JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **126**, 51–69 (1987)

On the Right Focal Point Boundary Value Problems for Integro-Differential Equations

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Submitted by V. Lakshmikantham

Received December 20, 1985

For the nth order nonlinear integro-differential equations subject to right focal point boundary conditions we provide necessary and sufficient conditions for the existence, uniqueness, and convergence of an approximate iterative method.

1. INTRODUCTION

In this paper we shall consider the following nth order equation

$$x^{(n)}(t) = f(t, \mathbf{x}(t), A\mathbf{x}(t)) \tag{1.1}$$

together with the right focal point (the nomenclature comes from polynomial interpolation) boundary conditions

$$\begin{aligned} x^{(i)}(a) &= A_i, \qquad 0 \leqslant i \leqslant k-1 \\ x^{(i)}(b) &= B_i, \qquad k \leqslant i \leqslant n-1, \end{aligned} \tag{1.2}$$

where $n \geqslant 2$, $1 \leqslant k \leqslant n-1$ is fixed. In (1.1), $\mathbf{x}(t)$ stands for $(x(t), x'(t),..., x^{(n-1)}(t))$ and A is a continuous operator which maps $C^{(n-1)}[a, b]$ into $C[a, b]$. The function f is assumed to be continuous in all of its arguments.

The problem (1.1), (1.2) includes several particular cases, for example, the boundary value problem for differential equations if $A = 0$, considered

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in [6, 7, 9–19, 22, 23], for the integro-differential equations of Volterra type if

$$A\mathbf{x}(t)=\int_a^t g(t, s, \mathbf{x}(s))\, ds$$

for the integro-differential equations of Fredholm type if

$$A\mathbf{x}(x)=\int_a^b g(t, s, \mathbf{x}(s))\, ds$$

and so on. For the related problems also see [1–4, 8, 20 and references therein].

The plan of this paper is as follows: In Section 2, we state some lemmas which are needed in Section 3 to obtain necessary and sufficient conditions for the existence and uniqueness of the solutions of (1.1), (1.2). In Section 3, we also provide a priori conditions so that the sequence $\{x_m(t)\}$ generated from Picard's iterative method (3.13) converges to the unique solution $x^*(t)$ of the boundary value problem (1.1), (1.2). In practical evaluation of this sequence only an approximate sequence $\{y_m(t)\}$ is constructed which depends on approximating f and A by some simpler function and operator. In Section 4, we shall find $y_{m+1}(t)$ by approximating f and A by f_m and A_m following relative and absolute error criterian and obtain necessary and sufficient conditions for the convergence of $\{y_m(t)\}$ to the solution. In Section 5, we consider several examples which dwell upon the importance of our results.

2. Some Basic Lemmas

Lemma 2.1 [6]. *The Green's function $g_k(t, s)$ of the boundary value problem*

$$x^{(n)}=0, \tag{2.1}$$

$$x^{(i)}(a)=0, \qquad 0\leqslant i\leqslant k-1 \tag{2.2}$$

$$x^{(i)}(b)=0, \qquad k\leqslant i\leqslant n-1 \tag{2.3}$$

can be written as

$$g_k(t, s)=\frac{1}{(n-1)!}\begin{cases}\displaystyle\sum_{i=0}^{k-1}\binom{n-1}{i}(t-a)^i(a-s)^{n-i-1}, & s\leqslant t\\ \displaystyle -\sum_{i=k}^{n-1}\binom{n-1}{i}(t-a)^i(a-s)^{n-i-1}, & s\geqslant t\end{cases} \tag{2.4}$$

and

$$(-1)^{n-k} g_k^{(i)}(t, s) \geq 0, \qquad 0 \leq i \leq k-1 \tag{2.5}$$

$$(-1)^{n-i} g_k^{(i)}(t, s) \geq 0, \qquad k \leq i \leq n-1 \tag{2.6}$$

on $[a, b] \times [a, b]$, *where* $g_k^{(i)}(t, s)$ *denotes the* ith *derivative* $(\partial^i/\partial t^i)\, g_k(t, s)$.

LEMMA 2.2 [6]. *Let* $x(t) \in C^{(n)}[a, b]$, *satisfying* (2.2), (2.3) *then*

$$|x^{(i)}(t)| \leq C_{n,i}^k (b-a)^{n-i} \max_{a \leq t \leq b} |x^{(n)}(t)|, \qquad 0 \leq i \leq n-1, \tag{2.7}$$

where

$$C_{n,i}^k = \frac{1}{(n-i)!} \left| \sum_{j=0}^{k-i-1} \binom{n-i}{j} (-1)^{n-i-j} \right|, \qquad 0 \leq i \leq k-1$$
$$= \frac{1}{(n-i)!}, \qquad k \leq i \leq n-1.$$

In (2.7) *the constants* $C_{n,i}^k$, $0 \leq i \leq n-1$ *are the best possible as they are exact for the function*

$$x(t) = \frac{1}{n!} \sum_{i=k}^{n} \binom{n}{i} (a-b)^{n-i} (t-a)^i$$

and only for this function up to a constant factor.

LEMMA 2.3. *The unique polynomial of degree* $(n-1)$ *satisfying* (1.2) *is*

$$P_{n-1}(t) = \sum_{i=0}^{k-1} \frac{(t-a)^i}{i!} A_i + \sum_{j=0}^{n-k-1} \left(\sum_{i=0}^{j} \frac{(t-a)^{k+i}(a-b)^{j-i}}{(k+i)!\,(j-i)!} \right) B_{k+j}.$$

LEMMA 2.4 [5, 21]. *Let* B *be a Banach space and let* $r > 0$; $\bar{S}(x_0, r) = \{x \in B\colon \|x - x_0\| \leq r\}$. *Let* $T\colon \bar{S}(x_0, r) \to B$ *and*

(i) *for all* $x, y \in \bar{S}(x_0, r)$, $\|Tx - Ty\| \leq \alpha \|x - y\|$, *where* $0 \leq \alpha < 1$,

(ii) $r_0 = (1-\alpha)^{-1} \|Tx_0 - x_0\| \leq r$. *Then,* (1) T *has a fixed point* x^* *in* $\bar{S}(x_0, r_0)$, (2) x^* *is the unique fixed point of* T *in* $\bar{S}(x_0, r)$, (3) *the sequence* $\{x_m\}$ *defined by* $x_{m+1} = Tx_m$; $m = 0, 1, 2,...$ *converges to* x^* *with* $\|x^* - x_m\| \leq \alpha^m r_0$, (4) *for any* $x \in \bar{S}(x_0, r_0)$, $x^* = \lim_{m \to \infty} T^m x$, (5) *any sequence* $\{\bar{x}_m\}$ *such that* $\bar{x}_m \in \bar{S}(x_m, \alpha^m r_0)$; $m = 0, 1, 2,...$ *converges to* x^*.

