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Lattices over Subhereditary Orders and Socle-Projective Modules

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INTRODUCTION

It was noted in [11, 12] that there is a close interrelation between the local theory of lattices over a classical order and torsion-free modules over an artinian algebra with respect to a hereditary torsion theory. This connection is most apparent if the order Λ is a *subhereditary order*. This means that there exists a hereditary order Γ such that

$$\mathrm{rad}(\Gamma) \subset \Lambda \subset \Gamma.$$

In this case the study of Λ-lattices is closely related to the study of the socle-projective modules over the algebra

$$\mathscr{D} = \begin{bmatrix} \Gamma/\mathrm{rad}(\Gamma) & \Gamma/\mathrm{rad}(\Gamma) \\ 0 & \Lambda/\mathrm{rad}(\Gamma) \end{bmatrix}.$$

(These modules were also studied by Simson [13] and Nishida [6], and in [7].) The aim of this note is to elaborate on this connection.

The most prominent examples are the generalized Bäckström-orders which were treated extensively in [7]; in this case $\mathscr{D}$ is a hereditary algebra. These generalized Bäckström-orders seem to play the same role from the point of view of integral representations as do the hereditary algebras in the artinian situation. In the artinian case the theory of simply connected algebras of finite representation type, i.e., the Auslander–Reiten quiver is connected, and every τ-orbit passes through a projective module is via the tilting theory [5, 3] closely related to the representation theory of hereditary algebras. We hope that a similar connection can be established between generalized Bäckström-orders of finite representation type and orders, whose Auslander–Reiten quiver is connected, and every τ-orbit either passes through a projective lattice or through a Γ-lattice.

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The paper is organized as follows. In the first section we study a very general situation: R is a complete local noetherian integral domain with field of fractions K. A is a finite dimensional K-algebra, and $\operatorname{rad}(\Gamma) \subset \Lambda \subset \Gamma$ are R-orders in A (i.e., Λ is finitely generated as R-module, and spans A). In this situation the category ${}_{\Lambda}\mathcal{M}^0(\Gamma)$ of R-torsion-free finitely generated Λ-modules X with $\Gamma \cdot X$ projective over Γ is representation equivalent to the category of socle-projective modules M over the algebra $\mathcal{D}$ above, which do not have simple direct summands (Theorem I).

In Section 2 we study irreducible maps and almost split sequences in $\mathcal{S}(\mathcal{D})$ and ${}_{\Lambda}\mathcal{M}^0(\Gamma)$ resp., extending results from [7] (Theorem II). In this very general situation we prove Brauer-Thrall $1\frac{1}{2}$ for the category ${}_{\Lambda}\mathcal{M}^0(\Gamma)$ (Theorem III).

In Section 3 we turn to the classical situation, where R is Dedekind, A is separable and Γ is hereditary, and we describe the Auslander–Reiten quiver of Λ; in this case ${}_{\Lambda}\mathcal{M}^0(\Gamma)$ comprises all Λ-lattices. One of the main results, Theorem IV, is that indecomposable Γ-lattices have Λ-projective successors in the Auslander–Reiten quiver of Λ. Finally, in Theorem V, we show that the orders Λ of finite lattice type, which are subhereditary and for which $\mathcal{S}(\mathcal{D})$ has preprojective components are precisely those possessing indecomposable lattices $Q_i, ..., Q_s$, which have only projective successors and such that every oriented cycle in the Auslander–Reiten quiver $A(\Lambda)$ of Λ passes through one of the Q_i (equivalently, $A(\Lambda)\backslash\{Q_i\}_{1 \leqslant i \leqslant s}$ has no oriented cycles). As A. Wiedemann has noted, the second condition is satisfied, provided $K \cdot \bigoplus_{i=1}^{s} Q_i$ is a faithful Λ-module.

Some of the results here (Theorems II and IV) were proved in [7] for generalized Bäckström-orders and the proofs carry over for subhereditary orders. This was done in the "Diplomarbeit" of Th. Weichert [15]. Weichert has also supplied the example in Section 3.

We finally point out that the results of Sections 1 and 2 also *apply in the following situation*: Let k be a field and Λ a finite dimensional k-algebra, contained in a semi-simple k-algebra Γ (e.g., via the regular representation). In this case ${}_{\Lambda}\mathcal{M}^0(\Gamma)$ consists of the finitely generated Λ-modules V, contained in a Γ-module W with $\Gamma \cdot V = W$.

1. A Categorical Equivalence between Lattices and Arthinian Modules

The result in this section was proved for classical orders in [11, 12] (see also [7]). We shall review it here in the following more general situation:

Let R be a commutative noetherian complete local domain with field of quotients K and let A be a finite dimensional K-algebra. An R-order Λ in A is a subring of A containing the same identity as A such that

(i) Λ is finitely generated as R-module,

(ii) $K \cdot \Lambda = A$; i.e., Λ contains a K-basis for A.

Given two orders $\Lambda \subset \Gamma$ we shall consider the full subcategory ${}_{\Lambda}\mathscr{M}^0(\Gamma)$ of the category of left Λ-modules, which satisfy

$$\operatorname{ob}({}_{\Lambda}\mathscr{M}^0(\Gamma)) = (X = R\text{-torsion-free finitely generated left } \Lambda\text{-module with } \Gamma \cdot X \text{ projective over } \Gamma).$$

Note. We identify $K \otimes_R X = K \cdot X = A \cdot X$, X being R-torsion-free, and hence we can form $\Gamma \cdot X$ inside $A \cdot X$; however, one should observe that $\Gamma \cdot X$ is in general different from $\Gamma \otimes_\Lambda X$: as a matter of fact, $\Gamma \cdot X$ is the quotient of $\Gamma \otimes_\Lambda X$ modulo its R-torsion submodule; note that the latter is a Γ-submodule. One sees this by considering the natural map $\Gamma \otimes_\Lambda X \to \Gamma \otimes_\Lambda K \cdot X \cong K \cdot X$, induced from the inclusion $X \to K \cdot X$. In particular every Λ-homomorphism in ${}_{\Lambda}\mathscr{M}^0(\Gamma)$ between X and Y gives rise to a Γ-homomorphism between $\Gamma \cdot X$ and $\Gamma \cdot Y$. Conceptually one can view X as a form of the projective Γ-module $\Gamma \cdot X$. It should be noted that this concept can be applied in the more general situation to any ring Γ and a subring Λ, if one considers only those left Λ-modules which are Λ-submodules of a free Γ-module.

Without loss of generality we can always assume that Λ is indecomposable as a ring.

Remarks. (1) In the classical situation, where R is a Dedekind domain and A is separable, Λ is any Rorder in A and Γ is a hereditary R-order containing Λ, then ${}_{\Lambda}\mathscr{M}^0(\Gamma)$ is just the category of all Λ-lattices.

(2) In the algebraic geometric situation, where R is a regular, Λ is the local ring of dimension d of an isolated singularity. In this case Γ is the normalization of Λ; then ${}_{\Lambda}\mathscr{M}^0(\Gamma)$ contains just the Λ-modules X which become projective when extended to Γ; i.e., $\Gamma \cdot X$ is Γ-projective. (The same applies if Γ is any ring between Λ and its normalization.)

Choose now a two-sided Λ-ideal I such that

(i) I is also a two-sided Γ-ideal,

(ii) $I \subset \operatorname{rad}(\Lambda)$, the Jacobson radical of Λ.

We observe that then automatically

$$I \subset \operatorname{rad}(\Gamma).$$

In fact, $I \subset \operatorname{rad}(\Lambda)$ and so I is nilpotent modulo

$$\operatorname{rad}(R) \cdot \Lambda \subset \operatorname{rad}(R) \cdot \Gamma \subset \operatorname{rad}(\Gamma); \qquad \text{i.e., } I \subset \operatorname{rad}(\Gamma).$$

With this notation we put

$$\mathcal{A} = \Lambda/I \qquad \text{and} \qquad \mathcal{B} = \Gamma/I.$$

Then $\mathcal{A}$ and $\mathcal{B}$ are finitely generated algebras over the commutative local ring $\bar{R} = R/(R \cap I)$. Moreover, the inclusion $\Lambda \to \Gamma$ induces an $\bar{R}$-algebra injection $\mathcal{A} \to \mathcal{B}$, and we identify $\mathcal{A}$ as a subring of $\mathcal{B}$. We now construct the *pair category* $\mathcal{E}$ as follows: An object consists of a finitely generated left $\mathcal{A}$-module U and a finitely generated projective left $\mathcal{B}$-module V together an $\mathcal{A}$-monomorphism

$$\sigma: U \to V$$

such that

$$\mathcal{B} \cdot \mathrm{Im}(\sigma) = V.$$

Morphisms in $\mathcal{E}$ are commutative diagrams

$$\begin{array}{ccc} U & \xrightarrow{\sigma} & V \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ U' & \xrightarrow{\sigma'} & V', \end{array}$$

where α is $\mathcal{A}$-linear and β is $\mathcal{B}$-linear.

