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THE DIAGONAL COMULTIPLICATION ON HOMOLOGY

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1. Introduction

This paper describes the diagonal comultiplication (or cup coproduct) defined on integral homology modules of groups. Analysis of this coproduct should provide a new method of testing for non-isomorphism of groups which have isomorphic integral homology modules; here, the dimension two coproduct is applied to this problem.

The first part (Section 2) is couched in terms of groupnets (Brandt groupoids) and shows two things: that there exists a cup product defined on the integral cohomology of any groupnet, extending that for groups, and that there exists a comultiplication defined on the integral homology of any group, natural up to dimension two, which gives the homology modules the structure of a commutative graded co-ring.

In the second part (Sections 3 and 4), this diagonal comultiplication Ω is constructed to dimension two, and the information it carries about the lower central series of a group G is investigated. Modulo torsion in $H_1(G;\mathbf{Z})$, Ω_2 induces an abelian group homomorphism with cokernel G_2/G_3, which distinguishes between large classes of groups, in particular the one-relator groups with non-trivial multiplicator, and the finitely-generated nilpotent groups of class two whose relators are all in the commutator subgroup.

2. Diagonal maps

Groupnets generalise groups and provide a very natural means of describing groups which are graph products (fundamental groups of graphs of groups in the terminology of Bass and Serre).

For the groupnet theory involved see [1] and for exposition of the theory of ringnets and the homology theory of groupnets see [5]. The following notation is employed: for any groupnet A, TA is the trivial A-module and $\mathbf{Z}A$ is the

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groupringnet of A. If A is *connected*, its (co)homology with coefficients in TA is the same as the integral (co)homology of the loop group of A.

2.1. Definition. For a groupnet A, the *diagonal map* $\Delta: A \to A \times A$ is defined as $\Delta(a) = (a, a)$ for a in A. It associates, so that the induced ringnet morphism also does:

$$\begin{array}{ccc} \mathbf{Z}A & \xrightarrow{\Delta} & \mathbf{Z}A \otimes \mathbf{Z}A \\ {\scriptstyle \Delta}\downarrow & \circ & \downarrow{\scriptstyle 1 \otimes \Delta} \\ \mathbf{Z}A \otimes \mathbf{Z}A & \xrightarrow[\Delta \otimes 1]{} & \mathbf{Z}A \otimes \mathbf{Z}A \otimes \mathbf{Z}A \end{array}$$

If C is a free A-resolution of TA, then $C \otimes_{\mathbf{Z}} C$ is a free $A \times A$-resolution of $TA \otimes_{\mathbf{Z}} TA$. The next result is an immediate consequence of the regular comparison theorem [5, 6.1]. (The definition of $\chi(\Delta)$-homotopy of chain maps, where $\chi(\Delta)$ is the constant ringnet homotopy induced by Δ, is given in [5, 4.2]. It extends the usual definition of homotopy of chain maps between complexes over a ring by accounting for chain maps between complexes over different rings.)

2.2. Lemma. *If C is a free A-resolution of TA for a groupnet A, there exists a Δ-chain map $\omega: C \to C \otimes_{\mathbf{Z}} C$ such that $\partial\omega_0 = \bar{\Delta}\partial$, where $\bar{\Delta}$ is the induced Δ-morphism $\bar{\Delta}: TA \to TA \otimes_{\mathbf{Z}} TA$. Moreover, any two such chain maps are $\chi(\Delta)$-homotopic.*

Any Δ-chain map $\omega: C \to C \otimes_{\mathbf{Z}} C$ lifting $\bar{\Delta}: TA \to TA \otimes_{\mathbf{Z}} TA$ which satisfies $\omega(c) = c \otimes c$ for all generators c of the A-module C_0 is called a *diagonal approximation*; clearly such maps exist.