3. Existence and Uniqueness

THEOREM 3.1. *Suppose that*

(i) $K_i > 0$, $0 \leqslant i \leqslant n-1$ *be given real numbers and* $D_0 = \{x(t) \in C^{(n-1)}[a, b]: |x^{(i)}(t)| \leqslant 2K_i, 0 \leqslant i \leqslant n-1\}$ *also,*

$$|f(t, \mathbf{x}(t), A\mathbf{x}(t))| \leqslant Q \qquad \text{on} \quad [a, b] \times D_0,$$

(ii) $\max_{a \leqslant t \leqslant b} |P_{n-1}^{(i)}(t)| \leqslant K_i$, $0 \leqslant i \leqslant n-1$,

(iii) $(b-a) \leqslant (K_i/QC_{n,i}^k)^{1/n-i}$, $0 \leqslant i \leqslant n-1$.

Then, the boundary value problem (1.1), (1.2) *has a solution in* D_0.

Proof. The set

$$B[a, b] = \{x(t) \in C^{(n-1)}[a, b]: \|x^{(i)}\| \leqslant 2K_i, 0 \leqslant i \leqslant n-1\},$$

where $\|x^{(i)}\| = \max_{a \leqslant t \leqslant b} |x^{(i)}(t)|$ is a closed convex subset of the Banach space $C^{(n-1)}[a, b]$. Consider an operator T: $C^{(n-1)}[a, b] \to C^{(n)}[a, b]$ as

$$(Tx)(t) = P_{n-1}(t) + \int_a^b g_k(t, s) f(s, \mathbf{x}(s), A\mathbf{x}(s))\, ds. \tag{3.1}$$

Obviously any fixed point of (3.1) is a solution of (1.1), (1.2).

We note that $(Tx)(t) - P_{n-1}(t)$ satisfies the conditions of Lemma 2.2 and $(Tx)^{(n)}(t) - P_{n-1}^{(n)}(t) = (Tx)^{(n)}(t) = f(t, \mathbf{x}(t), A\mathbf{x}(t))$. Thus, for all $x(t) \in B[a, b]$ it follows that $\|(Tx)^{(n)} - P_{n-1}^{(n)}\| \leqslant Q$. Hence, we find

$$\|(Tx)^{(i)} - P_{n-1}^{(i)}\| \leqslant C_{n,i}^k Q(b-a)^{n-i}, \qquad 0 \leqslant i \leqslant n-1$$

which also implies that

$$\begin{aligned} \|(Tx)^{(i)}\| &\leqslant \|P_{n-1}^{(i)}\| + C_{n,i}^k Q(b-a)^{n-i} \\ &\leqslant K_i + K_i = 2K_i, 0 \leqslant i \leqslant n-1. \end{aligned} \tag{3.2}$$

Thus, T maps $B[a, b]$ into itself. Further, the inequalities (3.2) imply that the sets $\{(Tx)^{(i)}(t): x(t) \in B[a, b]\}$, $0 \leqslant i \leqslant n-1$ are uniformly bounded and equicontinuous on $[a, b]$. Hence $\overline{TB}[a, b]$ is compact follows from the Ascoli–Arzela theorem. The Schauder fixed point theorem is applicable and a fixed point of (3.1) in D_0 exists.

COROLLARY 3.2. *Assume that the function* $f(t, \mathbf{x}(t), A\mathbf{x}(t))$ *on* $[a, b] \times C^{n-1}[a, b]$ *satisfies the condition*

$$|f(t, \mathbf{x}(t), A\mathbf{x}(t))| \leqslant c_0 + \sum_{i=0}^{n-1} c_{i+1} |x^{(i)}(t)|^{\alpha(i)} + \sum_{i=0}^{n-1} \int_a^b h_i(t, s) |x^{(i)}(s)|^{\beta(i)} ds, \tag{3.3}$$

where $0 \leqslant \alpha(i) < 1$, $0 \leqslant \beta(i) < 1$, $0 \leqslant i \leqslant n-1$, c_i, $0 \leqslant i \leqslant n-1$ *are nonnegative constants*; $h_i(t, s)$, $0 \leqslant i \leqslant n-1$, *are nonnegative integrable functions and* $\sup_{a \leqslant t \leqslant b} \sum_{i=0}^{n-1} \int_a^b h_i(t, s)\, ds < \infty$. *Then, the boundary value problem* (1.1), (1.2) *has a solution.*

Proof. Since f satisfies (3.3), it follows that on D_0,

$$|f(t, \mathbf{x}(t), A\mathbf{x}(t))| \leqslant c_0 + \sum_{i=0}^{n-1} c_{i+1}(2K_i)^{\alpha(i)} + \sum_{i=0}^{n-1} (2K_i)^{\beta(i)} \sup_{a \leqslant t \leqslant b} \int_a^b h_i(t, s)\, ds = M \qquad \text{(say)}.$$

Now, Theorem 3.1 is applicable by choosing K_i, $0 \leqslant i \leqslant n-1$, sufficiently large so that

$$\max_{a \leqslant t \leqslant b} |P_{n-1}^{(i)}(t)| \leqslant K_i, \; MC_{n,i}^k(b-a)^{n-i} \leqslant K_i, \qquad 0 \leqslant i \leqslant n-1.$$

Theorem 3.1 is a local existence theorem whereas Corollary 3.2 does not require any condition on the length of the interval or the boundary conditions (global existence). The question: what happens if in the inequality (3.3) the constants $\alpha(i) = \beta(i) = 1$, $0 \leqslant i \leqslant n-1$? is considered in our next result.

THEOREM 3.3. *Suppose that the inequality* (3.3) *with* $\alpha(i) = \beta(i) = 1$, $0 \leqslant i \leqslant n-1$ *is satisfied on* $[a, b] \times D_1$, *where* $D_1 = \{x(t) \in C^{(n-1)}[a, b]$: $|x^{(i)}(t)| \leqslant \max_{a \leqslant t \leqslant b} |P_{n-1}^{(i)}(t)| + C_{n,i}^k(b-a)^{n-i}(c_0 + p)/(1-\alpha)$, $0 \leqslant i \leqslant n-1\}$ *and*

$$p = \max_{a \leqslant t \leqslant b} \sum_{i=0}^{n-1} c_{i+1} |P_{n-1}^{(i)}(t)| + \sup_{a \leqslant t \leqslant b} \sum_{i=0}^{n-1} \int_a^b h_i(t, s) |P_{n-1}^{(i)}(s)|\, ds,$$

$$\alpha = \sum_{i=0}^{n-1} \left(c_{i+1} + \sup_{a \leqslant t \leqslant b} \int_a^b h_i(t, s)\, ds \right) C_{n,i}^k(b-a)^{n-i} < 1.$$

Then, the boundary value problem (1.1), (1.2) *has a solution in* D_1.