It should be noted that $\mathcal{E}$ can be identified with a certain category of finitely generated modules over the artinian algebra

$$\mathcal{D} = \begin{bmatrix} \mathcal{B} & \mathcal{B} \\ 0 & \mathcal{A} \end{bmatrix}.$$

We now construct a natural functor

$$\mathbb{F}: {}_{\Lambda}\mathcal{M}^0(\Gamma) \longrightarrow \mathcal{E},$$

$$M \longmapsto (M/I \cdot M \xrightarrow{\sigma} \Gamma \cdot M/I \cdot M),$$

where σ is induced by the inclusion $\iota: M \to \Gamma \cdot M$. Moreover, if $\alpha_1: M \to M'$ is a Λ-homomorphism in ${}_{\Lambda}\mathcal{M}^0(\Gamma)$, then it induces a Γ-homomorphism $\beta_1: \Gamma \cdot M \to \Gamma \cdot M'$ rendering the following diagram commutative:

$$\begin{array}{ccc} M & \xrightarrow{\iota} & \Gamma \cdot M \\ {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\beta_1} \\ M' & \longrightarrow & \Gamma \cdot M'. \end{array}$$

Hence we obtain a morphism in $\mathscr{E}$

$$\begin{array}{ccc} M/I\cdot M & \longrightarrow & \Gamma\cdot M/I\cdot M \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ M'/I\cdot M & \longrightarrow & \Gamma\cdot M'/I\cdot M'. \end{array}$$

It should be noted that $\Gamma\cdot M/I\cdot M$ is $\mathscr{B}$-projective, $\Gamma\cdot M$ being Γ-projective. Moreover,

$$\mathscr{B}\cdot(M/I\cdot M)=(\Gamma/I)\cdot(M/I\cdot M)=\Gamma\cdot M/I\cdot M.$$

THEOREM I. *The functor* $\mathbb{F}$ *induces a representation equivalence between* ${}_{\Lambda}\mathscr{M}^0(\Gamma)$ *and* $\mathscr{E}$; *i.e.,* $\mathbb{F}$ *induces a bijection between the indecomposable objects in* ${}_{\Lambda}\mathscr{M}^0(\Gamma)$ *and in* $\mathscr{E}$.

Remark. The essential and important point of the theorem is that it allows one to compare the lattices in ${}_{\Lambda}\mathscr{M}^0(\Gamma)$ with the finitely generated modules in $\mathscr{E}$, which are modules over the artinian algebra $\mathscr{D}$. The application of this result in later sections is as follows: Under certain conditions, the category $\mathscr{E}$ is closely related to the category of finitely generated $\mathscr{D}$-modules, which have a projective socle. This category has almost split sequences, and the structure of its Auslander–Reiten quiver allows one to deduce structural results about ${}_{\Lambda}\mathscr{M}^0(\Gamma)$.

For the *proof* we have:

Claim 1. $\mathbb{F}$ is up to isomorphism surjective on objects.

Proof of Claim 1. Let $U \to^{\sigma} V$ be an object in $\mathscr{E}$. We may assume that σ is a set theoretic inclusion. Since V is a projective $\mathscr{B}$-module, let Q be the projective Γ-module reducing to V; i.e., $\pi: Q\to V$ is the induced epimorphism and put $M=\pi^{-1}(U)$. Then $M\in {}_{\Lambda}\mathscr{M}^0(\Gamma)$ by Nakayama's lemma. This proves Claim 1. ∎

Claim 2. $\mathbb{F}$ is surjective on morphisms.

Proof. Given a morphism (α,β) in $\mathscr{E}$, we may assume thanks to Claim 1 that we have a commutative diagram

$$\begin{array}{ccc} M/I\cdot M & \xrightarrow{\sigma} & \Gamma\cdot M/I\cdot M \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ M'/I\cdot M' & \xrightarrow{\sigma} & \Gamma\cdot M'/I\cdot M' \end{array}$$

with $M, M' \in {}_{\Lambda}\mathcal{M}^0(\Gamma)$. Since $\Gamma \cdot M$ is Γ-projective we can complete the following diagram commutatively by a Γ-homomorphism β_1:

$$\begin{array}{ccccc} \Gamma \cdot M & \longrightarrow & \Gamma \cdot M/I \cdot M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta} & & \\ \Gamma \cdot M' & \longrightarrow & \Gamma \cdot M'/I \cdot M' & \longrightarrow & 0. \end{array}$$

The given morphism (α, β) induces an $\mathscr{A}$-homomorphism viewing $\Gamma \cdot M/I \cdot M$ as an $\mathscr{A}$-module

$$\gamma: \mathrm{Coker}(\sigma) \to \mathrm{Coker}(\sigma').$$

Since $\mathrm{Coker}(\sigma) \simeq \Gamma \cdot M/M$, we find, using the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \Gamma \cdot M & \longrightarrow & \Gamma \cdot M/I \cdot M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & M' & \longrightarrow & \Gamma \cdot M' & \longrightarrow & \Gamma \cdot M'/I \cdot M' & \longrightarrow & 0, \end{array}$$

that $\alpha_1 = \beta_1|_M: M \to M'$ is a Λ-homomorphism, which under $\mathbb{F}$ gives rise to (α, β). This proves Claim 2. ∎

Claim 3. $\mathbb{F}$ reflects decompositions; i.e., if for $\alpha: M \to M'$, the map $\mathbb{F}(\alpha)$ is a split epimorphism, then α was a split epimorphism to start with. (Similarly for split monomorphisms.)

Proof. In view of Claim 2 it suffices to show that $\mathbb{F}$ reflects isomorphisms. With the notation of the proof of Claim 2, let (α, β) be an isomorphism in $\mathscr{E}$. We have to show that $\alpha_1: M \to M'$ is an isomorphism. From Nakayama's lemma it follows that α_1 is surjective. By using the inverse of (α, β) we conclude that we also have a surjective map $\alpha_2: M' \to M$. Passing to the corresponding Λ-modules we conclude, counting K-dimensions, that α_1 must be injective; whence an isomorphism.

This proves Claim 3 and completes the proof of Theorem I. ∎

Remark. The above situation is most transparent, if $I = \mathrm{rad}(\Gamma)$; i.e., if we have an inclusion

$$\mathrm{rad}(\Gamma) \subset \mathrm{rad}(\Lambda) \subset \Lambda \subset \Gamma.$$

We shall assume this from now on. In this case $\mathscr{B}$ is semi-simple over $k = R/\mathrm{rad}(R)$ and $\mathscr{A}$ is a finite dimensional k-algebra. Thus, if we consider the algebra

$$\mathscr{D} = \begin{bmatrix} \mathscr{B} & \mathscr{B} \\ 0 & \mathscr{A} \end{bmatrix},$$

then $\mathscr{D}$ has a projective socle. We denote by $\mathscr{S}(\mathscr{D})$ the full subcategory of finitely generated left $\mathscr{D}$-modules which have a projective socle. Then we have:

LEMMA 1. *Let $S_1, \ldots, S_t$ be all the simple non-isomorphic $\mathscr{B}$-modules, then*

$$\mathrm{ob}(\mathscr{E}) = \{ U \in \mathscr{S}(\mathscr{D}) \colon U \text{ has no simple direct summand} \}.$$

Proof. This is just a restatement of [7, Proposition I, p. 8, and pp. 26–27].

The k-algebras $\mathscr{D}$ that arise in the above construction can easily be described as follows:

LEMMA 2. *A k-algebra $\mathscr{D}_0$ is Morita-equivalent to an algebra*

$$\mathscr{D} = \begin{bmatrix} \Gamma/\mathrm{rad}(\Gamma) & \Gamma/\mathrm{rad}(\Gamma) \\ 0 & \Lambda/\mathrm{rad}(\Gamma) \end{bmatrix}$$

for R-orders Λ, Γ with

$$\mathrm{rad}(\Gamma) \subset \Lambda \subset \Gamma,$$

if and only if $\mathscr{D}_0$ has a projective left socle, and no simple ring direct factor.

Proof. Obviously $\mathscr{D}$ has a projective left socle and no simple ring direct summand. Conversely, let the socle S of $\mathscr{D}_0$ be projective as a left module. Let P be the direct sum of the non-isomorphic non-simple projective left $\mathscr{D}_0$-modules. There is no loss of generality if we assume $\mathscr{D}_0$ to be indecomposable as a ring. Then

$$\mathrm{soc}(P) \oplus P$$

is progenerator for the category of left $\mathscr{D}_0$-modules. Hence $\mathscr{D}_0$ is a Morita-equivalent to

$$\mathscr{D} = \begin{bmatrix} \mathrm{End}_{\mathscr{D}_0}(\mathrm{Soc}(P)) & \mathrm{End}_{\mathscr{D}_0}(\mathrm{soc}(P)) \\ 0 & \mathrm{End}_{\mathscr{D}_0}(P) \end{bmatrix}.$$

In fact, $\mathrm{Hom}_{\mathscr{D}_0}(\mathrm{Soc}(P), P) = \mathrm{End}_{\mathscr{D}_0}(\mathrm{Soc}(P))$ and $\mathrm{Hom}_{\mathscr{D}_0}(P, \mathrm{Soc}(P)) = 0$, $\mathrm{Soc}(P)$ being projective, and P having no simple direct summand.

