Elements of $q$-Harmonic Analysis

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Measures that act as $q$-analogs of Lebesgue measure are introduced. Some $q$-analogs of Fourier and Laplace transforms are studied. Last, the connection between certain spaces of $q$-exponential type and discrete $q$-Fourier transforms is discussed. © 1989 Academic Press, Inc.

I. INTRODUCTION

This study is in the spirit of those current "generalizations" of classical harmonic analysis that do not neatly fit in with the "neo-classical" approach of group representation theory. Such examples include the theory of hypergroups [J] and the study of positive definite functions on co-algebras [Sc]. In the first part we present some background material and then some $q$-analogs of Fourier and Laplace transforms. The second part of the study discusses "spaces of $q$-exponential type." Much of this work, particularly the second part, no doubt is subsumed in principle by the classic work of [BB], e.g., and one should also look at [W]. However, what makes "$q$-mathematics" often so interesting are the striking parallels with their "ordinary" analogs. For this reason the explicit formulations below appear to me to be worthwhile.

Another motivating point that plays a role behind this work is that in many cases $q$-analogs can be constructed more or less canonically when there is an underlying operator algebra or Lie group structure. If one can find a $q$-analog of the underlying structure, then $q$-analogs of related structures, e.g., representations, can be found. Thus, we discuss, somewhat briefly, the $q$-Heisenberg algebra below. See [F2, F4] for this point of view.

Works on $q$-series and related topics that provide background and motivation for this study are, e.g., [A, AW, C1, C2, C3, E, Sl]. Also see the references in [F2].
II. Background

There are three ideas behind this study:

1. \( q \)-series and basic hypergeometric functions;
2. random variables, particularly discrete random variables concentrated on a cyclic group (or semigroup) with one generator, \( q \);
3. the \( q \)-Heisenberg algebra.

Topic (1) has been mentioned above and noting the references will suffice here. Before taking up (2) we establish some useful notations:

(a) Throughout, \( q \) is fixed, \( 0 < q < 1 \). \( Q \) denotes \( q^{-1} \).
(b) For \( x \in \mathbb{C} \), \( (x)_n = (1 - x)(1 - qx) \cdots (1 - q^{n-1}x) \), \( (x)_\infty = \prod_{n=0}^{\infty} (1 - q^n) \). For \( a, x \in \mathbb{C} \), \( (x)_a = (x)_\infty / (xq^n)_\infty \).
(c) The \( q \)-binomial coefficients: \( \binom{a}{n} = (q)_n / (q)_k (q)_{n-k} \).
(d) The \( q \)-exponential functions: \( e(x) = (x)_\infty \), \( E(x) = (x)_{-1} \).

Note that these \( e, E \) functions are analogs of the exponential functions \( e^{ax}, e^x \). For example, the eigenfunctions of the \( q \)-difference operator \( f(x) \rightarrow (f(x) - f(qx))/(1 - q) x \) are \( E(ax) \).
(e) Whenever unspecified, all integer indices run from 0 to \( \infty \).

Now to topic (2) (see \[ F1, F3 \]). Let \( X \) be a random variable taking values \( q^n, n \geq 0 \), with \( P(X = q^n) = p_n \). Consider the \( E \)-transform \( \langle E(tX) \rangle = \sum_{n=0}^{\infty} p_n E(tq^n) = E(t) \sum_{n=0}^{\infty} p_n(t) n \), where the angle brackets denote expected value. This suggests that the class of transforms \( \{ p_n \} \rightarrow \{ \varphi(t) : \varphi(t) = \sum_{n=0}^{\infty} p_n(t) n \} \) plays a role analogous to the usual generating functions for random variables taking integer values \( n \geq 0 \). Define the class

\[ \Phi = \{ \varphi(t) : \varphi(t) = \sum_{n=0}^{\infty} p_n(t) n, \]

\[ p_n \text{ a discrete probability distribution} \} \].

Note that we can write

\[ \varphi(t) = \langle (t)_X \rangle \]

for the corresponding random variable \( X \). One result of the work here is to provide some information regarding the class \( \Phi \) (also see \[ F1 \]).

To invert the transform, introduce the \( q \)-difference operator \( W \):

\[ Wf(x) = (f(x) - f(qx))/Qx. \quad (2.1) \]
Noting that $W(x)_n = (1 - q^n)(x)_{n-1}$ and that $(x)_n$ vanishes at $x = 1$ for $n > 0$ gives the inversion formula

$$p_n = (q)_n^{-1} W^n \varphi(1). \quad (2.2)$$

We can define $W$ on entire functions and define the operator $P$ mapping entire functions to entire functions by

$$Pf(x) = \sum_{0}^{\infty} (x)_n (q)_n^{-1} W^n f(1). \quad (2.3)$$

Denote by $\mathcal{D}(P)$ the domain of $P$, i.e., $\{ f: Pf \text{ is entire} \}$. Notice that $P$ has nontrivial kernel, namely, since $W^n(x)_{\infty} = 0$ at $x = 1$, $n \geq 0$, $Pe(x) = 0$.