2.3. Lemma. *The diagonal map $\Delta: A \to A \times A$ induces a homology map*

$$H_*(\Delta): H_*(A; TA) \to H_*(A \times A; TA \otimes_{\mathbf{Z}} TA)$$

and a cohomology map $H^*(\Delta): H^*(A \times A; L) \to H^*(A, L)$ for any left $A \times A$-module L.

Proof. Let C be any free A-resolution of TA, and let $\omega: C \to \otimes_{\mathbf{Z}} C$ be any Δ-chain map lifting $\bar{\Delta}: TA \to TA \otimes_{\mathbf{Z}} TA$. Then

$$\bar{\Delta} \otimes \omega: TA \otimes_A C \to (TA \otimes_{\mathbf{Z}} TA) \otimes_{A \times A} (C \otimes_{\mathbf{Z}} C)$$

determines the homology map and $\langle \omega, L \rangle: \hom_{A \times A}(C \otimes_{\mathbf{Z}} C, L) \to \hom_A(C, L)$ determines the cohomology map. □

The diagonal map induces a cohomology map

$$H^*(A; L) \otimes_{\mathbf{Z}} H^*(A; L) \to H^*(A; L \otimes_{\mathbf{Z}} L)$$

for any left A-module L. It is the composite map

$$H^*(A;L)\otimes_{\mathbf{Z}} H^*(A;L) \xrightarrow{p} H^*(\hom_A(C,L)\otimes_{\mathbf{Z}} \hom_A(C,L)) \xrightarrow{H^*(\eta)}$$
$$H^*(A\times A; L\otimes_{\mathbf{Z}} L) \xrightarrow{H^*(\Delta)} H^*(A; L\otimes_{\mathbf{Z}} L),$$

where $H^*(\eta)$ is the homology map induced from

$$\eta:\hom_A(C,\ L)\otimes_{\mathbf{Z}} \hom_A(C,L)\to \hom_{A\times A}(C\otimes_{\mathbf{Z}} C, L\otimes_{\mathbf{Z}} L),$$

with $[\eta(f\otimes g)](c\otimes c^*)=f(c)\otimes g(c^*)$, for a projective A-resolution C of TA. The map p is a specific example of the *external homology product*, defined for any standard right R-complex K and any standard left R-complex L to be the map

$$p: H(K)\otimes_R H(L)\to H(K\otimes_R L)$$

given by tensor extension of

$$p([u]\otimes[v])=[u\otimes v] \qquad \forall [u]\in H_k(K),\quad [v]\in H_l(L).$$

When $L=TA$, $H^*(A;L\otimes_{\mathbf{Z}} L)$ may be replaced by $H^*(A;L)$ and the diagonal map induces the *cup product*

$$\cup: H^*(A;TA)\otimes_{\mathbf{Z}} H^*(A;TA)\to H^*(A;TA).$$

As in the case when A is a group, the cup product induces a commutative graded ring structure on the cohomology module $H^*(A;TA)$.

The next two results restate those for groups (see, for example, [7, V.10.4, V.8.6]).

2.4. Theorem (the Künneth formula). *If A is a connected groupnet there is a split short exact sequence*

$$0\to[H(A;TA)\otimes H(A;TA)]_n \xrightarrow{\bar p} H_n(A\times A; TA\otimes_{\mathbf{Z}} TA)$$
$$\to[\mathrm{Tor}_1(H(A;TA),H(A;TA))]_{n-1}\to 0 \qquad \text{(D2.1)}$$

for each n in $\mathbf{Z}$ (although the splitting is not natural).

The map $\bar p$ is the composition of the external homology product with the 'middle four interchange' isomorphism:

$$(TA\otimes_A C_p)\otimes_{\mathbf{Z}} (TA\otimes_A C_q)\cong (TA\otimes_{\mathbf{Z}} TA)\otimes_{A\times A}(C_p\otimes_{\mathbf{Z}} C_q)$$

for all nonnegative integers p and q.