Moreover, the natural map

$$\Phi \colon \mathrm{End}_{\mathscr{D}_0}(P) \to \mathrm{End}_{\mathscr{D}_0}(\mathrm{Soc}(P))$$

is injective, $\mathrm{Soc}(P)$ being projective.

Now $\mathscr{D}$ can even be realized with R-orders Λ and Γ over a Dedekind domain by choosing Γ such that $\Gamma/\text{rad}(\Gamma) = \text{End}_{\mathscr{D}_0}(\text{Soc}(P))$. We then take Λ to be the pullback of the diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & \text{End}_{\mathscr{D}_0}(\text{Soc}(P)) \\ \uparrow & & \uparrow \\ \Lambda & \longrightarrow & \text{End}_{\mathscr{D}_0}(P). \end{array}$$

This proves Lemma 2. ▮

Note. In [12; 7, 1.11] a necessary and sufficient condition was given for the category $\mathscr{S}(\mathscr{D})$ to have finitely many indecomposable objects provided $\mathscr{D}$ was the tensor algebra of a multivalued oriented graph [4], in particular $\mathscr{D}$ is a hereditary algebra. In this case Λ is called a generalized Bäckström-order [7], the representation theory of which is well understood.

2. Irreducible Maps

Let $\mathscr{C}$ be either ${}_\Lambda\mathscr{M}^0(\Gamma)$, or $\mathscr{S}(\mathscr{D})$ for a left socle-projective k-algebra $\mathscr{D}$, and recall, that we always assume

$$\text{rad}(\Gamma) \subset \text{rad}(\Lambda) \subset \Lambda \subset \Gamma.$$

Definitions. (1) A morphism $X \to^{\varphi} Y$ in $\mathscr{C}$ is said to be *irreducible* if

(i) $\varphi \in \text{rad}(\mathscr{C}(X, Y))$, where $\mathscr{C}(X, Y)$ is the set of morphisms between X and Y in $\mathscr{C}$, and $\text{rad}(\mathscr{C}(X, Y))$ are the non-split maps between X and Y;

(ii) for every factorization

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \| & & \uparrow{\scriptstyle\beta} \\ X & \xrightarrow{\alpha} & Z \end{array}$$

either α is a split monomorphism or β is a split epimorphism.

(i) and (ii) can be summarized by saying

$$\varphi \in \text{rad}(\mathscr{C}(X, Y))/\text{rad}^2(\mathscr{C}(X, Y)).$$

The R-module (k-module)

$$\text{rad}(\mathscr{C}(X, Y))/\text{rad}^2(\mathscr{C}(X, Y))$$

is called *the "module" of irreducible maps.*

(2) An exact sequence

$$\mathbb{E}: 0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Y \longrightarrow Z \longrightarrow 0$$

in $\mathscr{C}$ is said to be an *almost split sequence*, provided

(i) $\mathbb{E}$ is not split exact,

(ii) every $\chi: Z' \to Z$, which is not a split epimorphism, factorizes via ψ. Every $\chi': X \to X'$, which is not a split monomorphism, factorizes via φ.

A map $\varphi(\psi)$ which is not a split monomorphism (split epimorphism), is called a *source map* (*sink map*), if it satisfies 2 (ii).

(3) $\mathscr{C}$ is said to *have almost split sequences*, if for every indecomposable X, which is not ext-injective in $\mathscr{C}$; i.e., the functor $\mathrm{Ext}^1_{\mathscr{C}}(-, X)$ is not zero (Y, which is not ext-projective in $\mathscr{C}$; i.e., the functor $\mathrm{Ext}^1_{\mathscr{C}}(Y, -)$ is not zero) there exists an almost split sequence in $\mathscr{C}$, beginning with X (ending in Y).

Note. If X is not ext-injective in $\mathscr{S}(\mathscr{D})$, then in the almost split sequence in $\mathscr{S}(\mathscr{D})$

$$0 \longrightarrow X \xrightarrow{\oplus \varphi_i} \bigoplus^{n} E_i \xrightarrow{\oplus \psi_i} Y \longrightarrow 0,$$

where E_i, $1 \leqslant i \leqslant n$, are indecomposable, the $(\varphi_i)_{1 \leqslant i \leqslant n}$ form a basis of the "space" of the irreducible maps leaving X, and dually the $(\psi_i)_{1 \leqslant i \leqslant n}$ form a basis of the "space" of the irreducible maps ending in Y [10].

We first turn to $\mathscr{S}(\mathscr{D})$, where $\mathscr{D}$ has a left-projective socle. For a finitely generated left $\mathscr{D}$-module M we denote by tM the maximal submodule, the socle of which has no projective submodule, and put $fM = M/tM$. The modules tM are the torsion-modules in a hereditary torsion-theory [14]. The category $\mathscr{S}(\mathscr{D})$ has almost split sequences as was observed in [2, 8]. In [7, 2.6] the following result of C. M. Ringel and the author was proved (in much more generality):

THEOREM II. *If $\mathscr{D}$ is left socle-projective, then $\mathscr{S}(\mathscr{D})$ has almost split sequences, which are constructed as follows. Given X indecomposable in $\mathscr{S}(\mathscr{D})$ which is not an injective $\mathscr{D}$-module.*

(i) *X is ext-injective in $\mathscr{S}(\mathscr{D})$ if and only if in the almost split sequence for X in the category of all finitely generated left $\mathscr{D}$-modules*

$$\tilde{\mathbb{E}}: 0 \longrightarrow X \longrightarrow \tilde{Y} \longrightarrow \tilde{Z} \longrightarrow 0$$

we have $f\tilde{Z} = 0$.

(ii) *If X is not ext-injective in $\mathscr{S}(\mathscr{D})$, then*

$$0 \longrightarrow X \longrightarrow \tilde{Y}/t\tilde{Z} \longrightarrow f\tilde{Z} \longrightarrow 0$$

is the almost split sequence in $\mathscr{S}(\mathscr{D})$.

We now turn to $\mathscr{C} = {}_{\Lambda}\mathscr{M}^0(\Gamma)$. For arbitrary R it is not known whether $\mathscr{C}$ has almost split sequences. Only in the following situations:

(1) R is Dedekind and Γ hereditary [7], i.e., the classical situation.

(2) In case Λ is the coordinate ring of an isolated singularity. (M. Auslander [1] has shown that in these cases the category of Cohen–Macauly modules has almost split sequences; but our category is different from that of the Cohen–Macauly modules.)

${}_{\Lambda}\mathscr{M}^0(\Gamma)$ has almost split sequences. (The existence in the second case was proved in [9].)

Because of the connection between irreducible maps and almost split sequences we shall have a close look at irreducible maps and our functor

$$\mathbb{F}\colon {}_{\Lambda}\mathscr{M}^0(\Gamma) \longrightarrow \mathscr{S}(\mathscr{D}) \qquad \text{(as introduced in Section 1).}$$

LEMMA 3. *Let $\varphi\colon M \longrightarrow N$ be a map between indecomposable modules in ${}_{\Lambda}\mathscr{M}^0(\Gamma)$. Assume $\mathbb{F}(\varphi) \neq 0$.* If *$\mathbb{F}(\varphi)$ is irreducible in $\mathscr{S}(\mathscr{D})$, then φ was irreducible to start wth.*

Proof. Since $\mathscr{E} = \operatorname{Im}(\mathbb{F}) \subset \mathscr{S}(\mathscr{D})$, $\mathbb{F}(\varphi)$ is irreducible in $\mathscr{E}$. Surely $\varphi \in \operatorname{rad}(\operatorname{Hom}_{\Lambda}(M, N))$ by Theorem I. Given now a factorization in ${}_{\Lambda}\mathscr{M}^0(\Gamma)$

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \| & & \uparrow{\scriptstyle\beta} \\ M & \xrightarrow{\alpha} & L, \end{array}$$

then we have $\mathbb{F}(\varphi) = \mathbb{F}(\alpha) \cdot \mathbb{F}(\beta)$ and so $\mathbb{F}(\alpha)$ is a split monomorphism or $\mathbb{F}(\beta)$ is a split epimorphism. But by Theorem I, split maps lift to split maps. This proves Lemma 3. ▮

We pause a moment to draw some interesting consequences from Lemma 3. Let $\mathscr{C}$ be either ${}_{\Lambda}\mathscr{M}^0(\Gamma)$ or $\mathscr{S}(\mathscr{D})$.

DEFINITIONS. *The Auslander–Reiten quiver of $\mathscr{C}$* is the oriented graph, which has as vertices the isomorphism classes of indecomposable objects in $\mathscr{C}$, and there is an arrow from $[X]$ to $[Y]$, provided, there is an irreducible map in $\mathscr{C}$ from X to Y.