**Proposition 2.1.**

$$W^n f(x) = q^n x^{-n} \sum_{0}^{\infty} \binom{n}{k}_q (-1)^k q^{\frac{k}{2}} f(xQ^k). \quad (2.4)$$

**Proof.** Recall the Cauchy formula $(a)_n = \sum \binom{n}{k}_q q^{\frac{k}{2}} (-a)^k$. For our formula it is enough to check for $f(x) = x^m$. The LHS (left-hand side) gives $q^n(Q^n)_n x^{m-n}$. The RHS yields

$$q^n \sum \binom{n}{k}_q (-1)^k q^{\frac{k}{2}} Q^{km} x^{m-n} = q^n(Q^n)_n x^{m-n}.$$

For characterizing the class $\Phi$, if one knew that, say, $f = Pf$, then it would suffice to know that $W^n f(1) > 0$, $n \geq 0$, and that $f(0) = 1$, analogously as for usual generating functions. The problem here is to determine conditions on $f$ that guarantee $f = Pf$. This is studied in Section IV below.

At this point it is useful to summarize some formulas that will be of use:

$$(Q^n)_n = (-1)^n Q^n q^{\frac{n}{2}}(q)_k/(q)_{k-n} \quad (2.5)$$

$$(v)_n = \sum \binom{n}{k}_q (-1)^k q^{\frac{k}{2}} v^k \quad (2.6)$$

$$v^n = \sum (Q^n)_k q^k (v)_k/(q)_k \quad (2.7)$$

$$v^n = \sum \binom{n}{k}_q (-qQ^n)_k q^{\frac{k}{2}} (v)_k \quad (2.8)$$

$$E(x) = \sum_{0}^{\infty} x^n/(q)_n, \quad |x| < 1 \quad (2.9)$$
\[ e(x) = \sum_{0}^{\infty} (-1)^n q^{\frac{n}{2}} x^n/(q)_n \]  

(2.10)

\[ (tx)_x/(x)_x = \sum_{0}^{\infty} x^n(t)_n/(q)_n, \quad |x| < 1. \]  

(2.11)

**Remarks.** Equation (2.7) follows from (2.3), and (2.8) is a rearrangement of (2.7). Equations (2.9) and (2.10) are Euler’s formulas. Equation (2.11) is the “q-binomial theorem.”

The Measures \( d\psi, d\psi, d\eta \)

Another way to view the function \( \varphi(t) = \sum_{0}^{\infty} p_n(t)_n \) is as follows. Define the measures

\[ d\psi(x) = \sum_{0}^{\infty} q^n \delta(x - q^n), \quad d\psi(x) = \sum_{0}^{\infty} \delta(x - q^n). \]  

(2.12)

Then \( E(t) \varphi(t) = \int E(tx) p(x) d\psi(x), \) \( p_n = p(q^n). \) Notice that the family of measures \( (1 - q) d\psi(x) \) converges weakly to Lebesgue measure on \((0, 1)\) as \( q \to 1. \)

The measure \( d\eta \) arises as follows. Let \( Y = \sum_{0}^{\infty} X_k \) be the sum of independent exponentially distributed random variables with \( \langle X_k \rangle = q^k. \) Then the characteristic function of \( Y \) is

\[ \langle e^{itY} \rangle = \prod_{0}^{\infty} (1 - itq^k)^{-1} = E(it). \]  

(2.13)

Now define

\[ d\eta(y) = \sum_{0}^{\infty} \eta_k \delta(y - Q^k), \quad \eta_k = (-1)^k q^{\frac{k}{2}}/(q)_{k}(q)_{\infty}. \]  

(2.14)

**Proposition 2.2.** \( Y \) has density \( dG(u) = \sum_{0}^{\infty} \eta_k e^{-uQ^k} du \) on \((0, \infty). \)

**Proof.** Observe that \( \int_{0}^{\infty} e^{iu} dG(u) \) is analytic in the strip \( \text{Re } z < 1. \) Thus we check \( \int_{0}^{\infty} e^{iu} dG(u), \) for \( |t| < 1: \)

\[ \sum_{0}^{\infty} \eta_k q^k (1 - itq^k)^{-1} = \sum_{0}^{\infty} \sum_{n} \eta_k (it)^n q^{kn} q^k \]

\[ = \sum_{n} (it)_n (qq^n)_\infty/(q)_\infty = \sum_{n} (it)_n/(q)_n \]

\[ = E(it). \] 

\( \blacksquare \)
Remarks. (a) $d\eta$ is of bounded variation:

$$\| \eta \|_{\text{var}} = \sum_{0}^{\infty} |\eta_k| = e(-1) E(q). \tag{2.15}$$

(b) Thus, by $L^1(d\eta)$ we mean $L^1(|d\eta|)$.

