10, 15),	(24, 6, 8, ∞_0),	(∞_1, 19, 3, 23),	(26, 9, 27, ∞_2),
	(2, 17, 25, ∞_3),	(13, 10, 5, ∞_4),	(∞_5, 16, 22, 18),	(12, 11, 15, ∞_6),	(∞_7, 1, 4, 20).
8	(0, 26, 7, 17),	(0, 25, 14, 31),	(21, 18, 6, 17),	(31, 13, 11, ∞_0),	(∞_1, 5, 27, 7),
	(15, 10, 28, ∞_2),	(23, 25, 2, ∞_3),	(1, 29, 4, ∞_4),	(∞_5, 14, 20, 19),	(22, 9, 12, ∞_6),
	(∞_7, 30, 3, 26).				
9	(0, 26, 11, 33),	(0, 23, 17, 22),	(1, 34, 22, 30),	(2, 31, 12, 28),	(29, 8, 6, ∞_0),
	(∞_1, 7, 35, 5),	(15, 19, 20, ∞_2),	(∞_3, 17, 13, 25),	(32, 21, 16, ∞_4),	(∞_5, 4, 24, 26),
	(11, 10, 23, ∞_6),	(∞_7, 3, 14, 33).			
10	(0, 9, 15, 38),	(0, 15, 6, 1),	(24, 28, 1, 12),	(37, 35, 2, 16),	(17, 11, 8, 9),
	(13, 29, 14, ∞_0),	(∞_1, 31, 15, 18),	(4, 36, 32, ∞_2),	(∞_3, 25, 3, 22),	(39, 21, 7, ∞_4),
	(∞_5, 33, 38, 26),	(19, 6, 23, ∞_6),	(∞_7, 5, 34, 27).		

Lemma 3.8. *A 3PDTWh-frame of type* $g^u m^1$ *exists for all* $(g, u, m) \in \{(6, 9, 0), (8, 5, 0), (8, 5, 4), (8, 5, 8), (24, 7, 36)\}$.

Proof. For type 6^9, the given design is over Z_{54}. Develop (mod 54) the following blocks, which form a partial parallel class missing the group $\{0, 9, 18, \ldots, 45\}$:

(2, 49, 19, 50),	(25, 33, 53, 20),	(7, 3, 17, 51),	(38, 13, 16, 46),
(26, 23, 30, 15),	(28, 12, 35, 24),	(48, 42, 10, 47),	(52, 40, 21, 11),
(41, 6, 22, 8),	(4, 32, 44, 43),	(31, 29, 1, 14),	(5, 37, 39, 34).

For type 8^5, the required design is over $(Z_4 \times Z_8) \cup \{\infty_0, \infty_1, \ldots, \infty_7\}$ and the groups on the non-infinite points are $Z_4 \times \{i, i+4\}$ for $0 \le i \le 7$. The second coordinates of the points in each of the first two blocks are all distinct (mod 4); hence these blocks each generate 4 partial parallel classes missing the infinite points. The others form a partial parallel class missing the group $Z_4 \times \{0, 4\}$.

((2, 0), (0, 1), (0, 3), (3, 2)),	((3, 0), (3, 6), (3, 7), (2, 1)),	((3, 5), (1, 2), (2, 3), ∞_0),
(∞_1, (1, 7), (3, 1), (2, 2)),	((2, 6), (2, 1), (3, 7), ∞_2),	(∞_3, (0, 6), (1, 3), (2, 5)),
(∞_4, (3, 6), (1, 5), (3, 3)),	((0, 7), (3, 2), (0, 5), ∞_5),	(∞_6, (2, 7), (0, 2), (0, 1)),
(∞_7, (1, 1), (1, 6), (0, 3)).		