Proof. The boundary value problem (1.1), (1.2) is equivalent to the problem

$$y^{(n)}(t)=f(t, \mathbf{y}(t)+\mathbf{P}_{n-1}(t), A(\mathbf{y}(t)+\mathbf{P}_{n-1}(t))) \tag{3.4}$$

$$\begin{aligned} &y^{(i)}(a)=0, \qquad 0\leqslant i\leqslant k-1 \\ &y^{(i)}(b)=0, \qquad k\leqslant i\leqslant n-1. \end{aligned} \tag{3.5}$$

We define M as the set of functions n times continuously differentiable on $[a, b]$ and satisfying the boundary conditions (3.5). If we introduce in M the norm $\| y\| = \max_{a\leqslant t\leqslant b} |y^{(n)}(t)|$ then, M becomes a Banach space. We shall show that the mapping T: $M\to M$ defined by

$$(Ty)(t)=\int_a^b g_k(t, s) f(s, \mathbf{y}(s)+\mathbf{P}_{n-1}(s), A(\mathbf{y}(s)+\mathbf{P}_{n-1}(s)))\, ds \tag{3.6}$$

maps the ball $S=\{y(t)\in M\colon \| y\| \leqslant (c_0+p)/(1-\alpha)\}$ into itself. For this, let $y(t)\in S$ then from Lemma 2.2, we have

$$|y^{(i)}(t)|\leqslant C^k_{n,i}(b-a)^{n-i}\frac{c_0+p}{1-\alpha}, \qquad 0\leqslant i\leqslant n-1$$

and hence

$$|y^{(i)}(t)+P^{(i)}_{n-1}(t)|\leqslant |P^{(i)}_{n-1}(t)|+C^k_{n,i}(b-a)^{n-i}\frac{c_0+p}{1-\alpha}, \qquad 0\leqslant i\leqslant n-1$$

which implies that $(t, \mathbf{y}(t)+\mathbf{P}_{n-1}(t))\in [a, b]\times D_1$.

Further, from (3.6) we have

$$\begin{aligned} \|(Ty)\| &= \max_{a\leqslant t\leqslant b} |f(t, \mathbf{y}(t)+\mathbf{P}_{n-1}(t), A(\mathbf{y}(t)+\mathbf{P}_{n-1}(t)))| \\ &\leqslant c_0+\sum_{i=0}^{n-1} c_{i+1} |y^{(i)}(t)+P^{(i)}_{n-1}(t)| \\ &\quad +\sum_{i=0}^{n-1}\int_a^b h_i(t, s)|y^{(i)}(s)+P^{(i)}_{n-1}(s)|\, ds \\ &\leqslant c_0+\sum_{i=0}^{n-1} c_{i+1}\left[|P^{(i)}_{n-1}(t)|+C^k_{n,i}(b-a)^{n-i}\frac{c_0+p}{1-\alpha}\right] \\ &\quad +\sum_{i=0}^{n-1}\int_a^b h_i(t, s)\left[|P^{(i)}_{n-1}(s)|+C^k_{n,i}(b-a)^{n-i}\frac{c_0+p}{1-\alpha}\right] ds \\ &\leqslant c_0+p+\alpha\frac{c_0+p}{1-\alpha} \\ &=\frac{c_0+p}{1-\alpha}. \end{aligned}$$

Thus, it follows from Schauder's fixed point theorem that T has a fixed point in S. This fixed point $y(t)$ is a solution of (3.4), (3.5) and hence the boundary value problem (1.1), (1.2) has a solution $x(t) = y(t) + P_{n-1}(t)$.

THEOREM 3.4. *Suppose that the boundary value problem* (1.1), (2.2), (2.3) *has a nontrivial solution* $x(t)$ *and the condition* (3.3) *is satisfied with* $c_0 = 0$, $\alpha(i) = \beta(i) = 1$, $0 \leqslant i \leqslant n-1$, *on* $[a, b] \times D_2$, *where*

$$D_2 = \{x(t) \in C^{(n-1)}[a, b] \colon |x^{(i)}(t)| \leqslant C^k_{n,i}(b-a)^{n-i} m, 0 \leqslant i \leqslant n-1\}$$

and $m = \max_{a \leqslant t \leqslant b} |x^{(n)}(t)|$. *Then, it is necessary that* $\alpha \geqslant 1$.

Proof. Since $x(t)$ is a nontrivial solution of the boundary value problem (1.1), (2.2), (2.3) it is necessary that $m \neq 0$. Further, Lemma 2.2 implies that $(t, \mathbf{x}(t)) \in [a, b] \times D_2$. Thus, we have

$$\begin{aligned} m &= \max_{a \leqslant t \leqslant b} |x^{(n)}(t)| = \max_{a \leqslant t \leqslant b} |f(t, \mathbf{x}(t), A\mathbf{x}(t))| \\ &\leqslant \max_{a \leqslant t \leqslant b} \left[\sum_{i=0}^{n-1} c_{i+1} |x^{(i)}(t)| + \sum_{i=0}^{n-1} \int_a^b h_i(t, s) |x^{(i)}(s)| \, ds \right] \\ &\leqslant \left(\sum_{i=0}^{n-1} (c_{i+1} + \sup_{a \leqslant t \leqslant b} \int_a^b h_i(t, s) \, ds) \, C^k_{n,i}(b-a)^{n-i} \right) m \\ &= \alpha m \end{aligned}$$

and hence $\alpha \geqslant 1$.

The conditions of Theorem 3.4 ensure that in (3.3) at least one of the c_{i+1} or $h_i(t, s)$, $0 \leqslant i \leqslant n-1$, will not be zero, otherwise $x(t)$ will coincide on $[a, b]$ with a polynomial of degree at most $(n-1)$ and will not be a nontrivial solution of (1.1), (2.2), (2.3). Further, $x(t) \equiv 0$ is obviously a solution of (1.1), (2.2), (2.3) and if $\alpha < 1$ then, it is also unique.