For $M \in {}_\Lambda\mathcal{M}^0$ we define the *size of M*, sz(M), to be the number of composition factors of $K \cdot M$ as A-module.

THEOREM III (Brauer-Thrall $1\frac{1}{2}$). *Let Δ be a connected component of the Auslander–Reiten quiver $A({}_\Lambda\mathcal{M}^0(\Gamma))$ of ${}_\Lambda\mathcal{M}^0(\Gamma)$ such that:*

(i) *The vertices of Δ have bounded size; i.e., the sizes of the modules in Δ have a uniform bound.*

(ii) *$\mathbb{F}(\Delta)$ contains at least one module for each ring direct factor of $\mathcal{D}$.*

Then

(α) $A({}_\Lambda\mathcal{M}^0(\Gamma)) = \Delta$,

(β) *$A({}_\Lambda\mathcal{M}^0(\Gamma))$ is finite; i.e., ${}_\Lambda\mathcal{M}^0(\Gamma)$ has only finitely many indecomposable lattices.*

Remarks. (1) The hypothesis (ii) is satisfied if, for example, Δ contains all indecomposable projective Λ-lattices or if Δ contains all indecomposable projective Γ-lattices.

(2) The hypothesis (ii) is superfluous if dim $R = 1$ and Γ is hereditary in a separable algebra A, since in this case ${}_\Lambda\mathcal{M}^0(\Gamma)$ is the category of all Λ-lattices, and then one knows the result.

(3) It is likely that the hypothesis (ii) is superfluous in general, cf. [9].

Proof. Because of Lemma 3 and Theorem I, $\mathbb{F}(\Delta)$ is a union of connected components of the Auslander–Reiten quiver of $\mathcal{E} = \text{Im}(\mathbb{F})$. (The notion of the Auslander–Reiten quiver of $\mathcal{E}$ should be self-explanatory.) It follows from the proof of [7, 4.4] that an irreducible map in $\mathcal{E}$ is also irreducible in $\mathcal{S}(\mathcal{D})$. Thus the Auslander–Reiten quiver $A(\mathcal{E})$ of $\mathcal{E}$ is obtained from $A(\mathcal{S}(\mathcal{D}))$ by omitting the points corresponding to the simple projectives in $\mathcal{S}(\mathcal{D})$ and omitting all arrows leaving the simple projectives. We show next that the vertices in $\mathbb{F}(\Delta)$ have uniformly bounded composition lengths. To see this we note that there exists $n \in \mathbb{N}$ such that for every $[M] \in \Delta$, $\Gamma \cdot M$ has at most n indecomposable summands. Hence all modules in $\mathbb{F}(\Delta)$ have their number of composition factors in the socle uniformly bounded by n. Thus the number of indecomposable summands in the injective envelope of all modules in $\mathbb{F}(\Delta)$ is uniformly bounded. Consequently, the composition length of all modules in $\mathbb{F}(\Delta)$ are uniformly bounded. Thanks to Lemma 3, $\mathbb{F}(\Delta)$ decomposes into a finite number of connected components Δ_i, $1 \leq i \leq n$ each of which has uniformly bounded composition length. Now these are components in the Auslander–Reiten quiver of $\mathcal{E}$. Let Δ_{n+1} be the Auslander–Reiten quiver of $\mathcal{E}$ without the components Δ_i, $1 \leq i \leq n$. Note that in $A(\mathcal{S}(\mathcal{D}))$ there are no irreducible maps between Δ_i and Δ_j for $i \neq j$. Let now S be a simple projective $\mathcal{D}$-module. Note that, because of

Lemma 2, S cannot be injective, and so S has an almost split sequence in $\mathscr{S}(\mathscr{D})$

$$0 \longrightarrow S \longrightarrow \bigoplus E_i \longrightarrow T \longrightarrow 0,$$

with E_i indecomposable. Assume that $E_1 \in \Delta_j$. Since $T, E_i \in \mathscr{E}$ we conclude that $T, E_i \in \Delta_j$ for all i, since there are no irreducible maps between Δ_j and Δ_k for $k \neq j$. Let $\tilde{\Delta}_j$ be the component in $A(\mathscr{S}(\mathscr{D}))$ generated by Δ_j, $1 \leqslant j \leqslant n+1$. Then $\tilde{\Delta}_j$, $1 \leqslant j \leqslant n$, are connected components in $A(\mathscr{S}(\mathscr{D}))$ all modules of which have uniformly bounded composition length. By hypothesis, $\bigcup_{i=1}^n \tilde{\Delta}_i$ contains at least one module from each ring direct summand of $\mathscr{D}$, and so we can invoke Brauer-Thrall $1\frac{1}{2}$ for $\mathscr{S}(\mathscr{D})$ [7, 2.14], to conclude $A(\mathscr{S}(\mathscr{D})) = \bigcup_{i=1}^n \tilde{\Delta}_i$ is finite and the statement of the theorem follows. ∎

In order to discuss the irreducible maps φ in ${}_\Lambda\mathscr{M}^0(\Gamma)$ we have to restrict the morphisms in ${}_\Lambda\mathscr{M}^0(\Gamma)$ considerably; since we are mainly interested in indecomposable objects, this is not a severe restriction.

Definition. ${}_\Lambda\mathscr{M}^0(\Gamma)_s$ has as objects the same objects as ${}_\Lambda\mathscr{M}^0(\Gamma)$, but we allow only morphisms $\varphi: M \to N$, M, $N \in \text{ob}({}_\Lambda\mathscr{M}^0(\Gamma))$ such that $\Gamma \cdot \text{Im}(\varphi)$ is Γ projective.

Remarks. (1) We still have a representation equivalence between ${}_\Lambda\mathscr{M}^0(\Gamma)_s$ and $\mathscr{E}$; i.e., Theorem I remains valid for ${}_\Lambda\mathscr{M}^0(\Gamma)_s$. *To see this* we make the following observation:

(i) Let

$$\begin{array}{ccc} U & \xrightarrow{\sigma} & V \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ U' & \xrightarrow{\sigma'} & V' \end{array}$$

be a morphism in $\mathscr{E}$. Since $\mathscr{B} = \Gamma/\text{rad}(\Gamma)$ is semi-simple, we have a decomposition $V' = \text{Im}(\beta) \oplus V_0$. Hence we can lift V' to a projective Γ-lattices $Q' = Q_1 \oplus Q_2$, where Q_1 reduces to $\text{Im}(\beta)$. Now we lift β to $\tilde{\beta}_1$: $Q \to Q_1$, where Q is a lift of V. If now

$$\beta_1 : Q \xrightarrow{\tilde{\beta}_1} Q_1 \longrightarrow Q_1 \oplus Q_2,$$

then the proof of Claim 2 shows that (α, β) can be lifted to a morphism in ${}_\Lambda\mathscr{M}^0(\Gamma)_s$.

(ii) If we have a composition $\lambda\mu$ of the morphisms of $\mathscr{E}$, then λ can be lifted to φ and μ to ψ; φ, ψ morphisms in ${}_\Lambda\mathscr{M}^0(\Gamma)_s$ such that the composition is a morphism in ${}_{\Lambda}\mathcal{M}^0(\Gamma)_s$. (Note that in general the composition of morphisms in ${}_{\Lambda}\mathcal{M}^0(\Gamma)_s$ is not a morphism in ${}_{\Lambda}\mathcal{M}^0(\Gamma)_s$.)

The idea of a proof is the same as in (i), and we indicate it symbolically:

$$\begin{array}{ccccc} V & \xrightarrow{\lambda} & \mathrm{Im}(\lambda)\oplus V' & \xrightarrow{\mu} & W_1\oplus V_2\oplus W_3 \\ \uparrow & & \uparrow \quad \uparrow & & \uparrow \quad \uparrow \quad \uparrow \\ Q & \longrightarrow & Q_1\oplus Q_2 & \longrightarrow & Q_1'\oplus Q_2'\oplus Q_3'. \end{array}$$

(2) Thanks to (ii) above, Lemma 3 carries over to ${}_{\Lambda}\mathcal{M}^0(\Gamma)_s$, and consequently Theorem III holds in ${}_{\Lambda}\mathcal{M}^0(\Gamma)_s$.

(3) In the classical situation, where $\dim R = 1$ and Γ is hereditary. ${}_{\Lambda}\mathcal{M}^0(\Gamma) = {}_{\Lambda}\mathcal{M}^0(\Gamma)_s$.