For types 8^5m^1, $m = 4, 8$, the given designs are over $Z_{40} \cup \{\infty_0, \infty_1, \ldots, \infty_{m-1}\}$. Develop the blocks below (mod 40). The partial parallel classes missing the infinite points here are obtained by adding $i, i+4, \ldots, i+36$ to the first block (for $0 \le i \le 3$) when $m = 4$, or by adding $i, i+8, \ldots, i+32$ to the first two blocks (for $0 \le i \le 7$) when $m = 8$. The other base blocks form a partial parallel class missing the group $\{0, 5, 10, \ldots, 35\}$.

m	Initial games				
4	(0, 26, 29, 23),	(22, 19, 23, 1),	(38, 34, 2, 11),	(33, 16, 7, 9),	(18, 6, 12, 4),
	(8, 24, 17, 36),	(21, 28, 29, ∞_0),	(32, 31, 13, ∞_1),	(∞_2, 3, 14, 27),	(∞_3, 39, 37, 26).
8	(0, 12, 29, 33),	(19, 22, 10, 23),	(3, 29, 7, 16),	(33, 12, 26, 34),	(∞_0, 17, 9, 11),
	(∞_1, 4, 1, 2),	(∞_2, 32, 13, 24),	(∞_3, 14, 38, 31),	(36, 19, 8, ∞_4),	(23, 39, 37, ∞_5),
	(6, 28, 22, ∞_6),	(21, 27, 18, ∞_7).			

For type $24^7 36^1$, the given design is over $Z_{168} \cup \{\infty_0, \infty_1, \ldots, \infty_{35}\}$. Multiply the following blocks by 1, 25 and $25^2 = 121$, then develop the resulting 54 blocks (mod 168). Also, for $i = 0, 1, 2, \ldots, 11$, replace ∞_i by ∞_{12+i} and ∞_{24+i} when multiplying a block by 25 and 25^2 respectively. The 36 partial parallel classes missing the infinite points are obtained by adding $i, i+4, \ldots, i+164$ (for $0 \le i \le 3$) to each of the first three blocks and their multiples. The other base blocks and their multiples form a partial parallel class missing the group $\{0, 7, 14, \ldots, 161\}$.

(0, 71, 150, 69),	(0, 13, 163, 26),	(0, 114, 27, 29),	(44, 59, 144, 110),	(128, 5, 106, 94),
(4, 80, 138, 36),	(∞_0, 67, 142, 12),	(∞_1, 68, 29, 2),	(∞_2, 57, 151, 24),	(∞_3, 26, 123, 3),
(∞_4, 87, 71, 25),	(∞_5, 10, 158, 19),	(45, 61, 37, ∞_6),	(73, 141, 89, ∞_7),	(55, 114, 54, ∞_8),
(33, 65, 52, ∞_9),	(16, 15, 40, ∞_{10}),	(155, 30, 143, ∞_{11}). □		

4. Recursive constructions: The case $v \le 253$

In Section 3, some direct constructions were given for $3PDTWh(v)$ where $25 \le v \le 241$, with some possible exceptions. In this section, we shall provide recursive constructions for some of these possible exceptions. Throughout the rest of this paper, we shall let V denote the set $\{v$: a $3PDTWh(v)$ exists$\}$.

Lemma 4.1. $\{161, 201, 205, 217, 221, 225, 237, 245, 249, 253\} \subset V$.

Proof. The first four of these make use of frames obtained in Lemma 3.8. For $v = 161$, 201 and 217, we apply Construction 2.3 inflating $3PDTWh$-frames of types 8^5, 8^5 and 6^9 by 4, 5 and 4 respectively. This gives $3PDTWh$-frames of types 32^5, 40^5 and 24^9. Now adjoin an infinite point to these frames and fill in the groups of sizes 32, 40 and 24 with a $3PDTWh(t)$ for $t = 33, 41$ or 25. For 205, we adjoin an infinite point to a 3PDTWh-frame of type $24^7\ 36^1$, and fill in the groups with a $3PDTWh(25)$ or a $3PDTWh(37)$. For $221 \le v \le 253$, we start with a TD(8, 7) and apply Construction 2.2. In the first seven groups, we give all points a weight of four. In the last group, we give the points a weight of zero, 4 or 8 so that we have a total weight of $4t$ where $6 \le t \le 14$. Here $3PDTWh$-frames of type 4^n for $n \in \{7, 8\}$ are needed, as well as the $3PDTWh$-frame of type $4^7 8^1$. These all exist by Lemmas 3.6 and 3.7. We then adjoin one infinite point to the resulting $3PDTWh$-frame of type $28^7(4t)^1$ by using a $3PDTWh(29)$ or $3PDTWh(4t+1)$ for $6 \le t \le 14$ to fill in the holes. This completes the proof. □

Summarizing the foregoing, we have now proved the following:

Lemma 4.2. *If $v \equiv 1 \pmod 4$ and $25 \le v \le 253$, then a $3PDTWh(v)$ exists, except possibly for $v \in \{117, 129, 141, 145, 153, 165, 177, 185, 189, 209, 213\}$.*

5. Recursive constructions: The case $v \ge 253$

All designs in this section are obtained using the $3PDTWh$-frames of type 4^n for $n \in \{7, 8, 9, 10, 11\}$ given in Lemma 3.6. First, we will need the following working lemma.