Finally, before proceeding to the transform theory, we consider briefly the $q$-Heisenberg algebra [F2]. Define the shift operator $R$ by $R(v)_n = (v)_{n+1}$. One readily checks that $[W, R] = WR - RW = Z$, where $Z(v)_n = (1 - q) q^n(v)_n$. It follows that $WZ = qZW$, $ZR = qRZ$. That is, $Z$ is "$q$-central," i.e., defining the $q$-commutator by $[A, B]_q = AB - qBA$ we have $[W, Z]_q = [Z, R]_q = 0$, analogously as for the usual Heisenberg algebra. In the case $q = +1$ we have "bosons," $q = -1$ "fermions" in physics terminology [F4]. The $q$-Hilbert space with basis $\psi_n = (v)_n$ is isomorphic to the $q$HS in [F2], the $q$-Fock space. $W$ and $R$ are adjoints when the inner product is defined by

$$(\psi_n, \psi_m) = (q)_n \delta_{nm}. \tag{2.16}$$

It is easy to check that $W$ and $R$ are bounded operators on this space, in fact, contractions. This is one reason why the "harmonic analysis" of this work is much simpler, at least initially, than for the usual case, namely we have bounded operators underlying the analysis. Contrast this with the unbounded operators $x, d/dx$ in the usual case.

III. Transform Theory

First, recall the measures $d\psi, d\eta$:

$$\text{supp}(d\psi) = \{q^n\}_{n \geq 0}, \quad \text{supp}(d\eta) = \{q^n\}_{n \leq 0}. \tag{3.1}$$

Introduce the notation

$x \leq d\psi$ means that the variable $x$ takes values in $\text{supp}(d\psi)$ \hspace{1cm} \text{(3.2)}$

and similarly for $d\eta$. Define the $E$-kernel

$$E(a \mid b) = E(ab)/E(b) = (b)_\infty/(ab)_\infty. \tag{3.3}$$

We have

$$E(q^n \mid b) = (b)_n, \quad E(a \mid q^k) = (q^k)_n \quad \text{for} \quad a = q^n, \quad n \in \mathbb{Z}, \quad 0 \text{ otherwise,} \tag{3.4}$$

$$\int E(x \mid t) p(x) d\psi(x) - \sum_{0}^{\infty} (t)_n p_n q^n, \quad p_n - p(q^n). \tag{3.5}$$
We will study the transform pairs ($p_n$ are not necessarily positive)

$$f(v) = \sum_{n=0}^{\infty} (v)_n p_n q^n = \int E(x \mid v) p(x) \, d\psi(x)$$

(3.6)

$$p(x) = \int e(x y q) f(y) \, d\eta(y), \quad x \ll d\psi.$$  

Introduce the kernels

$$H(x, y) = \int e(x s q) E(y \mid s) \, d\eta(s), \quad x, y \ll d\psi$$

(3.7)

$$K(x, y) = \int E(s \mid x) e(s y q) \, d\psi(s), \quad y \ll d\eta.$$  

We will see that these act as $\delta$-functions as one might expect.

**First Inversion Formula**

We will see that for $p \in L^1(d\psi), p(x) = \int H(x, y) p(y) \, d\psi(y).$ We need the following:

**Lemma 3.1.**

$$(q)_n + i q^{\frac{l}{2}}(q)_l = (-1)^n q^n (Q^{n+l})_n q^{\frac{n+l}{2}}.$$  

The proof follows readily by induction on $l$, starting from $l = 0.$

**Proposition 3.2.**

$$H(x, y) = y^{-1} \int e(x s q y^{-1}) \, d\eta(s), \quad x, y \ll d\psi.$$  

**Proof.**  Let $y = q^n, n \geq 0.$ $H(x, q^n) = \sum_0^{\infty} (x q^k q)_{\infty} (Q^k)_n \eta_k.$ This should equal $Q^n \sum_0^{\infty} (x q Q^{l+n})_{\infty} \eta_l.$ Lemma 3.1 implies

$$Q^n \eta_l = (Q^{n+l})_n \eta_{n+l}.$$  

Multiply both sides by $(x q Q^{l+n})_{\infty}$ and sum over $l$:

$$Q^n \sum_0^{\infty} (x q Q^{l+n})_{\infty} \eta_l = \sum_0^{\infty} (x q Q^{l+n})_{\infty} (Q^{n+l})_n \eta_{n+l}.$$  

Now replace $l + n$ by $k$ to get $\sum_{k \geq n} (x q Q^k)_{\infty} (Q^k)_n \eta_k.$ Since $(Q^k)_n = 0$ for $k < n,$ the desired formula results.
PROPOSITION 3.3.