LEMMA 4. *Let $\varphi: M \to N$ be an irreducible morphism between indecomposable objects in ${}_{\Lambda}\mathcal{M}^0(\Gamma)_s$. If $\mathbb{F}(\varphi) = 0$, then $\Gamma \cdot M$ is an indecomposable projective Γ-lattice, and N is Λ-projective.* (The *converse is trivially true.*)

Proof. Since $\mathbb{F}(\varphi) = 0$, $\mathrm{Im}(\varphi) \subset \mathrm{rad}(\Gamma) \cdot N$ moreover $\Gamma \cdot \mathrm{Im}(\varphi)$ is Γ-projective, and we have the inclusions

$$\mathrm{Im}(\varphi) \subset \Gamma \cdot \mathrm{Im}(\varphi) \subset \mathrm{rad}(\Gamma) \cdot N \subset N \subset \Gamma \cdot N.$$

Since $\mathrm{Im}(\varphi) \in \mathrm{ob}({}_{\Lambda}\mathcal{M}^0(\Gamma))$, we have the factorization

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \| & & \uparrow \beta \\ M & \xrightarrow{\alpha} & \Gamma \cdot \mathrm{Im}\,\varphi \end{array} , \qquad \text{note } \Gamma \cdot \mathrm{Im}(\varphi) \in \mathrm{ob}({}_{\Lambda}\mathcal{M}^0(\Gamma)).$$

However, β is not a split epimorphism, and so α must be a split monomorphism. But M was indecomposable, and so $M \simeq \Gamma \cdot \mathrm{Im}(\varphi)$ is an indecomposable projective Γ-lattice.

Let now

$$P \xrightarrow{\kappa} N$$

be the projective cover of N as Λ-module, and note that κ is a morphism in ${}_{\Lambda}\mathcal{M}^0(\Gamma)_s$. κ induces a split morphism

$$\Gamma \cdot P \xrightarrow{\kappa} \Gamma \cdot N, \qquad \text{with} \quad \lambda \cdot \kappa = \mathrm{id}_{\Gamma \cdot N},$$

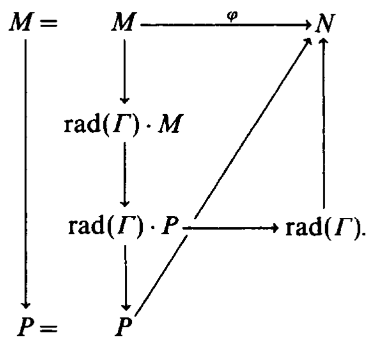
and hence a split epimorphism

$$\mathrm{rad}(\Gamma) \cdot P \xrightarrow{\kappa} \mathrm{rad}(\Gamma) \cdot N.$$

Then the following diagram is commutative:

$$\begin{array}{ccc} \operatorname{rad}(\Gamma)\cdot P & \xrightarrow{\kappa} & \operatorname{rad}(\Gamma)\cdot N \\ \downarrow & & \downarrow \\ P & \xrightarrow{\kappa} & N; \end{array}$$

and we have the commutative diagram



If N is not projective, then α must be a split monomorphism, but $M \subset \operatorname{rad}(\Gamma)\cdot P \subset \operatorname{rad}_\Lambda(P) \subset P$ cannot be a direct summand of P. Thus N is Λ-projective. This proves Lemma 4. ∎

In this general situation, I cannot say anything in case φ is irreducible in ${}_\Lambda\mathcal{M}^0(\Gamma)({}_\Lambda\mathcal{M}^0(\Gamma)_s)$ and $\mathbb{F}(\varphi)\neq 0$. I would need Lemma 4 for ${}_\Lambda\mathcal{M}^0$ (that the morphism sets in ${}_\Lambda\mathcal{M}^0(\Gamma)_s$ form abelian groups). However, Lemma 4 does not hold in ${}_\Lambda\mathcal{M}^0(\Gamma)$ and the morphism sets in ${}_\Lambda\mathcal{M}^0(\Gamma)_s$ do not form abelian groups in general. The remedy is to *turn to the classical situation*:

LEMMA 5. *Let* $\dim R = 1$ *and assume that* Γ *is hereditary and* A *is semisimple. Let* $\psi\colon M \to N$ *be an irreducible map between indecomposables in* ${}_\Lambda\mathcal{M}^0(\Gamma)$. *If* $\mathbb{F}(\varphi)\neq 0$, *then* $\mathbb{F}(\varphi)$ *is irreducible in* $\mathcal{S}(\mathcal{D})$.

Proof. If follows from the proof of [7, 4.4] that it is enough to show that $\mathbb{F}(\varphi)$ is irreducible in $\mathcal{E}$. Assume we have a factorization in $\mathcal{E}$

$$\begin{array}{ccc} \mathbb{F}(M) & \xrightarrow{\mathbb{F}(\varphi)} & \mathbb{F}(N) \\ \| & & \downarrow{\scriptstyle\beta} \\ \mathbb{F}(M) & \xrightarrow{\bar{\alpha}} & \mathbb{F}(X), \end{array}$$

where $X \in \text{ob}({}_{\Lambda}\mathscr{M}^0(\Gamma))$. $\bar{\alpha}$ and $\bar{\beta}$ can be lifted to morphisms in ${}_{\Lambda}\mathscr{M}^0(\Gamma)$; but in general $\varphi \neq \alpha\beta$. We show first that

$$\psi = \varphi - \alpha\beta$$

cannot be irreducible. In fact, assume that ψ is irreducible. Then by Lemma 4, M is a Γ-lattice and N is a projective Λ-lattice. Since φ was irreducible, it follows that $\text{Im}(\varphi)$ is a direct summand of $\text{rad}_{\Lambda}(N)$; but then also $\text{Im}(\varphi) \subset \text{rad}(\Gamma) \cdot N$ by [7, 1.4]. Thus $\mathbb{F}(\varphi) = 0$, a contradiction. Hence ψ is not irreducible, and so $\psi \in \text{rad}^2(\text{Hom}_{\Lambda}(M, N))$. But then φ is irreducible if and only if $\alpha\beta$ is irreducible. Thus $\mathbb{F}(\varphi)$ is irreducible. This proves Lemma 5. ∎

In the classical situation of generalized Bäckström-orders, the predecessors of an indecomposable Γ-lattice Q in the Auslander–Reiten quivers must be injective Λ-lattice, and the successors of Q are projective Λ-lattices [7, Sct. 4].

It is surpirsing that this result also holds in the very general situation considered here, for the successors.

LEMMA 6. *In ${}_{\Lambda}\mathscr{M}^0(\Gamma)$ let $\varphi: Q \to M$ be an irreducible map with Q an indecomposable projective Γ-lattice and M indecomposable in ${}_{\Lambda}\mathscr{M}^0(\Gamma)$. Then M is a projective Λ-lattice.*

Proof. We first show that $\mathbb{F}(\varphi) = 0$. If not, we have a non-zero map

$$\mathbb{F}(\varphi): \mathbb{F}(Q) \to \mathbb{F}(M),$$

which cannot be an isomorphism, φ being irreducible—note that $\mathbb{F}(Q)$ and $\mathbb{F}(M)$ are indecomposable. Since $\mathbb{F}(Q)$ is an injective $\mathscr{D}$-module—not just an injective object in $\mathscr{E}$—the map $\mathbb{F}(\varphi)$ factors via $\mathbb{F}(Q)/\text{soc}(\mathbb{F}(Q))$, where $\text{soc}(\mathbb{F}(Q))$ denotes the socle of $\mathbb{F}(Q)$ as $\mathscr{D}$-module. But $\mathbb{F}(Q)/\text{soc}(\mathbb{F}(Q))$ is torsion, whereas $\mathbb{F}(M)$ is torsion-free. So there are no non-zero maps from $\mathbb{F}(Q)$ to $\mathbb{F}(M)$. Thus $\mathbb{F}(\varphi) = 0$. If one now reviews the proof of Lemma 4, one sees that then M must be an indecomposable projective Λ-lattice. (Note that in Lemma 4 the hypothesis $\varphi \in {}_{\Lambda}\mathscr{M}^0(\Gamma)_s$ was only used to ensure that M is an indecomposable projective Γ-lattice.) This proves Lemma 6. ∎

I can only prove the corresponding statement for the predecessors under additional assumptions.

LEMMA 7. *Assume that ${}_{\Lambda}\mathscr{M}^0(\Gamma)$ has left almost split sequences. (This is surely so, if* $\dim R = 1$ *and Γ is hereditary.) If Q is an indecomposable* *projective Γ-lattice and $M \in {}_{\Lambda}\mathcal{M}^0(\Gamma)$ is indecomposable with an irreducible map $\varphi: M \to Q$, then M is an ext-injective object in ${}_{\Lambda}\mathcal{M}^0(\Gamma)$.*

Proof. If M is not ext-injective, then it has an almost split sequence

$$\begin{array}{ccccccc} & & & Q & & & \\ & & \nearrow & & \searrow & & \\ 0 \longrightarrow & M & & \oplus & & N & \longrightarrow 0. \\ & & \searrow & & \nearrow & & \\ & & & E & & & \end{array}$$

According to Lemma 6, N is projective, a contradiction. This proves Lemma 7. ∎

Note. The lemma actually only needs $\mathbb{F}(\varphi)$ to be irreducible.