Lemma 5.1. *Suppose that $1 \le x \le 4$ and a TD $(7 + x, m)$ exists. Suppose also that there exists $3PDTWh(4t + 1)$ for $t = m, a_1, a_2, a_3, \ldots, a_x$ where $0 \le a_i \le m$ for $1 \le i \le x$. If $v = 28m + 4(a_1 + a_2 + \cdots + a_x) + 1$, then there exists a $3PDTWh(v)$.*

Proof. Truncate x groups in TD$(7+x, m)$ to sizes a_i, $1 \le i \le x$, to obtain a $\{7, 8, \ldots, 7+x\}$-GDD of type $m^7 a_1{}^1 a_2{}^1 \cdots a_x{}^1$. Applying Construction 2.2 with weight 4, we then adjoin one infinite point to the resulting $3PDTWh$-frame by using $3PDTWh(4t+1)$ for $t = m, a_1, \ldots, a_x$ to fill in the holes. This produces the desired designs. Here $3PDTWh$-frames of type 4^n for $n \in \{7, 8, \ldots, 7+x\}$ are needed as input designs; these all exist by Lemma 3.6. The proof is complete. □

Lemma 5.2. *If* $v \equiv 1 \pmod 4$ *and* $249 \le v \le 365$ *or* $v \ge 1369$*, then* $v \in V$.

Proof. Lemma 2.1 guarantees the existence of an $(n+1, \{7, 8, 9\})$-PBD for all n where $62 \le n \le 91$ or $n \ge 342$. By deleting one point from this PBD, we create a $\{7, 8, 9\}$-GDD of order n and of type $6^{u_1} 7^{u_2} 8^{u_3}$, where the exponents are non-negative integers not all equal to zero. Next we give all points of this GDD weight 4 and apply Construction 2.2 and the results of Lemma 3.6 to produce a $3PDTWh$-frame of order $4n$ and of type $24^{u_1} 28^{u_2} 32^{u_3}$. Now we have a $3PDTWh(t)$ for $t = 25, 29, 33$, as guaranteed by Lemma 3.5. So we apply Construction 2.4 to adjoin an infinite point and fill in the holes of the resulting $3PDTWh$-frame to obtain the desired $3PDTWh(v)$ for the stated values of $v \equiv 1 \pmod 4$. □

Lemma 5.3. *If* $v \equiv 1 \pmod 4$ *and* $333 \le v \le 1485$*, then* $v \in V$.

Proof. Apply Lemma 5.1 with the values of m and x shown below, $a_i = 0$ or $6 \le a_i \le m$, with $a_i \le 27$ if $m \ge 27$. This gives $v \in V$, for v within the intervals given. Here a $3PDTWh(4t+1)$ is required for $t \in \{m, 6, 7, \ldots, 27\}$ to fill in the holes and all of these exist by Lemma 4.2.

Range for v		
[333, 485]:	$m = 11$,	$x = 4$
[473, 705]:	$m = 16$,	$x = 4$
[669, 1013]:	$m = 23$,	$x = 4$
[781, 1189]:	$m = 27$,	$x = 4$
[1061, 1485]:	$m = 37$,	$x = 4$. □

Combining the results of Lemmas 4.2, 5.2 and 5.3, we have now established the following theorem.

Theorem 5.4. *There exists a* $3PDTWh(v)$ *for all* $v \ge 25$*, where* $v \equiv 1 \pmod 4$*, with the possible exceptions of* $v \in \{117, 129, 141, 145, 153, 165, 177, 185, 189, 209, 213\}$.