\[ \int e(qsxxy^{-1}) \, d\eta(s) = \delta(x - y), \quad x, y \ll d\psi. \]

Proof. Let \( x = q^m \), \( y = q^n \). As in the previous proposition we are considering \( \sum_0^\infty (xqQ^{-l-n})_\infty \eta_l \). If \( m < n \), then \( (q^mq^{-l-n})_\infty = 0, l \geq 0 \). So assume \( m \geq n \). Then we have

\[
\sum_0^\infty (xqQ^{l+n})_\infty \left( -1 \right)^l \frac{q(l)}{(q)_l} (q)_\infty (q)_\infty
\]

If \( m > n \), \( xQ^n < 1 \) and the \( q \)-binomial theorem yields

\[
(qxQ^n)_\infty (1)_\infty / (xQ^n)_\infty (q)_\infty = 0.
\]

On the other hand, if \( m = n \), only the term \( l = 0 \) remains and the result is 1.

Combining the above propositions yields

THEOREM 3.4.

\[
\int e(xsq) \, E(y \mid s) \, d\eta(s) = y^{-1} \delta(x - y), \quad x, y \ll d\psi.
\]

COROLLARY 3.5: FIRST INVERSION FORMULA. If \( p \in L^1(\partial \psi) \) and \( f(v) = \int E(x \mid v) \, p(x) \, d\psi(x) \), then \( f \in L^1(\partial \eta) \) and \( p(x) = \int e(xvq) \, f(y) \, d\eta(y), x \ll d\psi. \)

Proof. The integral for \( p \) exists since the sum \( \sum_0^\infty (xQ^kq)_\infty f(Q^k) \eta_k \) is finite for \( x \ll d\psi \). Let \( x = q^m \), \( t = q^n \), \( y = Q^k \):

\[
\int \left| e(xvq) \, E(t \mid y) \, p(t) \right| \, d\psi(t) \, d\eta(y)
\]

\[\leq \sum_{n \leq k \leq m} (q^n + 1 - k)_\infty |(Q^k)_n| q^n \, p_n \, |(Q^k)_n| \eta_k |]

\[\ll \sum_{n \leq k \leq m} |(Q^k)_n| \eta_k | q^n \, p_n |, \]
where this sum bounds the $L^1(\|d\eta\|)$ norm of $f$. From (2.5):

\[
\sum_{k \geq n} |(Q^k)_n| \eta_k \leq \sum_{k=n+1}^l Q^{n^2+n^2l} q^{(n)}_2 q^{(n)}_2 q^{(l)}_2 + n^l \\
\leq Q^n \sum_{l} q^{(l)}_2 = C_0 Q^n.
\]

Thus we have the bound $C_0 \sum |p_n| < \infty$ for the double integral. Then Theorem 3.4 yields the result. 

\textit{The K-Kernel and the Second Inversion Formula}

We will show that the $K$-kernel, (3.7), is the integral kernel for the operator $P$ (recall (2.3)). First, we see that $K$ can be expressed as a "Cauchy kernel."

\textbf{PROPOSITION 3.6.}

\[ K(x, y) = (x)_{\infty} (y-x)^{-1}, \quad y \ll d\eta. \]

\textbf{Proof.} $K(x, Q^k) = \sum_{0}^{\infty} (x)_n q^n(q^{n+1-k})_{\infty}$. Substituting $n = k + l$:

\[
\sum_{0}^{\infty} (x)_{k+l} q^{k+l}(q^{l+1})_{\infty} \\
= (x)_k q^k \sum_{0}^{\infty} (xq^k)_l q(q)_{\infty}/(q)_l \\
= (x)_k q^k (xq^k+1)_{\infty}, \quad \text{by the } q\text{-binomial theorem}, \\
= q^k(x)_{\infty} (1-xq^k)^{-1} = (x)_{\infty} (y-x)^{-1}. \]

Next we show that $K$ is the integral kernel for the operator $P$. In particular, $K$ acts as a reproducing kernel on the set \{f= Pf\}.