3. The Classical Situation: Auslander–Reiten Quivers and Examples

We assume henceforth that R is one dimensional, that A is separable, and that Λ, Γ are R-orders in A with Γ hereditary, such that

$$\operatorname{rad}(\Gamma) \subset \Lambda \subset \Gamma.$$

The notations from the previous sections are retained. In that case ${}_{\Lambda}\mathcal{M}^0(\Gamma) = {}_{\Lambda}\mathcal{M}^0$ is just the category of all left Λ-lattices. And so the Auslander–Reiten quiver $A(\Lambda) = A({}_{\Lambda}\mathcal{M}^0)$ of all Λ-lattices carries an additional structure, namely the partially defined translation coming from almost split sequences, together with a valuation on the arrows. The same holds for $A(\mathcal{S}(\mathcal{D}))$. (For details we refer to [7, 2.12].) The structure of the Auslander–Reiten quivers $A(\Lambda)$ and $A(\mathcal{S}(\mathcal{D}))$ are intimately related.

Let us recall the definition of the *permutation associated to a hereditary order Γ*. Let $(Q_i)_{1 \leqslant i \leqslant m}$ be the non-isomorphic indecomposable Γ lattices. Since Γ is hereditary, these Γ-lattices are at the same time injective Γ-lattices; thus Q_i has a unique minimal over-lattice $S(Q_i)$, which is again a projective Γ-lattice, and hence

$$S(Q_i) \simeq Q_{\sigma(i)}$$

for some $\sigma(i) \in (1, 2, ..., m)$. This map $\sigma: (1, ..., m) \to (1, ..., m)$ is a permutation, called *the permutation of Γ*.

THEOREM IV. (i) *If M and N are indecomposable Λ-lattices, and M is not a Γ-lattice, then we have for the spaces of irreducible maps*

$$\mathrm{Irr}_\Lambda(M, N) \simeq \mathrm{Irr}_{\mathscr{S}(\mathscr{D})}(\mathbb{F}(M), \mathbb{F}(N)).$$

(ii) *If M is a Γ-lattice, say $M = Q_i$, then for an indecomposable projective Λ-lattice P*

$$\mathrm{Irr}_\Lambda(Q_i, P) \simeq \mathrm{Irr}_{\mathscr{S}(\mathscr{D})}(Q_{\sigma(i)}/\mathrm{rad}_\Gamma(Q_{\sigma(i)}), \mathbb{F}(P)).$$

(iii) *The Auslander–Reiten quiver of Λ is obtained from that of $\mathscr{S}(\mathscr{D})$ by identifying the injective $\mathscr{D}$-module*

$$E_i = \begin{bmatrix} Q_i/\mathrm{rad}_\Gamma(Q_i) \\ Q_i/\mathrm{rad}_\Gamma(Q_i) \end{bmatrix}$$

with the simple projective $\mathscr{D}$-module

$$S_{\sigma(i)} = \begin{bmatrix} Q_{\sigma(i)}/\mathrm{rad}_\Gamma(Q_{\sigma(i)}) \\ 0 \end{bmatrix}.$$

Proof. The first statement is just a summary of what was proved in the Lemmata 3, 4, and 5. We also know from Lemma 6 that the only irreducible maps from Q_i are of the form

$$\varphi: Q_i \to P,$$

where P is indecomposable projective over Λ.

We prove (ii) in the form of

LEMMA 8. *There is a natural bijection between the irreducible maps $Q_i \to P$ in ${}_\Lambda\mathscr{M}^0$ and the irreducible maps*

$$S_{\sigma(i)} := \begin{bmatrix} Q_{\sigma(i)}/\mathrm{rad}(\Gamma) \cdot Q_{\sigma(i)} \\ 0 \end{bmatrix} \to \mathbb{F}(P).$$

Proof. This will be established if we can show $Q_i^{(s)}$—i.e., s copies of Q_i—is a direct summand of $\mathrm{rad}_\Lambda(P)$ if and only if $(S_{\sigma(i)})^{(s)}$ is a direct summand of $\mathrm{rad}_{\mathscr{D}}(\mathbb{F}(P))$.

By definition of σ, $Q_i^{(s)}$ is a direct summand of $\mathrm{rad}(\Gamma) \cdot P$ if and only if $(Q_{\sigma(i)})^{(s)}$ is a direct summand of $\Gamma \cdot P$. We write

$$\mathrm{rad}_\Lambda(P) = X \oplus Q,$$

where Q is a Γ-lattice and X does not have any Γ-direct summands. Then by [7, 1.4] we have

$$\mathrm{rad}_{\Lambda}(P)/\mathrm{rad}(\Gamma)\cdot P \simeq X/\mathrm{rad}(\Gamma)\cdot X,$$

the isomorphism being induced from the inclusion $X \to \mathrm{rad}_{\Lambda}(P)$. In particular $Q \subset \mathrm{rad}(\Gamma)\cdot P$.

Hence

$$\mathrm{rad}(\Gamma)\cdot P \supset \mathrm{rad}(\Gamma)\cdot X \oplus Q.$$

However, the above equality shows

$$\mathrm{rad}(\Gamma)\cdot P = \mathrm{rad}(\Gamma)\cdot X \oplus Q.$$

If $Q = \bigoplus_{i=1}^{m} Q_i^{(n_i)}$, we put

$$S(Q) = \bigoplus_{i=1}^{m} (Q_{\sigma(i)})^{(n_i)}.$$

With this notation we have

$$\Gamma\cdot P = \Gamma\cdot X \oplus S(Q).$$

We now can put

$$\begin{aligned}\mathrm{rad}(\mathbb{F}(P)) &= \begin{bmatrix} \Gamma\cdot P/\mathrm{rad}(\Gamma)\cdot P \\ \mathrm{rad}(P)/\mathrm{rad}(\Gamma)\cdot P \end{bmatrix} \\ &= \begin{bmatrix} (\Gamma\cdot X \oplus S(Q))/(\mathrm{rad}(\Gamma)\cdot X \oplus Q) \\ (X \oplus Q)/(\mathrm{rad}(\Gamma)\cdot X \oplus Q) \end{bmatrix} \\ &\simeq \begin{bmatrix} \Gamma\cdot X/\mathrm{rad}(\Gamma)\cdot X \\ X/\mathrm{rad}(\Gamma)\cdot X \end{bmatrix} \oplus \begin{bmatrix} S(Q)/Q \\ 0 \end{bmatrix}.\end{aligned}$$

Hence in $\mathscr{S}(\mathscr{D})$ we have the irreducible map

$$\begin{bmatrix} S(Q)/Q \\ 0 \end{bmatrix} \to \mathbb{F}(P).$$

We shall show next that

$$\begin{bmatrix} S(Q)/Q \\ 0 \end{bmatrix}$$

is the maximal semi-simple summand of rad($\mathbb{F}(P)$). Let $X = \bigoplus_{i=1}^{t} X_i$ be the decomposition of X into indecomposables and let $\psi_i : X_i \to P$ be the natural inclusions. Then the ψ_i are irreducible and since no X_i is a Γ-module, we conclude with Lemmata 3, 4, 5 that

$$\mathbb{F}(\psi_i) : \mathbb{F}(X_i) \to \mathbb{F}(P)$$

is irreducible, whence the statement. ∎

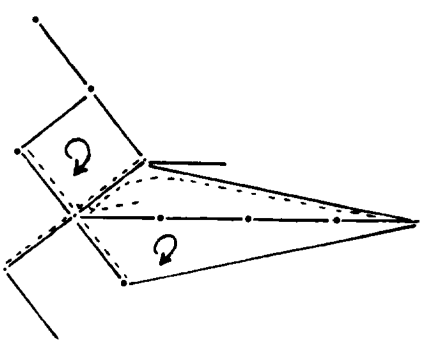
This proves Lmma 8 and Theorem IV. ∎

Example [15, p. 85]. Let $\Gamma = (R)_4 \Pi (R)_4 \Pi (R)_7$, then the permutation of Γ is the identity. Let Λ be defined as follows:

$$\begin{bmatrix} R & R & R & R \\ \pi & R & R & R \\ \pi & \pi & R & \pi \\ \pi & \pi & \pi & R \end{bmatrix} \quad \begin{bmatrix} R & \pi & R & R \\ \pi & R & R & R \\ \pi & \pi & R & R \\ \pi & \pi & \pi & R \end{bmatrix} \quad \begin{bmatrix} R & R & R & R & R & R & \pi \\ \pi & R & R & R & \pi & R & \pi \\ \pi & \pi & R & R & \pi & R & \pi \\ \pi & \pi & \pi & R & \pi & R & \pi \\ \pi & \pi & \pi & \pi & R & R & \pi \\ \pi & \pi & \pi & \pi & \pi & R & \pi \\ \pi & \pi & \pi & \pi & \pi & \pi & R \end{bmatrix}$$

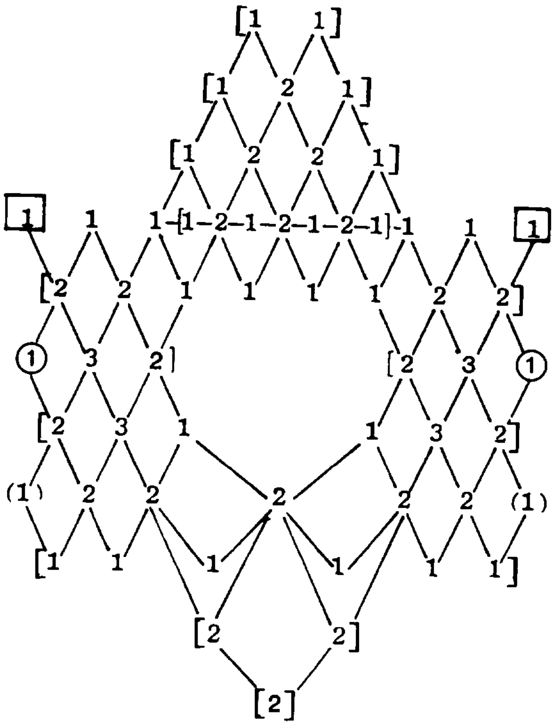
Here the matrix entries linked by a line are congruent modulo π, where π is the maximal ideal in R.