2.5. Corollary. *If A is a connected groupnet there is a natural isomorphism*

$$[H(A;TA)\otimes_{\mathbf{Z}} H(A;TA)]_n\cong H_n(A\times A; TA\otimes_{\mathbf{Z}} TA),\quad 0\le n\le 2.$$

2.6. Corollary. *The diagonal map $\Delta: A \to A \times A$ for a connected groupnet A induces $\Omega: H_*(A; TA) \to H_*(A; TA) \otimes_{\mathbf{Z}} H_*(A; TA)$. In dimensions* 0, 1 *and* 2 *it is unique; in higher dimensions it is unique to within the splitting isomorphism of the Künneth Formula.*

Such a homology map is called a *diagonal comultiplication*. It induces a graded co-ring structure on the homology module $H_*(A; TA)$ which commutes (that is, $a \cdot b = (-1)^{\deg a \deg b} b \cdot a$ always) and associates to within the splitting isomorphism of Theorem 2.4, by [4, 5.2.12].

As usual with proofs involving the comparison theorem, the existence of a required chain map is comparatively easy to demonstrate but its construction is often difficult. There is always a diagonal approximation for the bar resolution of a group [7, VIII.9, Exercise 1]; this construction is, of course, extremely cumbersome to manipulate on the homology level. Explicit construction to dimension two of a simpler diagonal approximation is given in Section 3.

3. The Gruenberg approximation

In this section a diagonal approximation for any group is evaluated to dimension 2, using the Gruenberg resolution [3, p. 218] and thus requiring knowledge of a free presentation of the group. The corresponding diagonal comultiplication is then calculated.

3.1. The Gruenberg approximation. Let $1 \to R \to F \overset{\pi}{\to} G \to 1$ be a free presentation of a group G and let X and Y be free generating sets for F and R respectively. If

$$\mathscr{G} = \mathbf{Z}G \otimes_F \mathscr{F} \otimes R/R_2 \to \mathbf{Z}G \otimes R/R_2 \xrightarrow{\partial_2} \mathbf{Z}G \otimes_F \mathscr{F} \xrightarrow{\partial_1} \mathbf{Z}G \to \mathbf{Z}$$

is the left version of the Gruenberg resolution define the maps $\omega: \mathscr{G} \to \mathscr{G} \otimes \mathscr{G}$ on its free generators as follows:

$$\omega_0(1) = 1 \otimes 1;$$

$$\omega_1(1 \otimes (x-1)) = 1 \otimes (1 \otimes (x-1)) + (1 \otimes (x-1)) \otimes x, \quad \forall x \in X;$$

and

$$\omega_2(1 \otimes (y + R_2)) = 1 \otimes (1 \otimes (y + R_2)) + (1 \otimes (y + R_2)) \otimes 1 + \sum_{i=1}^{n} \varepsilon_i \left\{ \left(1 \otimes \left(\sum_j \frac{\partial u_i}{\partial x_j}(x_j - 1)\right)\right) \otimes (1 \otimes u_i(x_i - 1)) \right\},$$

$$\forall y = \prod_{i=1}^{n} x_i^{\varepsilon_i} \in Y.$$

Here u_i is Fox's ith initial section and

$$y-1=\sum_{i=1}^{n}\varepsilon_i u_i(x_i-1)=\sum_j \frac{\partial y}{\partial x_j}(x_j-1)$$

as an element of $\mathscr{F}$ (see [2, Section 2]).