DEFINITION. The function $f(t, \mathbf{x}(t), A\mathbf{x}(t))$ is said to be of Lipschitz class, if for all $(t, x(t))$, $(t, y, (t)) \in [a, b] \times C^{(n-1)}[a, b]$ the following inequality is satisfied

$$\begin{aligned} &|f(t, \mathbf{x}(t), A\mathbf{x}(t)) - f(t, \mathbf{y}(t), A\mathbf{y}(t))| \\ &\quad \leqslant \sum_{i=0}^{n-1} L_i |x^{(i)}(t) - y^{(i)}(t)| \\ &\quad\quad + \sum_{i=0}^{n-1} \int_a^b h_i(t, s) |x^{(i)}(s) - y^{(i)}(s)| \, ds. \end{aligned} \tag{3.7}$$

THEOREM 3.5. *Let the function $f(t, \mathbf{x}(t), A\mathbf{x}(t))$ be of Lipschitz class. Then, if*

$$\theta = \sum_{i=0}^{n-1} \left(L_i + \sup_{a \leqslant t \leqslant b} \int_a^b h_i(t, s)\, ds \right) C_{n,i}^k (b-a)^{n-i} < 1 \tag{3.8}$$

the boundary value problem (1.1), (1.2) *has a unique solution for any* A_i, $0 \leqslant i \leqslant k-1$, *and* B_i, $k \leqslant i \leqslant n-1$.

Proof. We shall show that the mapping T defined on the Banach space M in Theorem 3.3 is contracting. For this, let $y(t)$, $z(t) \in M$ then, from (3.6), (3.7), and Lemma 2.2, we have

$$\begin{aligned}(Ty)^{(n)}(t) - (Tz)^{(n)}(t) = f(t, \mathbf{y}(t) + \mathbf{P}_{n-1}(t), A(\mathbf{y}(t) \\ + \mathbf{P}_{n-1}(t))) - f(t, \mathbf{z}(t) \\ + \mathbf{P}_{n-1}(t), A(\mathbf{z}(t) + \mathbf{P}_{n-1}(t)))\end{aligned}$$

and hence

$$\begin{aligned}\|(Ty) - (Tz)\| &= \max_{a \leqslant t \leqslant b} |f(t, \mathbf{y}(t) + \mathbf{P}_{n-1}(t), A(\mathbf{y}(t) + \mathbf{P}_{n-1}(t))) \\ &\quad - f(t, \mathbf{z}(t) + \mathbf{P}_{n-1}(t), A(\mathbf{z}(t) + \mathbf{P}_{n-1}(t)))| \\ &\leqslant \max_{a \leqslant t \leqslant b} \left[\sum_{i=0}^{n-1} L_i \,|y^{(i)}(t) - z^{(i)}(t)| \right. \\ &\quad \left. + \sum_{i=0}^{n-1} \int_a^b h_i(t, s)\, |y^{(i)}(s) - z^{(i)}(s)|\, ds \right] \\ &\leqslant \sum_{i=0}^{n-1} \left(L_i + \sup_{a \leqslant t \leqslant b} \int_a^b h_i(t, s)\, ds \right) \\ &\quad \times \max_{a \leqslant t \leqslant b} |y^{(i)}(t) - z^{(i)}(t)| \\ &\leqslant \theta \,\|y - z\|.\end{aligned}$$

Thus, the mapping T in M has a unique fixed point and this is equivalent to the existence and uniqueness of the solutions for the boundary value problem (1.1), (1.2).

If the function $f(t, \mathbf{x}(t), A\mathbf{x}(t))$ satisfies the Lipschitz condition (3.7) only over a compact region then Theorem 3.5 cannot be applied. To deal with such a situation we need the following:

DEFINITION. A function $\bar{x}(t) \in C^{(n)}[a, b]$ is called an approximate solution of the boundary value problem (1.1), (1.2) if there exist δ and ε nonnegative constants such that

$$\max_{a \leqslant t \leqslant b} |\bar{x}^{(n)}(t) - f(t, \bar{\mathbf{x}}(t), A\bar{\mathbf{x}}(t))| \leqslant \delta \tag{3.9}$$

and

$$\max_{a \leqslant t \leqslant b} |P_{n-1}^{(i)}(t) - \bar{P}_{n-1}^{(i)}(t)| \leqslant \varepsilon C_{n,i}^{k} (b-a)^{n-i}, \qquad 0 \leqslant i \leqslant n-1, \tag{3.10}$$

where $\bar{P}_{n-1}(t)$ is the $(n-1)$th degree polynomial satisfying

$$\bar{P}_{n-1}^{(i)}(a) = \bar{x}^{(i)}(a), \qquad 0 \leqslant i \leqslant k-1$$
$$\bar{P}_{n-1}^{(i)}(b) = \bar{x}^{(i)}(b), \qquad k \leqslant i \leqslant n-1,$$

i.e.,

$$\bar{P}_{n-1}(t) = \sum_{i=0}^{k-1} \frac{(t-a)^i}{i!} \bar{x}^{(i)}(a) + \sum_{j=0}^{n-k-1} \left(\sum_{i=0}^{j} \frac{(t-a)^{k-i}(a-b)^{j-i}}{(k+i)!\,(j-i)!} \right) \bar{x}^{(k+j)}(b).$$

The approximate solution $\bar{x}(t)$ can be expressed as

$$\bar{x}(t) = \bar{P}_{n-1}(t) + \int_a^b g_k(t, s)[f(s, \bar{\mathbf{x}}(s), A\bar{\mathbf{x}}(s)) + \eta(s)]\, ds, \tag{3.11}$$

where $\eta(t) = \bar{x}^{(n)}(t) - f(t, \bar{\mathbf{x}}(t), A\bar{\mathbf{x}}(t))$ and $\max_{a \leqslant t \leqslant b} |\eta(t)| \leqslant \delta$.

THEOREM 3.6. *Let there exists an approximate solution* $\bar{x}(t)$ *of the boundary value problem* (1.1), (1.2) *and*

(i) *the function* $f(t, \mathbf{x}(t), A\mathbf{x}(t))$ *satisfies the Lipschitz condition* (3.7) *on* $[a, b] \times D_3$, *where*

$$D_3 = \left\{ x(t) \in C^{(n-1)}[a, b] : |x^{(i)}(t) - \bar{x}^{(i)}(t)| \leqslant N \frac{C_{n,i}^k}{C_{n,0}^k (b-a)^i},\ 0 \leqslant i \leqslant n-1 \right\}.$$

(ii) $\theta < 1$ *and*

$$(1-\theta)^{-1}(\varepsilon + \delta)\, C_{n,0}^k (b-a)^n \leqslant N. \tag{3.12}$$

Then, the following holds:

(1) *There exists a solution* $x^*(t)$ *of the problem* (1.1), (1.2) *in* $\bar{S}(\bar{x}, N_0)$, *where* $\bar{S}(\bar{x}, N_0) = \{x(t) \in C^{(n-1)}[a, b]\colon \|x - \bar{x}\| \leqslant N_0\}$ *and* $N_0 = (1-\theta)^{-1} \|x_1 - \bar{x}\|$, *also*

$$\|x\| = \max_{0 \leqslant i \leqslant n-1} \left\{ \frac{C^k_{n,0}(b-a)^i}{C^k_{n,i}} \max_{a \leqslant t \leqslant b} |x^{(i)}(t)| \right\}.$$

(2) $x^*(t)$ *is the unique solution of the problem* (1.1), (1.2) *in* $\bar{S}(\bar{x}, N)$.