\textbf{Proposition 3.7.} For $f \in L^1(\eta)$, $Pf$ exists, i.e., $L^1(\eta) \subset \mathcal{D}(P)$, and

\[ Pf(x) = \int K(x, y) f(y) \, d\eta(y). \]

\textbf{Proof.}

\[ K(x, Q^k) = \sum (x)_n q^n(q^{n+1-k})_{\infty} = \sum (x)_n q^n(q)_{\infty}/(q)_{n-k}. \]
First we show that, for fixed \( x \), \( K \) is bounded for \( y \ll d\eta \):

\[
|K(x, Q^k)| = \left| \sum_{n \geq k} (x)_n q^n(q) \infty (q)_{n-k}^{-1} \right|
= \left| \sum_{l} (x)_{k+l} q^{k+l}(q) \infty (q)_{l}^{-1} \right|
\leq q^k e(-|x|).
\]

Thus we can interchange summations as follows:

\[
\sum K(x, Q^k) \eta_k f(Q^k)
= \sum \sum (x)_n q^n (-1)^k q^{(k\frac{5}{2})} f(Q^k)/(q)_{n-k} (q)_{k}
= \sum ((x)_n q^n/(q)_n) \sum_{k} \binom{n}{k} (-1)^k q^{(k\frac{5}{2})} f(Q^k)
= \sum (x)_n (q)_n^{-1} W^n f(1), \quad \text{by Proposition 2.1}
= Pf(x).
\]

**Proposition 3.8.** If \( f \in L^1(d\eta) \), then \( \left| f \right|_{\text{supp}(d\eta)} = Pf \big|_{\text{supp}(d\eta)} \).

**Proof.** From Proposition 2.1,

\[
W^n f(1) = q^n \sum_{k} \binom{n}{k} (-1)^k q^{(k\frac{5}{2})} f(Q^k)
= q^n(q) \infty \sum (q)_n(q)_{n-1}^{-1} \eta_l f(Q^l).
\]

Thus

\[
Pf(Q^k) = \sum_{n} (Q^k)_n q^n(q) \infty \sum_{n \geq l} \eta_l f(Q^l)/(q)_{n-l}, \quad \text{with } n = l + m,
\]

\[
= \sum \sum (Q^k)_{l+m} q^{m}(q)_{m}^{-1} (q) \infty q^l f(Q^l) \eta_l
= \sum \sum (Q^{k-l})_m q^{m}(q)_{m}^{-1} (q) \infty (Q^k), q^l \eta_l f(Q^l).
\]
where it is easily seen that the double sum is in fact finite. Applying the $q$-binomial theorem yields

$$\sum_{l} (Q^{k-l-1})_{\infty} (Q^{k})_{l} q^{l} \eta_{l} f(Q).$$

The first term forces $k-l-1 < 0$ and the second $k \geq l$, i.e., $l < k < l+1$ reducing to $(q)_{\infty} (Q^{k})_{k} q^{k} \eta_{k} f(Q^{k}) = f(Q^{k})$ by (2.5) and the definition of $\eta_{k}$. □

**Theorem 3.9. Second Inversion Formula.** Given $f \in L^{1}(d\eta)$, let $p(x) = \int e(xyq) f(y) \, d\eta(y), \, x \ll d\psi$. Then $p \in L^{1}(d\psi)$ and

$$f(v) = \int E(x \mid v) \, p(x) \, d\psi(x), \, v \ll d\eta.$$

**Proof.** Denote the $L^{1}(|d\eta|)$ norm of $f$ by $\|f\|$. Then

\[
\int \|p(x)\| \, d\psi(x)
\leq \sum \sum (q^{n+1-k})_{\infty} \|f(Q^{k})\| q^{n} |\eta_{k}|
\leq \sum_{k} |f(Q^{k})| |\eta_{k}| \sum_{n \geq k} q^{n}(q^{n-k+1})_{\infty}, \quad \text{with} \quad n = k + l,
\leq \|f\| \sum_{0}^{\infty} q^{l}(q^{l})_{\infty} \leq \|f\| (1 - q)^{-1}.
\]

Now use Proposition 3.7 and the definition of $K$ to find that $Pf(v) = \int E(x \mid v) \, p(x) \, d\psi(x)$. Then the result follows by Proposition 3.8. □

**Parseval-Plancherel Formulas**

The above discussion presents the basic transform theory. As corollaries of those results we have two Parseval-Plancherel formulas.

**Theorem 3.10.** If $f \in L^{1}(d\psi)$, $g \in L^{1}(d\psi)$, then

\[
\int f(x) \, g(x) \, d\psi(x) = \int d\eta(s) \left[ \int f(x) \, e(xsq) \, d\psi(x) \right] \left[ \int E(y \mid s) \, g(y) \, d\psi(y) \right].
\]

**Proof.** Denote by $\hat{g}$ the $E$-transform of $g$ on the RHS. Since $g \in L^{1}(d\psi)$, $g$ is bounded on $\text{supp}(d\psi)$ so the LHS exists. On the RHS we know that for $x \ll d\psi$, $s \ll d\eta$, $|e(xsq)| \leq 1$ since it takes the form $(q^{n+1-k})_{\infty}, \, x = q^{n},$
\[ s = Q^k. \] Thus we have the bound for the RHS: \[ \| \hat{g} \| \cdot \| f \|, \] where \( \hat{g} \in L^1(\eta) \) by Corollary 3.5. Now by Theorem 3.4 the RHS reduces to

\[
\int \int f(x) g(y) \delta(x-y) d\psi(y) d\psi(x) = \int f(x) g(x) d\psi(x).
\]