It may be checked that this is a diagonal approximation when extended freely by Δ-action. In particular,

$$(\omega_1\partial_2-(\partial\otimes\partial)_2\circ\omega_2)(1\otimes(y+R_2))=\sum_{i=1}^{n}\varepsilon_i\{(1\otimes u_i(x_i-1))\otimes(\pi(u_ix_i)-1)\}$$
$$+\sum_{i=1}^{n}\varepsilon_i\left\{\left(1\otimes\left(\sum_j\frac{\partial u_i}{\partial x_j}(x_j-1)\right)\right)\otimes\pi(u_i)(\pi(x_i)-1)\right\}$$

and this sum equals zero, for, by induction on n,

$$\sum_{i=1}^{n}\varepsilon_i\{u_i(x_i-1)\otimes(u_ix_i-1)\}+\sum_{i=1}^{n}\varepsilon_i\left\{\left(\sum_j\frac{\partial u_i}{\partial x_j}(x_j-1)\right)\otimes u_i(x_i-1)\right\}=$$
$$=(y-1)\otimes(y-1)$$

in $\mathscr{F}\otimes\mathbf{Z}F$.

Some notational considerations follow. Let w be a word in the free group F on generating set X. For each pair of generators (x, y) of F, the symbol $\langle w; x, y\rangle$ denotes the integer $\varepsilon(\partial^2 w/\partial x\,\partial y)$, where $\varepsilon:\mathbf{Z}F\to\mathbf{Z}$ is the augmentation map of the group ring $\mathbf{Z}F$. For each generator x of F, the symbol $\langle w, x\rangle$ denotes the integer $\varepsilon(\partial w/\partial x)$. That is, for $x\neq y$, $\langle w; x, y\rangle$ is the exponent sum in w of occurrences of x preceding each occurrence of y^{+1}, minus the exponent sum of occurrences of x preceding each occurrence of y^{-1}. For example,

$$\langle xy^{-2}x^2y; x, y\rangle=(-1)+(-1)+3=1.$$

By induction on the length of w, it may be shown that

$$\langle w; x, x\rangle=\tfrac{1}{2}\langle w, x\rangle(\langle w, x\rangle-1)$$

and

$$\langle w; x, y\rangle+\langle w; y, x\rangle=\langle w, x\rangle\langle w, y\rangle.$$

3.2. The diagonal comultiplication. Suppose $\Omega: H_*(G;\mathbf{Z})\to H_*(G;\mathbf{Z})\otimes_{\mathbf{Z}} H_*(G;\mathbf{Z})$ is induced from the Gruenberg approximation (3.1). In dimension 2 it is evaluated as follows. Let $\eta: R\cap F_2/[R,F]\cong H_2(G;\mathbf{Z})$ be the Hopf isomorphism, and suppose $r\in R\cap F_2$. The elements of X appearing in r may be ordered by correspondence with a finite subset of the integers. If

$$r\equiv\prod_{j=1}^{k} y_j^{\delta_j}\text{ modulo }[R,F]\text{ for }y_j\text{ in }Y\text{ and }\delta_j=\pm1,$$

then

$$\Omega_2\eta(r[R,F]) =$$
$$=\sum_y \sum_{x<y} \sum_{j=1}^{k} \delta_j\langle y_j;x,y\rangle\{\pi(x)G_2\otimes\pi(y)G_2-\pi(y)G_2\otimes\pi(x)G_2\}$$
$$=\sum_y \sum_{x<y} \langle r;x,y\rangle\{\pi(x)G_2\otimes\pi(y)G_2-\pi(y)G_2\otimes\pi(x)G_2\},$$

when the image of Ω_2 is restricted to $H_1(G;\mathbf{Z})\otimes H_1(G;\mathbf{Z})$.

4. The diagonal comultiplication and the lower central series

In this section the information carried by the diagonal comultiplication in dimension 2 is investigated, and its application to testing non-isomorphism for various large classes of groups is explained.

Because the restricted image of Ω_2 actually lies inside the symmetric difference $H_1(G;\mathbf{Z})\nabla H_1(G;\mathbf{Z})$ of $H_1(G;\mathbf{Z})$, for our purposes an induced abelian group homomorphism $\mathscr{D}(G)$ replaces Ω_2. Notation is that of Section 3.