(3) *The Picard's sequence* $\{x_m(t)\}$ *defined by*

$$x_{m+1}(t) = P_{n-1}(t) + \int_a^b g_k(t, s) f(s, x_m(s), Ax_m(s))\, ds$$
$$x_0(t) = \bar{x}(t);\ m = 0, 1, \ldots, \tag{3.13}$$

converges to $x^*(t)$ *with*

$$\|x^* - x_m\| \leqslant \theta^m N_0.$$

(4) *For* $x_0(t) = x(t) \in \bar{S}(\bar{x}, N_0)$ *the iterative process* (3.13) *converges to* $x^*(t)$.

(5) *Any sequence* $\{\bar{x}_m(t)\}$ *such that* $\bar{x}_m(t) \in \bar{S}(x_m, \theta^m N_0)$; $m = 0, 1, \ldots$ *converges to* $x^*(t)$.

Proof. We shall show that the operator $T\colon \bar{S}(\bar{x}, N) \to C^{(n)}[a, b]$ defined in (3.1) satisfies the conditions of Lemma 2.4. For this, let $x(t) \in \bar{S}(\bar{x}, N)$ then, from the definition of $\|\cdot\|$, we have

$$\frac{C^k_{n,0}(b-a)^i}{C^k_{n,i}} |x^{(i)}(t) - \bar{x}^{(i)}(t)| \leqslant \|x - \bar{x}\| \leqslant N$$

and hence

$$|x^{(i)}(t) - \bar{x}^{(i)}(t)| \leqslant \frac{C^k_{n,i}}{C^k_{n,0}(b-a)^i} N, \qquad 0 \leqslant i \leqslant n-1$$

which implies that $x(t) \in D_3$. Further, if $x(t)$, $y(t) \in D_3$ then, $(Tx)(t) - (Ty)(t)$ satisfies the conditions of Lemma 2.2, and we get

$$\begin{aligned} &|(Tx)^{(i)}(t) - (Ty)^{(i)}(t)| \\ &\quad \leqslant C^k_{n,i}(b-a)^{n-i} \max_{a \leqslant t \leqslant b} |f(t, \mathbf{x}(t), A\mathbf{x}(t)) - f(t, \mathbf{y}(t), A\mathbf{y}(t))| \\ &\quad \leqslant C^k_{n,i}(b-a)^{n-i} \max_{a \leqslant t \leqslant b} \left[\sum_{j=0}^{n-1} L_j |x^{(j)}(t) - y^{(j)}(t)| \right. \\ &\qquad \left. + \sum_{j=0}^{n-1} \int_a^b h_j(t, s) |x^{(j)}(s) - y^{(j)}(s)|\, ds \right] \end{aligned}$$

$$\leqslant C^k_{n,i}\,(b-a)^{n-i}\sum_{j=0}^{n-1}\left(L_j+\sup_{a\leqslant t\leqslant b}\int_a^b h_j(t,s)\,ds\right)$$

$$\times \max_{a\leqslant t\leqslant b}|x^{(j)}(t)-y^{(j)}(t)|$$

$$\leqslant C^k_{n,i}(b-a)^{n-i}\sum_{j=0}^{n-1}\left(L_j+\sup_{a\leqslant t\leqslant b}\int_a^b h_j(t,s)\,ds\right)$$

$$\times\frac{C^k_{n,j}}{C^k_{n,0}(b-a)^j}\,\|x-y\|$$

and hence

$$\frac{C^k_{n,0}(b-a)^i}{C^k_{n,i}}\,|(Tx)^{(i)}(t)-(Ty)^{(i)}(t)|$$

$$\leqslant\sum_{j=0}^{n-1}\left(L_j+\sup_{a\leqslant t\leqslant b}\int_a^b h_j(t,s)\,ds\right)$$

$$\times C^k_{n,j}(b-a)^{n-j}\,\|x-y\|,\qquad 0\leqslant i\leqslant n-1$$

from which it follows that

$$\|(Tx)-(Ty)\|\leqslant\theta\,\|x-y\|.$$

Further, from (3.1) and (3.11), we have

$$(T\bar{x})(t)-\bar{x}(t)=(Tx_0)(t)-x_0(t)$$

$$=P_{n-1}(t)-\bar{P}_{n-1}(t)-\int_a^b g_k(t,s)\,\eta(s)\,ds. \tag{3.14}$$

Obviously, the function $z(t)=-\int_a^b g_k(t,s)\,\eta(s)\,ds$ satisfies the conditions of Lemma 2.2 and $z^{(n)}(t)=-\eta(t)$, thus $\max_{a\leqslant t\leqslant b}|z^{(n)}(t)|=\max_{a\leqslant t\leqslant b}|\eta(t)|\leqslant\delta$, and hence $|z^{(i)}(t)|\leqslant C^k_{n,i}(b-a)^{n-i}\,\delta$, $0\leqslant i\leqslant n-1$. Using these inequalities and (3.10) in (3.14), we obtain

$$|(Tx_0)^{(i)}(t)-x_0^{(i)}(t)|\leqslant\varepsilon C^k_{n,i}(b-a)^{n-i}+\delta C^k_{n,i}(b-a)^{n-i}$$

and hence

$$\frac{C^k_{n,0}\,(b-a)^i}{C^k_{n,i}}\,|(Tx_0)^{(i)}(t)-x_0^{(i)}(t)|$$

$$\leqslant(\varepsilon+\delta)\,C^k_{n,0}(b-a)^n,\qquad 0\leqslant i\leqslant n-1$$

or

$$\|(Tx_0)-x_0\|\leqslant(\varepsilon+\delta)\,C^k_{n,0}(b-a)^n \tag{3.15}$$

which is from (3.12) same as

$$(1-\theta)^{-1} \| (Tx_0) - x_0 \| \leqslant N.$$

Thus the conditions of Lemma 2.4 are satisfied and conclusions (1)–(5) follow.