The algebra $\mathscr{D}$ is the path-algebra of the graph



where the dotted arrows are zero relation and ↻ indicates a commutativity relation.

The Auslander–Reiten quiver of $\mathscr{S}(\mathscr{D})$ is given as



According to Theorem III, the Auslander–Reiten quiver of ${}_{\Lambda}\mathscr{M}^0$ is obtained from $A(\mathscr{S}(\mathscr{D}))$ by identifying the modules ⊡, ..., ⊡ and ①, ① and (1), (1). The projective Λ-lattices are indicated by $[n$, and the injective Λ-lattices by $m]$.

Remark. The Auslander–Reiten quiver $A(\Lambda)$ of the category of Λ-lattices in the above example has the following property: If τ is the Auslander–Reiten translate, i.e., in an almost split sequence

$$0 \to M \to E \to N \to 0$$

$N = \tau^{-1}(M)$, then each τ-orbit in $A(\Lambda)$ contains either a projective Λ-lattice or a Γ-lattice. This is equivalent to the Auslander–Reiten quiver of $\mathscr{S}(\mathscr{D})$ having a *preprojective component.*

This phenomenon can be characterized by the internal structure of the order Λ. To do so we introduce some more notation:

DEFINITIONS. (i) Let Λ be an R-order in A, and let $Q_1, ..., Q_s$ be those indecomposable Λ-lattices which have only projective Λ-lattices as successors (equivalently, have only injective predecessors) in $A(\Lambda)$. Denote by $\mathbb{G}$ the full additive subcategory of ${}_{\Lambda}\mathcal{M}^0$ generated by $(Q_i)_{1 \leqslant i \leqslant s}$.

(ii) Let $\mathcal{K}$ be the following category: the indecomposable objects are the indecomposable Λ-lattices not in $\mathbb{G}$, and for each indecomposable $Q \in \mathbb{G}$, we introduce new objects Q^+ and Q^-. For X and Y indecomposable Λ-lattices not in $\mathbb{G}$ we put

$$\mathcal{K}(X, Y) = \mathrm{Hom}_{\Lambda}(X, Y)/\mathbb{G}(X, Y),$$

where $\mathbb{G}(X, Y)$ is the group of Λ-homomorphisms, factoring via an object in $\mathbb{G}$. Moreover, in addition we put

$$\mathcal{K}(Q^+, Y) = \mathrm{Hom}_{\Lambda}(Q, Y)/\mathrm{rad}_{\mathbb{G}}(Q, Y),$$

$$\mathcal{K}(X, Q^-) = \mathrm{Hom}_{\Lambda}(X, Q)/\mathrm{rad}_{\mathbb{G}}(X, Q),$$

$$\mathcal{K}(Q^+, Q_0^-) = \mathrm{rad}_{\mathbb{G}}(Q, Q_0)/\mathrm{rad}^2_{\mathbb{G}}(Q, Q_0),$$

$$\mathcal{K}(Q^-, X) = \mathcal{K}(X, Q^+),$$

where X, Y are indecomposable objects but $X \neq Q^+$ and $Y \neq Q_0^-$ for indecomposable objects $Q, Q_0 \in \mathbb{G}$. Here $\mathrm{rad}_{\mathbb{G}}(\ ,\ )$ are maps which factorize via the radical in $\mathbb{G}$.

Remark. It should be noted that in case Λ is a subhereditary order, $\mathcal{K}$ is just the category $\mathcal{S}(\mathcal{D})$, which is by the above definition characterized internally.

We next define the *separated Auslander–Reiten quiver* of Λ.

DEFINITION. The *separated Auslander–Reiten quiver of* Λ, $A^s(\Lambda)$, has as vertices

(i) The indecomposable Λ-lattices, which are not in $\mathbb{G}$,

(ii) for each $Q_i \in \mathbb{G}$, two new vertices $[Q_i^+]$ and $[Q_i^-]$.

The spaces of irreducible maps between two vertices $[X]$ and $[Y]$ are

(i) $\mathrm{Irr}_{\Lambda}(X, Y)$, if X and Y are indecomposable Λ-lattices not in $\mathbb{G}$,

(ii) $\mathrm{Irr}_{A^s(\Lambda)}(X, Q_i^+) = 0$, $\mathrm{Irr}_{A^s(\Lambda)}(Q_i^-, Y) = 0$,

(iii) $\mathrm{Irr}_{A^s(\Lambda)}(Q_i^+, X) = \mathrm{Irr}_{\Lambda}(Q_i, X)$, $\mathrm{Irr}_{A^s(\Lambda)}(Y, Q_i^-) = \mathrm{Irr}_{\Lambda}(Y, Q_i)$.

THEOREM V. *For an R-order* Λ *of finite lattice type, the following are equivalent*:

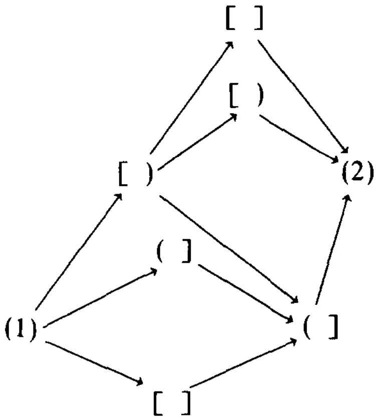
(i) *Every oriented cycle in* $A(\Lambda)$ *passes through a lattice in* $\mathbb{G}$.

(ii) *Λ is subhereditary and $A(\mathscr{S}(\mathscr{D}))$ has preprojective components. In this case $\mathscr{K}$ has almost split sequences and $A(\mathscr{K}) = A(\mathscr{S}(\mathscr{D}))$.*

If Λ satisfies one of these equivalent conditions, we shall call Λ an almost directed order.

Remark. The conditions (i) cannot be replaced by the condition that every Auslander–Reiten orbit contains a projective Λ-lattice or an object in $\mathbb{G}$:

(1) This cannot be done just by combinatorial arguments, as the following example of A. Wiedemann shows:



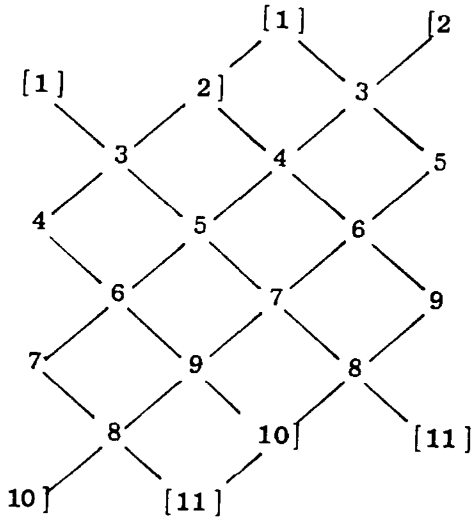
Here [ ) denotes projective objects, ( ] injective objects, and (1) and (2) have to be identified. Then every τ-orbit contains a projective object, but the conclusions of the theorem are false. On the other hand, rank-considerations show that the above picture cannot arise as Auslander–Reiten quiver of an order. (We point out that this example shows that some of the fundamental properties of lattices cannot be detected from the purely combinatorial structure of the Auslander–Reiten quiver.)

(2) Our computer has found an example of an order Λ such that every τ-orbit of $A(\Lambda)$ contains a projective Λ-lattice, $\mathbb{G} = \varnothing$, but Λ is not subhereditary. This shows that (i) is necessary:

EXAMPLE. The order

$$\Lambda = \begin{bmatrix} R & \pi & \pi^5 \\ R & R & \pi^5 \\ R & R & R \end{bmatrix}$$

has Auslander–Reiten quiver



Modules having the same numbers have to be identified. This shows that every τ-orbit contains a projective lattice. But there is no vertex with only projective successors and also Λ is not a subhereditary order.

Proof. (ii) $\Rightarrow$ (i) is a consequence of Theorem IV. Thus it remains to show (i) $\Rightarrow$ (ii). This will be done in several steps.

Claim 1. Let $\varphi: Q \to M$ be an epimorphism, where $Q \in \mathbb{G}$ and M is a Λ-lattice. Then φ is a split epimorphism. (This holds without any of the above assumptions.)