4.1. Definition. The map $\mathscr{D}(G):H_2(G;\mathbf{Z})\to H_1(G;\mathbf{Z})\wedge H_1(G;\mathbf{Z})$ is given by

$$\mathscr{D}(G)\circ\eta(r[R,F])=\sum_y \sum_{x<y} \langle r;x,y\rangle\pi(x)G_2\wedge\pi(y)G_2,$$

for $r\in R\cap F_2$.

If $\phi:H_1(G;\mathbf{Z})\nabla H_1(G;\mathbf{Z})\to H_1(G;\mathbf{Z})\wedge H_1(G;\mathbf{Z})$ is the homomorphism which maps $x\otimes y-y\otimes x$ to $x\wedge y$, then $\mathscr{D}(G)$ is induced from Ω_2 *modulo torsion in* $H_1(G;\mathbf{Z})$. If G is any group with a presentation in which $R\subseteq F_2$, then $H_1(G;\mathbf{Z})$ is torsion-free and ϕ is an isomorphism.

That $\mathscr{D}(G)$ is a homomorphism, is an incidental result of the lemma below.

4.2. Lemma. Coker $\mathscr{D}(G)=G_2/G_3$ *for any group* G.

Proof. There are isomorphisms

$$\mu:F_2/F_3[R,F]\cong H_1(G;\mathbf{Z})\wedge H_1(G;\mathbf{Z})$$

induced by $[f,f^*]F_3[R,F]\mapsto\pi(f)G_2\wedge\pi(f^*)G_2$ and

$$F_2/F_3(R\cap F_2)\cong G_2/G_3$$

induced by $[f,f^*]F_3(R\cap F_2)\mapsto[\pi(f),\pi(f^*)]G_3$. Consider the diagram

$$\begin{array}{ccccccc}
R\cap F_2/[R,F] & \xrightarrow{\alpha} & F_2/F_3[R,F] & \longrightarrow & F_2/F_3(R\cap F_2) & \longrightarrow & 1\\
\downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\mu} & & \downarrow & & \\
H_2(G;\mathbf{Z}) & \xrightarrow[\mathscr{D}(G)]{} & H_1(G;\mathbf{Z})\wedge H_1(G;\mathbf{Z}) & \xrightarrow[{[,]}]{} & G_2/G_3 & \longrightarrow & 1
\end{array}$$

where the group morphisms in the top row are induced by inclusion and $[\,,]\,(gG_2\wedge g^*G_2)=[g,g^*]G_3$. The top row is exact and it is apparent the right hand square commutes. Let r be as above. It is possible to write $r=r^*f^*$, where $f^*\in F_3$,

$$r^*=\prod_{i=1}^{n}[x_i,y_i]^{\varepsilon_i},$$

$\{x_i,y_i: 1\leqslant i\leqslant n\}\subseteq X$, $\varepsilon_i=\pm 1$ and $x_i<y_i$, $1\leqslant i\leqslant n$. Clearly $\mu\alpha(r[R,F])=\sum_{i=1}^{n}\varepsilon_i\{\pi(x_i)G_2\wedge\pi(y_i)G_2\}$, so that for any distinct pair of generators x, y appearing in r with $x<y$, the coefficient of $\pi(x)G_2\wedge\pi(y)G_2$ under $\mu\alpha$ is $\langle r^*,[x,y]\rangle$, the exponent sum of commutator $[x,y]$ in r^*. A counting argument considering the contribution of each appearance of y in f^* shows that

$$\langle r;x,y\rangle=\langle r^*;x,y\rangle$$

and an inductive argument on n shows that

$$\langle r^*;x,y\rangle=\langle r^*;[x,y]\rangle.$$

Thus $\mathscr{D}(G)\circ\eta=\mu\circ\alpha$. □

Hence Ω_2 is connected with the third term of the lower central series of G. However, analysis of the diagonal comultiplication provides a finer classification of groups than is given by the first two factor groups of their lower central series, that is, it gives a new method of testing for non-isomorphism of groups with isomorphic first and second integral homology modules.