6. Directed triplewhist tournaments and SOLSSOMs

A *quasigroup* is an ordered pair $(Q, \cdot)$, where Q is a set and $(\cdot)$ is a binary operation on Q such that the equations

$$a \cdot x = b \quad \text{and} \quad y \cdot a = b \tag{1}$$

are uniquely solvable for every pair of elements $a, b \in Q$. A quasigroup is called *idempotent* if the identity

$$x \cdot x = x \tag{2}$$

is satisfied for all $x \in Q$. If the identity

$$(x \cdot y) \cdot (y \cdot x) = x \tag{3}$$

holds for all $x, y \in Q$, then it is called a *Schröder quasigroup*. If the identity

$$(x \cdot y) \cdot (y \cdot x) = y \tag{4}$$

holds for all $x, y \in Q$, then the quasigroup is said to satisfy *Stein's third law*.

For a finite set Q, it is well known that the multiplication table of the quasigroup defines a Latin square; that is, a Latin square can be viewed as the multiplication table of the quasigroup with the headline and sideline removed. The *order* of the quasigroup is $|Q|$. Two quasigroups of the same order are *orthogonal* if when the two corresponding Latin squares are superposed, each symbol in the first square meets each symbol in the second square exactly once. A quasigroup (Latin square) is called *self-orthogonal* if it is orthogonal to its transpose. For more information on Latin squares, the interested reader may refer to the book of Dénes and Keedwell [16].

We define a t-SOLSSOM of order v (t-SOLSSOM(v)) to be a set of $2t+1$ mutually orthogonal Latin squares $A_1, A_2, \ldots, A_t$, $B_1, B_2, \ldots, B_t, C$ such that $A_i = B_i^T$ and $C = C^T$. Here SOLSSOM stands for self-orthogonal Latin squares with a symmetric orthogonal mate. When $t = 1$, the term SOLSSOM (rather than 1-SOLSSOM) is more commonly used. The existence problem for SOLSSOM(v) has been investigated for many years and the solution now is almost complete. More specifically we have the following result [3]:

Theorem 6.1. *If v is a positive integer, then a* SOLSSOM(v) *exists, except for $v \in \{2, 3, 6\}$ and possibly for $v \in \{10, 14\}$.*

More generally, we also have the following result on 2-SOLSSOMs from [1].

Theorem 6.2. *A 2-*SOLSSOM(v) *exists for all $v \geq 701$, with at most 183 possible exceptions below this value. Further, if $v \equiv 0 \pmod 8$, then a 2-SOLSSOM (v) exists except possibly for $v \in \{24, 40, 48\}$, and if $v \equiv 1 \pmod 2$, then a 2-SOLSSOM (v) exists except for $v = 3, 5$ and possibly for $v \in \{15, 21, 33, 35, 39, 51, 65, 87, 123, 135\}$.*

Schröder quasigroups and quasigroups satisfying Stein's third law are well known to be self-orthogonal (see, for example, [33]). Moreover, it is also known that the existence of a $DWh(v)$ implies the existence of an idempotent quasigroup satisfying Stein's third law, and which has a symmetric orthogonal mate, that is, a SOLSSOM(v). Similarly, the existence of a $TWh(v)$ implies the existence of an idempotent Schröder quasigroup of order v with a symmetric orthogonal mate, that is, a different SOLSSOM(v). Evidently, the existence of a DTWh(v) implies the existence of two different SOLSSOMs sharing the same symmetric mate. For more information relating to these interesting associations, the reader is referred to [8,10,12]. One immediate consequence of Theorem 5.4 is the following result:

Theorem 6.3. *For all $v \geq 25$, where $v \equiv 1 \pmod 4$, with the possible exceptions of $v \in \{105, 117, 129, 141, 145, 153, 165, 177, 185, 189, 205, 209, 213\}$, there exists a $DTWh(v)$, which implies the existence of two different SOLSSOM (v) with the same mate.*

We wish to remark that Theorem 6.3 should be viewed more as an interesting observation, as was done in [8], rather than providing any significant improvements to Theorems 6.1 and 6.2. Clearly, the 2 SOLS(v) with the same symmetric orthogonal mate (SOM) arising from a DTWh(v) cannot provide a 2-SOLSSOM(v), since approximately half of their entries are identical. If one wishes to combine the results of Theorems 6.2 and 6.3, we obviously have the following:

Theorem 6.4. *For all $v \geq 5$, where $v \equiv 1 \pmod 4$, with the exception of $v = 5$ and the possible exception of $v = 21$, there exist two different SOLSSOM (v) with the same mate.*

7. Concluding remarks

As already mentioned in the Introduction, the results for the existence of a $DWh(v)$ whenever $v \equiv 0 \pmod 4$ are still not conclusive, and the existence of a $3PDWh(v)$ is an even more open problem for this case. It is fairly well known [35] that a Z-cyclic $(v, 4, 1)$-perfect Mendelsohn design does not exist whenever $v \equiv 0 \pmod 4$. Consequently, a Z-cyclic $(v, 4, 1)$-RPMD or equivalently a Z-cyclic $DWh(v)$ does not exist whenever $v \equiv 0 \pmod 4$. So it would seem natural to consider the existence of Z-cyclic $3PDWh(v)$s for $v \equiv 1 \pmod 4$. The results in [8] essentially provided an infinite class of Z-cyclic $DTWh(v)$ with $v = p$, a prime, $p \geq 29$ and $p \equiv 5 \pmod 8$. It was also shown in [8] that there is a Z-cyclic $3PDTWh(p)$ for $p = 29, 37$. In [17], it was shown that a Z-cyclic $3PTWh(p)$ exists for any prime $p \equiv 1 \pmod 4$, with the exception of $p = 5, 13, 17$. In Lemma 3.4 of this paper, it was shown that there exists a Z-cyclic $3PDTWh(p)$ with p a prime, $29 \leq p \leq 241$ and $p \equiv 1 \pmod 4$. It is conceivable that a Z-cyclic $3PDTWh(p)$ exists for any prime $p \equiv 1 \pmod 4$, with the exception of $p = 5, 13, 17$. However, there appears to be no easy way of obtaining a Z-cyclic $DTWh(p)$ for p a prime of the form $2^m + 1$, since there is then no multiplier of odd order in Z_p that can be used. The most recent results and progress on this problem can be found in [39].