4. Convergence of the Approximate Iterates

In Theorem 3.6 conclusion (3) ensures the convergence of the sequence $\{x_m(t)\}$ obtained from the iterative scheme (3.13) to the unique solution $x^*(t)$ of the boundary value problem (1.1), (1.2). However, in practical evaluation this theoretical sequence $\{x_m(t)\}$ is approximated by the computed sequence, say, $\{y_m(t)\}$. To find $y_{m+1}(t)$; the function f and operator A are approximated by some simpler f_m and A_m. Therefore, the computed sequence $\{y_m(t)\}$ satisfies the iterative process

$$\begin{aligned} y_{m+1}(t) &= P_{n-1}(t) + \int_a^b g_k(t, s) f_m(s, \mathbf{y}_m(s), A_m \mathbf{y}_m(s))\, ds \\ y_0(t) &= x_0(t) = \bar{x}(t), \qquad m = 0, 1, \ldots \end{aligned} \tag{4.1}$$

For all $y_m(t)$ obtained from (4.1) we shall assume that the inequalities

$$\begin{aligned} &\max_{a \leqslant t \leqslant b} | f_m(t, \mathbf{y}_m(t), A_m \mathbf{y}_m(t)) - f(t, \mathbf{y}_m(t), A_m \mathbf{y}_m(t)) | \\ &\qquad \leqslant \Delta \max_{a \leqslant t \leqslant b} | f(t, \mathbf{y}_m(t), A_m \mathbf{y}_m(t)) | \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\max_{a \leqslant t \leqslant b} | f(t, \mathbf{y}_m(t), A_m \mathbf{y}_m(t)) - f(t, \mathbf{y}_m(t), A \mathbf{y}_m(t)) | \\ &\qquad \leqslant \nabla \max_{a \leqslant t \leqslant b} | f(t, \mathbf{y}_m(t), A \mathbf{y}_m(t)) | \end{aligned} \tag{4.3}$$

are satisfied, where Δ and ∇ are nonnegative constants. Inequalities (4.2) and (4.3) corresponds to the relative error in approximating f and A by f_m and A_m, respectively.

THEOREM 4.1. *Let there exist an approximate solution* $\bar{x}(t)$ *of the boundary value problem* (1.1), (1.2) *and the inequalities* (4.2), (4.3) *be satisfied, also*

(i) *the function* $f(t, \mathbf{x}(t), A\mathbf{x}(t))$ *satisfies the Lipschitz condition* (3.7) *on* $[a, b] \times D_3$

(ii) $\theta_1 = (1 + \Delta + \nabla + \Delta\nabla)\theta < 1$ *and*

$$N_1 = (1-\theta_1)^{-1}(\varepsilon + \delta + (\Delta + \nabla + \Delta\,\nabla)F)\, C^k_{n,0}(b-a)^n \leqslant N,$$

where $F = \max_{a \leqslant t \leqslant b} |f(t, \bar{\mathbf{x}}(t), A\bar{\mathbf{x}}(t))|$.

Then, the following holds:

(1) *all the conclusions* (1)–(5) *of Theorem* 3.6 *hold,*

(2) *the sequence* $\{y_m(t)\}$ *obtained from* (4.1) *remains in* $\bar{S}(\bar{x}, N_1)$,

(3) *the sequence* $\{y_m(t)\}$ *converges to* $x^*(t)$ *the solution of* (1.1), (1.2) *if and only if*

$$\lim_{m \to 0} a_m = 0, \tag{4.4}$$

where

$$a_m = \left\| y_{m+1}(t) - P_{n-1}(t) - \int_a^b g_k(t, s) f(s, \mathbf{y}_m(s), A\mathbf{y}_m(s))\, ds \right\|,$$

(4) *a bound on the error is given by*

$$\|x^* - y_{m+1}\| \leqslant (1-\theta)^{-1}[\theta\, \|y_{m+1} - y_m\| + (\Delta + \nabla + \Delta\,\nabla) \times \max_{a \leqslant t \leqslant b} |f(t, \mathbf{y}_m(t), A\mathbf{y}_m(t))|\, C^k_{n,0}(b-a)^n]. \tag{4.5}$$

Proof. Since $\theta_1 < 1$ implies $\theta < 1$ and $N_1 \geqslant (1-\theta)^{-1}(\varepsilon+\delta)\, C^k_{n,0}(b-a)^n$ the conclusions of Theorem 3.6 are satisfied and part (1) follows.

To prove (2), we note that $\bar{x}(t) \in \bar{S}(\bar{x}, r_1)$, and if $y_1(t), y_2(t), \ldots, y_m(t)$ are in $\bar{S}(\bar{x}, r_1)$, then it suffices to show that $y_{m+1}(t) \in \bar{S}(\bar{x}, r_1)$. For this, from (4.1) and (3.11), we have

$$y_{m+1}(t) - \bar{x}(t) = P_{n-1}(t) - \bar{P}_{n-1}(t) + \int_a^b g_k(t, s)[f_m(s, \mathbf{y}_m(s), A_m\mathbf{y}_m(s)) - f(s, \bar{\mathbf{x}}(s), A\bar{\mathbf{x}}(s)) - \eta(s)]\, ds$$

and hence from Lemma 2.2, we get

$$|y^{(i)}_{m+1}(t) - \bar{x}^{(i)}(t)| \leqslant (\varepsilon+\delta)\, C^k_{n,i}(b-a)^{n-i} + C^k_{n,i}(b-a)^{n-i} \max_{a \leqslant t \leqslant b} |f_m(t, \mathbf{y}_m(t), A_m\mathbf{y}_m(t)) - f(t, \bar{\mathbf{x}}(t), A\bar{\mathbf{x}}(t))|$$

$$\leqslant (\varepsilon+\delta)\, C^k_{n,i}(b-a)^{n-i}$$

$$+ C^k_{n,i}(b-a)^{n-i}\Big[\max_{a\leqslant t\leqslant b} |f_m(t, \mathbf{y}_m(t), A_m\mathbf{y}_m(t))$$

$$- f(t, \mathbf{y}_m(t), A_m\mathbf{y}_m(t))|$$

$$+ \max_{a\leqslant t\leqslant b} |f(t, \mathbf{y}_m(t), A_m\mathbf{y}_m(t)) - f(t, \mathbf{y}_m(t), A\mathbf{y}_m(t))|$$

$$+ \max_{a\leqslant t\leqslant b} |f(t, \mathbf{y}_m(t), A\mathbf{y}_m(t)) - f(t, \bar{\mathbf{x}}(t), A\bar{\mathbf{x}}(t))|\Big]$$

$$\leqslant (\varepsilon+\delta)\, C^k_{n,i}(b-a)^{n-i}$$

$$+ C^k_{n,i}(b-a)^{n-i}\Big[(1+\Delta+\nabla+\Delta\,\nabla)\max_{a\leqslant t\leqslant b} |f(t, \mathbf{y}_m(t), A\mathbf{y}_m(t))$$