As mentioned earlier, $\mathscr{D}(G)$ carried the same information as the diagonal comultiplication if G has a free presentation $1\to R\to F\to G\to 1$ with $R\subseteq F_2$. Such groups include the one-relator groups with non-trivial multiplicator: their diagonal comultiplication is completely analysed elsewhere [6]. Here we analyse the finitely generated nilpotent class 2 groups of this kind.

4.3. Lemma. *Let G be a finitely generated nilpotent group of class 2 which has a free resolution $1\to R\to F\to G\to 1$ with $R\subseteq F_2$. Then G is determined solely by $\mathscr{D}(G)$.*

Proof. Let $F^*=F/F_3$ and $R^*=R/F_3$ so that $1\to R^*\to F^*\xrightarrow{\pi}G\to 1$ is exact, $F_2^*=F_2/F_3$ and $R^*\subseteq F_2^*$. There is a diagram

$$\begin{array}{ccccccc}
H_2(G;\mathbf{Z}) & \xrightarrow{\mathscr{D}(G)} & \bigwedge_2 H_1(G;\mathbf{Z}) & \xrightarrow{[\,,]} & G_2 & \longrightarrow & 1\\
\uparrow & & \uparrow\cong & & \uparrow & & \\
H_2(F^*;\mathbf{Z}) & \xrightarrow[\mathscr{D}(F^*)]{} & \bigwedge_2 H_1(F^*;\mathbf{Z}) & \xrightarrow[{[\,,]}]{} & F_2^* & \longrightarrow & 1
\end{array}$$

with exact rows, and vertical maps all induced from π. The central one is an isomorphism. Clearly the right-hand square commutes. For any group H, there is an isomorphism $\psi:\bigwedge_2 H_1(H/H_2;\mathbf{Z})\to H_2(H/H_2,\mathbf{Z})$, and the homomorphism $H\to H/H_2$ induces $\psi\circ\mathscr{D}(H)$, so that the lefthand square is induced from

$$\begin{array}{ccc} G & \longrightarrow & G/G_2 \\ \uparrow{\scriptstyle\pi} & \circ & \uparrow{\scriptstyle\pi} \\ F^* & \longrightarrow & F^*/F_2^* \end{array}$$

and must commute. Since [,] is an isomorphism, $\mathscr{D}(F^*)=0$. Consequently, $\operatorname{Im}\mathscr{D}(G)=\operatorname{Ker}[\,,]\cong\operatorname{Ker}(F_2^*\to G_2)=R^*$. $\square$

To characterise these groups completely, it is necessary to determine the invariants of $\mathscr{D}(G)$, that is, to abstract that information contained in the diagonal comultiplication which is independent of ismorphisms of the first two homology modules. This poses a difficult and as yet unsolved problem in matrix theory. Partial solutions are available on the assumption that $H_2(G;\mathbf{Z})$ is free abelian (for example if G in (4.3) is torsion-free and cd $G\leq 2$). For then, if G is a group with $H_2(G;\mathbf{Z})\cong\mathbf{Z}^n$ and $H_1(G;\mathbf{Z})\cong\mathbf{Z}^m$ and comultiplication $\Omega:\mathbf{Z}^n\to\mathbf{Z}^m\otimes\mathbf{Z}^m$, the problem is to determine the form of a canonical representative of the set of all skew-symmetric linear maps $\Gamma:\mathbf{Z}^n\to\mathbf{Z}^m\otimes\mathbf{Z}^m$ for which there exist isomorphisms $\alpha:\mathbf{Z}^n\cong\mathbf{Z}^n$ and $\beta:\mathbf{Z}^m\cong\mathbf{Z}^m$ such that $\Gamma\circ\alpha=\beta\otimes\beta\circ\Omega$. This canonical form has been established for all m with $n=1$ (see [6]) and for all n with $m=2$ and $m=3$. Work is current for $m=4$ and $n=2$.

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