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References

[1] R.J.R. Abel, F.E. Bennett, The existence of 2-SOLSSOMs, in: W.D. Wallis (Ed.), Designs 2002: Further Computational and Constructive Design Theory, Kluwer Academic Publishers, 2003, pp. 1–21.
[2] R.J.R. Abel, F.E. Bennett, M. Greig, PBD-Closure, in: C.J. Colbourn, J.H. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, 2nd ed., CRC Press, Boca Raton, FL, 2006, pp. 247–255.
[3] R.J.R. Abel, F.E. Bennett, H. Zhang, L. Zhu, A few more self-orthogonal Latin squares and related designs, Australas J. Combin. 21 (2000) 85–94.
[4] R.J.R. Abel, C.J. Colbourn, J.H. Dinitz, Mutually Orthogonal Latin Squares (MOLS), in: C.J. Colbourn, J.H. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, 2nd ed, CRC Press, Boca Raton, FL, 2006, pp. 160–193.
[5] R.J.R. Abel, G. Ge, Some difference matrix constructions and an almost completion for the existence of triplewhist tournaments TWh(v), European J. Combin. 26 (2005) 1094–1104.
[6] I. Anderson, A hundred years of whist tournaments, J. Combin. Math. Combin. Comput. 19 (1995) 129–150.
[7] I. Anderson, Combinatorial Designs and Tournaments, Oxford University Press, 1997.
[8] I. Anderson, N.J. Finizio, Triplewhist tournaments that are also Mendelsohn designs, J. Combin. Des. 5 (1997) 397–406.
[9] R.D. Baker, Factorization of graphs, Doctoral Thesis, Ohio State University, 1975.
[10] R.D. Baker, Quasigroups and tactical systems, Aequationes Math. 18 (1978) 296–303.
[11] F.E. Bennett, G. Ge, Existence of directedwhist tournaments with the three person property $3PDWh(v)$, Discrete Appl. Math. 154 (2006) 1939–1946.
[12] F.E. Bennett, L. Zhu, Conjugate-orthogonal Latin squares and related structures, in: J. Dinitz, D. Stinson (Eds.), Contemporary Design Theory: A Collection of Surveys, Wiley, New York, 1992, pp. 41–96.
[13] T. Beth, D. Jungnickel, H. Lenz, Design Theory, Cambridge University Press, Cambridge, UK, 1999.
[14] K. Chen, On the existence of super-simple $(v, 4, 3)$-BIBDs, J. Combin. Math. Combin. Comput. 17 (1995) 149–159.
[15] K. Chen, On the existence of super-simple $(v, 4, 4)$-BIBDs, J. Statist. Plann. Inference 51 (1996) 339–350.
[16] J. Dénes, A.D. Keedwell, Latin Squares and their Applications, Academic Press, New York, London, 1974.
[17] T. Feng, Y. Chang, Existence of Z-cyclic $3PTWh(p)$ for any prime $p \equiv 1 (\mathrm{mod}\ 4)$, Des. Codes Cryptogr. 39 (2006) 39–49.
[18] N.J. Finizio, Whist tournaments–Three Person Property, Discrete Appl. Math. 45 (1993) 125–137.
[19] N.J. Finizio, A few more Z-cyclic whist tournaments, J. Combin. Math. Combin. Comput. 19 (1995) 93–95.
[20] N.J. Finizio, J.T. Lewis, A criterion for cyclic whist tournaments with the three person property, Util. Math. 52 (1997) 129–140.
[21] S.C. Furino, Y. Miao, J. Yin, Frames and Resolvable Designs, CRC Press, Boca Raton, 1996.
[22] G. Ge, Triplewhist tournaments with the three person property, J. Combin. Theory Ser. A 114 (2007) 1438–1455.
[23] G. Ge, C.W.H. Lam, Super-simple resolvable balanced incomplete block designs with block size 4 and index 3, J. Combin. Des. 12 (2004) 1–11.
[24] G. Ge, C.W.H. Lam, Whist tournaments with the three person property, Discrete Appl. Math. 138 (2004) 265–276.
[25] G. Ge, C.W.H. Lam, Some new triplewhist tournaments $TWh(v)$, J. Combin. Theory Ser. A 101 (2003) 153–159.
[26] G. Ge, L. Zhu, Frame constructions for Z-cyclic triplewhist tournaments, Bull. Inst. Combin. Appl. 32 (2001) 53–62.
[27] H.-D.O.F. Gronau, R.C. Mullin, On super-simple 2-$(v, 4, \lambda)$ designs, J. Combin. Math. Combin. Comput. 11 (1992) 113–121.
[28] A. Hartman, Doubly and orthogonally resolvable quadruple systems, in: Combinatorial mathematics, VII (Proc. Seventh Australian Conf., Univ. Newcastle, Newcastle, 1979), in: Lecture Notes in Math., vol. 829, Springer, Berlin, 1980, pp. 157–164.
[29] Y. Lu, L. Zhu, On the existence of triplewhist tournaments TWh(v), J. Combin. Des. 5 (1997) 249–256.
[30] Y. Lu, S.Y. Zhang, Existence of whist tournaments with the three-person property 3PWh(v), Discrete Appl. Math. 101 (2000) 207–219.
[31] N.S. Mendelsohn, Combinatorial designs as models of universal algebras, in: Recent Progress in Combinatorics, Academic Press, New York, 1969, pp. 123–132.
[32] E.H. Moore, Tactical Memoranda I–III, Amer. J. Math. 18 (1896) 264–303.
[33] S.K. Stein, On the foundations of quasigroups, Trans. Amer. Math. Soc. 85 (1957) 228–256.
[34] R.M. Wilson, Constructions and uses of pairwise balanced designs, Math. Centre Tracts 55 (1974) 18–41.
[35] X. Zhang, Direct construction methods for incomplete perfect Mendelsohn designs with block size four, J. Combin. Des. 4 (1996) 117–134.
[36] X. Zhang, On the existence of $(v, 4, 1)$-RPMD, Ars Combin. 42 (1996) 3–31.
[37] X. Zhang, A few more RPMDs with $k = 4$, Ars Combin. 74 (2005) 187–200.
[38] X. Zhang, G. Ge, Super-simple resolvable balanced incomplete block designs with block size 4 and index 2, J. Combin. Des. 15 (2007) 341–356.
[39] X. Zhang, G. Ge, Existence of Z-cyclic 3PDTWh(p) for prime $p \equiv 1$ (mod 4), Des. Codes Cryptogr. 45 (2007) 139–155.