$$- f(t, \bar{\mathbf{x}}(t), A\bar{\mathbf{x}}(t))| + (\Delta+\nabla+\Delta\,\nabla)\max_{a\leqslant t\leqslant b} |f(t, \bar{\mathbf{x}}(t), A\bar{\mathbf{x}}(t))|\Big]$$

$$\leqslant (\varepsilon+\delta+(\Delta+\nabla+\Delta\,\nabla)F)\, C^k_{n,i}(b-a)^{n-i}$$

$$+ C^k_{n,i}(b-a)^{n-i}(1+\Delta+\nabla+\Delta\,\nabla)$$

$$\times\left[\sum_{j=0}^{n-1}\left(L_j + \sup_{a\leqslant t\leqslant b}\int_a^b h_j(t,s)\,ds\right)\frac{C^k_{n,j}}{C^k_{n,0}(b-a)^j}\right]\| y_m - \bar{x}\|$$

which is same as

$$\frac{C^k_{n,0}(b-a)^i\,|y^{(i)}_{m+1}(t) - \bar{x}^{(i)}(t)|}{C^k_{n,i}}$$

$$\leqslant (\varepsilon+\delta+(\Delta+\nabla+\Delta\,\nabla)F)\, C^k_{n,0}(b-a)^n$$

$$+ (1+\Delta+\nabla+\Delta\,\nabla)\,\theta N_1, \qquad 0\leqslant i\leqslant n-1$$

or

$$\| y_{m+1} - \bar{x}\| \leqslant (1-\theta_1)N_1 + \theta_1 N_1 = N_1.$$

This completes the proof of part (2).

Next, we shall prove part (3). From the definition of $x_{m+1}(t)$ and $y_{m+1}(t)$, we have

$$x_{m+1}(t) - y_{m+1}(t) = -y_{m+1}(t) + P_{n-1}(t) + \int_a^b g_k(t,s)$$

$$\times f(s, \mathbf{y}_m(s), A\mathbf{y}_m(s))\,ds$$

$$+ \int_a^b g_k(t,s)[f(s, \mathbf{x}_m(s), A\mathbf{x}_m(s))$$

$$- f(s, \mathbf{y}_m(s), A\mathbf{y}_m(s))]\,ds$$

and hence as earlier, we find

$$\| x_{m+1} - y_{m+1} \| \leqslant a_m + \theta \| x_m - y_m \|. \tag{4.6}$$

Now following inductive arguments inequality (4.6) provides that

$$\| x_{m+1} - y_{m+1} \| \leqslant \sum_{i=0}^{m} \theta^{m-i} a_i. \tag{4.7}$$

Using (4.7) in the triangle inequality, we get

$$\| x^* - y_{m+1} \| \leqslant \sum_{i=0}^{m} \theta^{m-i} a_i + \| x_{m+1} - x^* \|. \tag{4.8}$$

In the right side of (4.8), Theorem 3.6 ensures that $\lim_{m \to \infty} \| x_{m+1} - x^* \| = 0$. Thus, the condition (4.4) is necessary and sufficient for the convergence of the sequence $\{y_m(t)\}$ to $x^*(t)$ follows from Toeplitz lemma "for any $0 \leqslant \alpha < 1$, let $s_m = \sum_{i=0}^{m} \alpha^{m-i} \alpha_i$; $m = 0, 1, \ldots$ then, $\lim_{m \to \infty} s_m = 0$ if and only if $\lim_{m \to \infty} d_m = 0$".

Finally, we shall prove part (4). For this, we note that

$$\begin{aligned} x^*(t) - y_{m+1}(t) = \int_a^b g_k(t, s)[&f(s, \mathbf{x}^*(s), A\mathbf{x}^*(s)) - f(s, \mathbf{y}_m(s), A\mathbf{y}_m(s)) \\ &+ f(s, \mathbf{y}_m(s), A\mathbf{y}_m(s)) - f(s, \mathbf{y}_m(s), A_m\mathbf{y}_m(s)) \\ &+ f(s, \mathbf{y}_m(s), A_m\mathbf{y}_m(s)) - f_m(s, \mathbf{y}_m(s), A_m\mathbf{y}_m(s))] \, ds \end{aligned}$$

and hence, we find

$$\begin{aligned} \| x^* - y_{m+1} \| \leqslant (\varDelta + \nabla + \varDelta \nabla) \max_{a \leqslant t \leqslant b} | f(t, \mathbf{y}_m(t), A\mathbf{y}_m(t)) | \\ \times C_{n,0}^k (b-a)^n + \theta \| y_m - x^* \| \end{aligned}$$

from which (4.5) easily follows.

If instead of inequalities (4.2), (4.3) we assume

$$\max_{a \leqslant t \leqslant b} | f_m(t, \mathbf{y}_m(t), A_m\mathbf{y}_m(t)) - f(t, \mathbf{y}_m(t), A_m\mathbf{y}_m(t)) | \leqslant \varDelta \tag{4.9}$$

and

$$\max_{a \leqslant t \leqslant b} | f(t, \mathbf{y}_m(t), A_m\mathbf{y}_m(t)) - f(t, \mathbf{y}_m(t), A\mathbf{y}_m(t)) | \leqslant \nabla \tag{4.10}$$

which corresponds to an absolute error in approximating f and A by f_m and A_m then, we have

THEOREM 4.2. *Let there exists an approximate solution $\bar{x}(t)$ of the boundary value problem* (1.1), (1.2) *and the inequalities* (4.9), (4.10) *be satisfied, also*

(i) *the function $f(t, \mathbf{x}(t), A\mathbf{x}(t))$ satisfies the Lipschitz condition* (3.7) *on $[a, b] \times D_3$*

(ii) $\theta < 1$ *and*

$$N_2 = (1-\theta)^{-1}(\varepsilon + \delta + \Delta + \nabla)\, C^k_{n,0}(b-a)^n \leqslant N.$$

Then, the following hold:

(1) *all the conclusions of Theorem* 3.6 *hold,*

(2) *the sequence $\{y_m(t)\}$ obtained from* (4.1) *remains in $\bar{S}(\bar{x}, N_2)$,*

(3) *the conclusion* (3) *of Theorem* 4.1 *holds,*

(4) *a bound on the error is given by*

$$\| x^* - y_{m+1} \| \leqslant (1-\theta)^{-1}[\theta \, \| y_{m+1} - y_m \| + (\Delta + \nabla)\, C^k_{n,0}(b-a)^n].$$

Proof. The proof is contained in Theorem 4.1.

5. SOME EXAMPLES

Here, we shall provide few examples which are sufficient to convey the importance of our results.