Similarly, we have

**Theorem 3.11.** If \( \sum_{k} |f(Q^k)| < \infty, g \in L^1(\eta), \) then

\[
\int f(x) g(x) d\eta(x) = \int d\psi(s) \left[ \int e(y) g(y) d\eta(y) \right] \left[ \int E(s|y) f(x) d\eta(x) \right].
\]

**Proof.** With \( s = q^n, y = Q^k, x = Q^l, \) the RHS (with absolute values) is bounded by

\[
\sum_{l \geq n \geq k} \sum_{r} q^n (q^{n-k+1}) \sum_{s \geq r} |g(Q^k)| |f(Q^l)| |E(Q^l)| |\eta_s| |\eta_k|
\]

\[
\leq \| g \| \sum_{l \geq n \geq k} \sum_{r} q^n q^r \sum_{s \geq r} |f(Q^{n+r})|
\]

\[
\leq \| g \| C_0 \sum |f(Q^k)|, \quad \text{as in the proof of Corollary 3.5.}
\]

Now, by definition of the \( K \)-kernel the RHS equals

\[
\int \int K(x, y) f(x) g(y) d\eta(x) d\eta(y) = \int f(x) P g(x) d\eta(x)
\]

\[
= \int f(x) g(x) d\eta(x)
\]

by Propositions 3.7 and 3.8. This last integral exists since \( f \in L^\infty(\eta). \)

**Complementary Functions**

For \( f \in L^1(\eta), \) \( Pf \) exists and \( f - Pf \) vanishes identically on \( \text{supp}(\eta). \) We will find an integral representation for this complementary function below. First, let us look at some series representations of functions that vanish on \( \text{supp}(\eta). \)
Define the $\lambda$-transform

$$\lambda(f)(x) = \int f(qxy) \, d\eta(y), \quad \text{for } x \ll d\psi$$

and consider

$$f(v) = \int e(vx^{-1}) \, \lambda(x) \, d\psi(x)$$

for some $\lambda(x)$. This latter sum equals, with $\lambda_n = \lambda(q^n)$,

$$\sum_{n=0}^{x} (vQ^n)_n \lambda_n = (v)_x \sum_{n=0}^{x} (vQ^n)_n \lambda_n.$$  

Then $f$ has the following properties:

1. For $vQ \ll d\psi$, the sum in (3.9) is finite. That is, if $v = q^k$, $k \geq 1$,

$$\sum_{n=0}^{k-1} (q^kQ^n)_n \lambda_n = \sum_{n=0}^{k-1} (q^kQ^n)_n \lambda_n.$$  

So $f(v)$ always exists for $v = q^k$, $k \geq 1$.

2. For $v \ll d\eta$, $f(v) = 0$, since $(Q^m)_\infty = 0$, $m \geq 0$. Thus, $Pf$ exists and, by Proposition 3.8, is identically zero. The $\lambda$-transform $\lambda(f)$ exists on $\text{supp}(d\psi)$ since the sum in (3.8) will be finite.

For $y = q^n$,

$$\int f(qxy) \, d\eta(x) = \sum_{k=0}^{n} f(q^{n+1-k}) \eta_k$$

since $f$ vanishes on $\text{supp}(d\eta)$. In fact, $\lambda(f) = \lambda$.

**Proposition 3.12.** If

$$f(v) = \int e(vy^{-1}) \, \lambda(y) \, d\psi(y), \quad Qv \ll d\psi$$

then

$$\lambda(x) = \int f(qxs) \, d\eta(s), \quad x \ll d\psi.$$
Proof. Let $x = q^n$. We have

$$\int f(qxs) \, d\eta(s)$$

$$= \int \int e(qxsy^{-1}) \lambda(y) \, d\eta(s) \, d\psi(y)$$

$$= \int \delta(x - y) \lambda(y) \, d\psi(y) \quad \text{by Proposition 3.3}$$

$$= \lambda(x).$$

Observe that the sums are in fact finite:

$$\sum \sum (q^{n+1-k-l})_\infty \lambda_n \eta_k$$

reduces to a sum over $k + l \leq n$. 

(3) Observe that for $v \neq 0$,

$$f(v) = (v)_\infty \sum_0^\infty (v Q^n)_n \lambda_n \lambda_n = (v)_\infty \sum_0^\infty (-1)^n v^n Q^n \binom{n}{2} \lambda_n (q/v)_n.$$ (3.13)

So $f$ will be analytic for $|v| < R$ when $\lim_{n \to \infty} |\lambda_n|^{1/n} Q^{n/2} \leq \sqrt{q/R}$. Thus $f$ will be analytic in some disk, while $ Pf $ is identically zero.