EXAMPLE 5.1. Consider the boundary value problem

$$x^{(4)}(t) = t^2 x(t) \cos x(t) + \int_0^t \frac{x^2(s)}{1+t^2+s^2}\, ds \tag{5.1}$$

$$x(0) = 1, \qquad x'(0) = x''(1) = x'''(1) = 0. \tag{5.2}$$

Obviously, $P_3(t) = 1$ and $D_0 = \{x(t) \in C[a, b]\colon |x(t)| \leqslant 2K_0\}$ and hence $Q = 2K_0 + 4K_0^2$, $C^2_{4,0} = \frac{1}{8}$. Thus, from Theorem 3.1 the problem (5.1), (5.2) has a solution if

$$1 \leqslant K_0 \qquad \text{and} \qquad \tfrac{1}{8}(2K_0 + 4K_0^2) \leqslant K_0,$$

i.e., $1 \leqslant K_0 \leqslant 1.5$.

EXAMPLE 5.2. For the boundary value problem

$$x^{(4)}(t) = \frac{8}{81} x^5(t) + \int_0^1 \frac{x(s) \sin x(s)}{1+t+s}\, ds \tag{5.3}$$

$$x(1) = \frac{1}{3}, \qquad x'(1) = -\frac{1}{3}, \qquad x''(2) = \frac{1}{12}, \qquad x'''(2) = -\frac{1}{8} \tag{5.4}$$

we have $P_3(t)=\frac{1}{48}(38-29t+8t^2-t^3)$ and hence $\max_{1\leqslant t\leqslant 2}|P_3(t)|\leqslant\frac{1}{3}$. Thus, the problem (5.3), (5.4) has a solution in $D_0=\{x(t)\in C[a,b]: |x(t)|\leqslant 2K_0\}$ if

$$\tfrac{1}{3}\leqslant K_0 \qquad \text{and} \qquad \tfrac{1}{8}\left[\tfrac{8}{81}(2K_0)^5+2K_0\right]\leqslant K_0,$$

i.e., $\frac{1}{3}\leqslant K_0\leqslant 1.173813435....$

EXAMPLE 5.3. For the integro-differential equation

$$x^{(4)}(t)=\sin t+x^{3/4}(t)\cos(e^{-x(t)})-\int_a^b s^2\sin x(s)\,ds$$

together with the boundary conditions (1.2), Corollary 3.2 ensures the existence of at least one solution in $D_0=\{x(t)\in C[a,b]: |x(t)|<\infty\}$ as long as A_i, $0\leqslant i\leqslant k-1$; B_i, $k\leqslant i\leqslant 3$ ($k=1$, 2, or 3) and $(b-a)$ are finite.

EXAMPLE 5.4. Consider the integro-differential equation

$$x^{(4)}(t)=t^2x(t)\sin x(t)+e^{-t^2}+\sin t+\int_0^1\frac{x(s)}{1+t+s}\,ds \tag{5.5}$$

together with the boundary conditions (5.2). For the right side of (5.5) the inequality (3.3) with $\alpha(0)=\beta(0)=1$ is satisfied for all $x(t)\in C[0,1]$, and $c_0=2$, $c_1=1$, $\int_0^1 h_0(t,s)\,ds=\int_0^1 1/(1+t+s)\,ds\leqslant 1$. Thus, $\theta=\frac{1}{8}(1+1)<1$ and hence Theorem 3.3 implies that (5.5), (5.2) has at least one solution $x^*(t)$ in $\{x(t)\in C[0,1]: |x(t)|<\infty\}$. Further, since $P_3(t)=1$, we find $p\leqslant 2$ and the same theorem provides that

$$|x^*(t)|\leqslant 1+\tfrac{1}{8}(2+2)/(1-\tfrac{1}{4})=\tfrac{5}{3}.$$

EXAMPLE 5.5. For the boundary value problem

$$x^{(4)}(t)=2x(t)-\cosh t+2\int_0^t\sinh(t-s)\,x(s)\,ds-\int_0^t e^{-x^2(s)}ds \tag{5.6}$$

$$x(0)=1, \qquad x'(0)=0, \qquad x''\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}, \quad x'''\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} \tag{5.7}$$

we take $\bar{x}(t)=\cos t$ so that $\varepsilon=0$ and

$$\begin{aligned}\delta=\max_{0\leqslant t\leqslant\pi/4}\Bigg|&\cos t-2\cos t+\cosh t\\ &-2\int_0^t\sinh(t-s)\cos s\,ds+\int_0^t e^{-\cos^2 s}\,ds\Bigg|\\ =\max_{0\leqslant t\leqslant\pi/4}&\left|\int_0^t e^{-x^2(s)}\,ds\right|\leqslant 0.785398163...,\end{aligned}$$

thus, we can take $\delta = 0.786$. Further, in $D_3 = \{x(t) \in C[0, \pi/4]: |x(t) - \cos t| \leqslant N\}$, we find $L_0 = 2$ and

$$\sup_{0 \leqslant t \leqslant \pi/4} \int_0^{\pi/4} h_0(t, s)\, ds \leqslant \sup_{0 \leqslant t \leqslant \pi/4} \int_0^{\pi/4} [2 \sinh(t-s) + 2(N+1)]$$
$$\simeq 2(1.325 + N).$$

Hence, for the problem (5.6), (5.7) the conditions of Theorem 3.6 are satisfied provided

$$\theta = \frac{1}{8} [2 + 2(1.325 + N)] \left(\frac{\pi}{4}\right)^4 < 1 \tag{5.8}$$

and

$$(1 - \theta)^{-1} (0.786) \frac{1}{8} \left(\frac{\pi}{4}\right)^4 \leqslant N. \tag{5.9}$$

Both of these inequalities are satisfied if

$$1 \leqslant \frac{8N}{4.65N + 2N^2 + 0.786} \left(\frac{4}{\pi}\right)^4. \tag{5.10}$$

The inequality (5.10) easily provides that $0.048285554... \leqslant N \leqslant 8.139080272...$. Thus, the problem (5.6), (5.7) has a unique solution $x^*(t)$ in $D_3 = \{x(t) \in C[0, \pi/4]: |x(t) - \cos t| \leqslant 8.139080272...\}$ and the iterative scheme

$$x_{m+1}^{(4)}(t) = 2x_m(t) - \cosh t + 2 \int_0^t \sinh(t-s)\, x_m(s)\, ds - \int_0^t e^{-x_m^2(s)}\, ds$$

$$x_{m+1}(0) = 1, \quad x'_{m+1}(0) = 0, \quad x''_{m+1}\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \quad x'''_{m+1}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

converges to $x^*(t)$. Further, we conclude that

$$|x^*(t) - \cos t| \leqslant 0.048285554....$$

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