Now suppose $f \in L^1(d\eta)$ is entire. Then $f - Pf$ vanishes on $1, Q, Q^2, \ldots$, so that $\gamma(f)(v) = (v)^{-1} (f(v) - Pf(v))$ is holomorphic. $\gamma(f)$ is the "complementary function." It has an integral representation reminiscent of the Hilbert transform.

**Theorem 3.13.** Let $f \in L^1(d\eta)$ be entire. Define $\gamma(f)$ by the relation $f(v) = Pf(v) + (v)_\infty \gamma(f)(v)$. Then

$$-\gamma(f)(v) = \int \frac{f(y) - f(v)}{y - v} \, d\eta(y).$$

Proof. By Propositions 3.6 and 3.7,

$$\gamma(f)(v) = (v)^{-1} f(v) - \int f(y)(y - v)^{-1} \, d\eta(y).$$
We check that \( \int (y-v)^{-1} \, d\eta(y) = (v)^{-1} \). This follows from Proposition 3.7 by considering the function \( f_0(x) = 1 \), for all \( x \):

\[
1 = f_0(x) = Pf_0(x) = \int K(x,y) f_0(y) \, d\eta(y) = (x) \int (y-x)^{-1} \, d\eta(y).
\]

IV. SPACES OF \( q \)-EXPONENTIAL TYPE

This is the "second part" of the study. Here we give more precise information about the operator \( P \). For example, we give some conditions for \( f = Pf \) to hold. (Theorems 4.2, 4.7, and part of 4.6 appear in [F1] in the probabilistic context.)

**Proposition 4.1.** \( f(v) = \sum_{n=0}^{\infty} a_n(v) \) is an entire function if and only if the series \( \sum_{n=0}^{\infty} a_n \) converges.

**Proof.** The necessity follows by evaluating at \( v = 0 \). For the sufficiency: Let \( A_n = \sum_{k=0}^{n-1} a_k, A_0 = 0, A_{\infty} = \sum_{n=0}^{\infty} a_k \). Then (summation by parts)

\[
\sum A_{n+1}((v)_{n+1} - (v)_n) + (v)_n(A_{n+1} - A_n)
\]

\[
= A_{\infty}(v)_{\infty} = \sum A_{n+1}(-qv^n(v)_n) + \sum a_n(v)_n.
\]

That is,

\[
\sum a_n(v)_n = A_{\infty}(v)_{\infty} + v \sum A_{n+1}q^n(v)_n.
\]

Since \( \sup |A_n| < \infty \), this last series converges uniformly on compact sets.

Now we will look at some stronger conditions. We make the definitions:

For \( \delta \geq 0 \), \( Q_\delta = \{ f \text{ entire}: \exists \text{ constant } C, |f(v)| \leq Ce(-q^\delta |v|) \} \). \hspace{1cm} (4.1)

Notice that \( e(-q^\delta |v|) = \prod_{n=0}^{\infty} (1 + |v| q^{\delta-n}) \leq \exp(q^\delta |v|(1-q)^{-1}) \).

\[
Q_\sigma = \left\{ f \text{ defined on } \text{supp}(d\eta): \sum_{k=0}^{\infty} |f(Q^k)| E(-Q^k) < \infty \right\}.
\]
Define the maximal function on \((0, \infty)\):

\[
Mf(r) = \sup_{|v| = r} |f(v)|
\]

for entire functions \(f\).

We now come to the main theorems of this section.

**Theorem 4.2.** \(f \in Q_\sigma\) if and only if \(Pf\) exists with \(Pf(v) = \sum_0^\infty f_n(v)\), \(\{f_n\} \in l_1\).

**Proof:** Let \(e(k) = |f(Q^k)| E(-Q^k)\).

**Sufficiency.** First we remark that since the proof of Proposition 3.8 involves only finite sums, we really need just the existence of \(Pf\) to have \(f = Pf\) on \(\text{supp}(d\eta)\). Thus

\[
e(k) \leq E(-Q^k) \sum_n |f_n| |(Q^k)_n|, \quad \text{substituting } n = k - l,
\]

\[
= E(-Q^k) \sum_l |f_{k-l}| |(Q^k)_{k-l}|
\]

\[
\leq \sum_l |f_{k-l}| q^k q^{\frac{1}{2}}(q)_i^{-1}, \quad \text{using (2.5)}.
\]

Thus \(\sum e(k) \leq \left(\sum |f_n| \right) (q)_\infty\).

**Necessity.** By Proposition 2.1,

\[
|f_n| = q^n(q)_n^{-1} \left| \sum_k \binom{n}{k} (-1)^k q^{\frac{k}{2}} f(Q^k) \right|
\]

\[
\leq q^n(q)_n^{-1} \sum \binom{n}{k} q^{\frac{k}{2}} e(k) e(-Q^k).
\]

Replacing \(n\) by \(k + l\) and summing over \(l \geq 0\),

\[
\sum |f_n| \leq e(-1) E(q) \sum q^k q^{\frac{k}{2}} e(k)(-Q^k)_k/(q)_k
\]

\[
\leq C \sum e(k)
\]

since \(q^k q^{\frac{k}{2}}(-Q^k)_k(q)_k^{-1} = (-q)^k (q)_k^{-1} \leq (-q)_\infty(q)_\infty^{-1}\).

**Corollary 4.3.** \(f \in Q_\sigma\) implies \(Pf \in Q_0\).
Proof.
\[
\left| \sum f_n(v)_n \right| \leq \sum |f_n|(-|v|)_n \leq e(-|v|) \sum |f_n|.
\]

It now follows that

**COROLLARY 4.4.** If holomorphic, \( f \in Q_\sigma \), implies \( f = Pf + g \), where \( Pf \in Q_0 \) and \( g = (v) \gamma \), with \( \gamma \) holomorphic.

**THEOREM 4.5.** \( f \in Q_\delta, \delta \geq 0 \), if and only if \( f = \sum_{n=0}^{\infty} c_n v^n, |c_n| \leq C q^n \delta + (\frac{n}{2}) \).

(Remark: \( C \) is used here to denote a generic constant.)

Proof. Sufficiency: \( |f(v)| \leq C \sum |v|^{n} q^{n \delta} + (\frac{n}{2}) \leq Ce(-q^\delta |v|). \)

Necessity. \( \sup_{|v| = q^\delta} |f(v)| \leq Ce(-q^\delta Q_k). \) Thus, by Cauchy's estimates on \( |v| = Q'^{n} \),

\[
|c_n| \leq Ce(-q^\delta Q^n) / (Q'^{n})^n = Ce(-q^\delta) \frac{1 + q^\delta Q}{Q^n} \ldots \frac{1 + q^\delta Q^n}{Q^n}
\]

\[
= Ce(-q^\delta) q^{\frac{n}{2}} q^{n \delta} (-q q^{-\delta})_n
\]

\[
\leq Ce(-q^\delta) e(-q q^{-\delta}) q^{\frac{n}{2}} + n^\delta.
\]

We are now prepared to derive a condition for \( f = Pf \) in terms of the maximal function.

**THEOREM 4.6.** \( Mf \in Q_\sigma \) if and only if \( \sum Q^{(\frac{k}{2})} |c_n| < \infty \), where \( f(v) = \sum_{n=0}^{\infty} c_n v^n \) is entire.

Proof. Sufficiency: \( |f(v)| \leq \sum |c_n| |v|^n, Mf(r) \leq \sum |c_n| r^n \) with \( |c_n| = q^{(\frac{k}{2})} \epsilon(n), \sum \epsilon(n) < \infty \). Thus

\[
\sum Mf(Q^k) E(-Q^k)
\]

\[
\leq \sum \sum E(-Q^k) Q^{nk} q^{\frac{k}{2}} \epsilon(n)
\]

\[
\leq E(-1) \sum \sum q^k q^{\frac{k}{2}} Q^{nk} \frac{1}{k} q^{\frac{k}{2}} \epsilon(n),
\]

since \( (q)_k \leq (-Q_k) q^k + (\frac{1}{2}) \)

\[
= E(-1) \sum (-Q^n)^\infty q^{\frac{k}{2}} \epsilon(n) \leq e(-1) \sum \epsilon(n).
\]
Necessity. As in the proof of Theorem 4.5, with $\delta = 0$, Cauchy's estimates yield $|c_n| \leq Mf(Q^n)/(Q^n)^n = \delta(n) e(-Q^n) q^n \leq Cq^n(\frac{n}{2})\delta(n)$, where $\sum \delta(n) < \infty$.

In conclusion, we have

**Theorem 4.7.** $Mf \in Q_{\sigma}$ implies $f = Pf$.

**Proof.** Let $e(n) = |c_n| Q^\frac{n}{2}$ where $f(v) = \sum_0^{\infty} c_n v^n$. On $|v| = R$, using (2.8),

$$|v|^n \leq \sum \binom{n}{k} q^\frac{k}{2} Q^n q^k (-R)_k \leq (-R)^n (-qQ^n)_n, \quad \text{by (2.6).}$$

Thus,

$$\sum |c_n| |v|^n \leq (-R)^n \sum e(n) q^\frac{n}{2} (-qQ^n)_n \leq (-1)^n (-R)^n \sum e(n).$$

Then by Fubini's theorem we have, using (2.7),

$$f(v) = \sum c_n v^n = \sum c_n \sum (Q^n)_k q^k (q)^{-1} (v)_k = \sum f_k(v)_k = Pf(v).$$

**References**