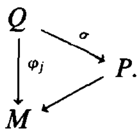
Proof. There is no loss of generality if we assume M to be indecomposable. If Q_i is an indecomposable direct summand of Q and $\varphi_i = \varphi|_{Q_i}$, then φ_i is not a split monomorphism, unless it is also a split epimorphism. If Q_i is not an injective Λ-lattice φ_i factorizes via the almost split sequence of Q_i:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_i & \xrightarrow{\sigma} & P & \longrightarrow & N & \longrightarrow & 0. \\ & & \downarrow{\scriptstyle \varphi_i} & \swarrow{\scriptstyle \psi_i} & & & & & \\ & & M & & & & & & \end{array}$$

But $\mathrm{Im}(\sigma)$ is a direct summand of $\mathrm{rad}_\Lambda(P)$, for the projective Λ-lattice P. Hence

$$\mathrm{Im}(\varphi_i) \subset \mathrm{rad}_\Lambda(M).$$

If Q_i is an injective Λ-lattice, then the unique minimal overmodule Q_i^- of Q_i is a projective Λ-lattice. However, φ_i is not a split monomorphism, and hence we get a factorization



Since σ is irreducible, $\mathrm{Im}(\sigma)$ is a direct summand of $\mathrm{rad}_\Lambda(P)$, and again we conclude

$$\mathrm{Im}(\varphi_i) \subset \mathrm{rad}_\Lambda(M).$$

But then $\mathrm{Im}(\varphi_i) \subset \mathrm{rad}_\Lambda(M)$, a contradiction. This proves Claim 1. ∎

Consequence. The Λ-lattices Q_i, $1 \leqslant i \leqslant s$ are irreducible, for otherwise they would have proper images.

Claim 2. $\mathbb{G}$ has kernels and cokernels of pure submodules. Every object in $\mathbb{G}$ is split projective, moreover $\tilde{\Gamma} = \mathrm{End}_\Lambda(\bigoplus_{i=1}^s Q_i)$ is a herediary order.

Proof. Let $Q_1, Q_2 \in \mathbb{G}$ and $\varphi \in \mathrm{Hom}_\Lambda(Q_1, Q_2)$. Because of Claim 1, $\mathrm{Im}(\varphi)$ is a direct summand of Q_1 and so

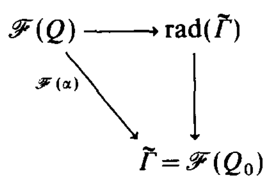
$$Q_1 \simeq \mathrm{Ker}(\varphi) \oplus \mathrm{Im}(\varphi),$$

and hence $\mathbb{G}$ has kernels. Since cokernels of pure submodules are again Λ-lattices, $\mathbb{G}$ has cokernels of pure submodules by Claim 1. Again by Claim 1, all objects in $\mathbb{G}$ are split projective. Finally $\mathbb{G}$ is equivalent to the category of projective $\tilde{\Gamma}$-lattices, via a functor

$$\mathbb{G} \xrightarrow{\mathscr{F}} \{\text{proj. } \tilde{\Gamma}\text{-modules}\},$$

where $\mathscr{F} = \mathrm{Hom}_\Lambda(\bigoplus_{i=1}^s Q_i, -)$.

In order to show that $\tilde{\Gamma}$ is hereditary we must prove that $\mathrm{rad}(\tilde{\Gamma})$ is projective. So let



be a projective cover for $\operatorname{rad}(\tilde{\Gamma})$. In $\mathbb{G}$ we have the situation

$$\begin{array}{ccc} Q & \longrightarrow & Q_0 \\ & {\scriptstyle\sigma}\searrow \quad \nearrow{\scriptstyle\tau} & \\ & \operatorname{Im}(\alpha). & \end{array}$$

Because of Claim 1, σ is a split emimorphism and $\operatorname{Im}(\alpha) \in \mathbb{G}$. For the $\tilde{\Gamma}$-modules we then have the situation

$$\begin{array}{ccc} \mathscr{F}(Q) & \longrightarrow & \operatorname{rad}(\tilde{\Gamma}) \\ {\scriptstyle\mathscr{F}(\sigma)}\downarrow & \nearrow{\scriptstyle\lambda} & \downarrow \\ \mathscr{F}(\operatorname{Im}(\alpha)) & \longrightarrow & \tilde{\Gamma}. \end{array}$$

Note that $\mathscr{F}(\sigma)$ is a split emimorphism and so $\operatorname{Im}(\mathscr{F}(\tau)) = \operatorname{rad}(\tilde{\Gamma})$. Since $\operatorname{rad}(\tilde{\Gamma})$ and $\mathscr{F}(\operatorname{Im}(\alpha))$ have the same R-rank, λ must be an isomorphism. This proves Claim 2. ∎

Claim 3. Λ is a subhereditary order with associated hereditary order Γ, which is Morita-equivalent to $\tilde{\Gamma}$.

Proof. We *define* the Λ-module T to be the image of the evaluation map

$$Q \otimes_{\operatorname{End}_\Lambda(Q)} \operatorname{Hom}_\Lambda(Q, \operatorname{rad}(\Lambda)) \to \Lambda,$$

where $Q = \bigoplus_{i=1}^{s} Q_i$.

Remark. Up to this stage we have not yet used that Λ is of finite lattice type; however, in the proof of Claim 3 we need it, but only to ensure that KT is a faithful Λ-module. Hence in the statement of the theorem, finite lattice type can be replaced by the condition that KT is faithful.

For any indecomposable Λ-lattice M, the multiplication with a parameter π of R factorizes via objects in $\mathbb{G}$, Λ being of finite representation type. Thus KT is faithful and hence by construction $KT = \Lambda$.

Moreover, since for each Q_i there exists a projective Λ-lattice P_i with $Q_i \oplus X_i = \operatorname{rad}_\Lambda(P_i)$, we conclude that

$$T \simeq \bigoplus_{i=1}^{s} Q_i^{(n_i)} \qquad \text{with } n_i > 0.$$

We now define

$$\Gamma = (a \in A : a \cdot T \subset T) = \operatorname{End}_\Lambda(T) \supset \Lambda.$$

This is then obviously Morita-equivalent to $\tilde{\Gamma}$ and whence hereditary by Claim 2.

In order to show that Λ is subhereditary with associated hereditary order Γ, we need to show that $\mathrm{rad}(\Gamma) \subset \Delta$.

Since Γ is hereditary, every Q_i has a unique minimal Γ-overlattice, Q_i^-. Let $\iota_i \colon Q_i \to Q_i^-$ be the natural injection. For any indecomposable Λ-lattice M and any morphism

$$\varphi : Q_i \to M$$

which does not factor properly via an other Γ-lattice, there exists—because of the structure of $A(\Lambda)$—an extension

$$Q_i \xrightarrow{\varphi} M \xrightarrow{\psi} Q_i^-$$

such that $\varphi \cdot \psi = \iota_i$.

Let $\psi \colon M \to \Gamma \cdot M$ be the natural injection. Then we can complete the following diagram commutatively:

$$\begin{array}{ccc} Q_i^- & \xrightarrow{\beta} & \Gamma \cdot M \\ \uparrow{\scriptstyle \iota_i} & & \uparrow{\scriptstyle \psi} \\ Q_i & \xrightarrow[\varphi]{} & M. \end{array}$$

In fact, if $\mathrm{Im}(\varphi \cdot \psi)$ would be a direct summand of $\Gamma \cdot M$, then M would be isomorphic to Q_i, and the statement is clear. Put $X = \mathrm{Im}(\beta) + \mathrm{Im}(\psi)$—we can identify $\mathrm{Im}(\beta)$ with Q_i^- as a submodule of $\Gamma \cdot M$.

Case 1. The map $Q_i^- \to X$ induced by β is a split monomorphism with inverse $\tau \colon X \to Q_i^-$. Then the map ψ factors via β and the statement follows.

Case 2. The map $Q_i^- \to X$ is not a split monomorphism. According to our hypothesis, the almost split sequence

$$0 \longrightarrow Q_i^- \xrightarrow{\gamma} P \longrightarrow \tau^{-1}(Q_i^-) \longrightarrow 0$$

has P projective and $\mathrm{Im}(\gamma) \subset \mathrm{rad}(P)$. By the universal property of almost split sequences, we get a factorization

$$\begin{array}{ccc} Q_i^- & \longrightarrow & P \\ & {\scriptstyle \beta}\searrow \quad \nearrow{\scriptstyle \delta} & \\ & X & \end{array} \quad .$$

In particular, $\mathrm{Im}(\tilde{\beta}) \subset \mathrm{rad}_\Lambda(X)$. Thus $\tilde{\psi}: M \to X$—$\tilde{\psi}$ is induced by φ—is an epimorphism. But then ψ is the identity and ψ factorizes via $\alpha_i: Q_i \to Q_i^-$, a contradiction to the hypothesis. Thus we get an extension

$$Q_i \xrightarrow{\varphi} M \xrightarrow{\psi} Q_i^- .$$

Let $\varphi_{ij}: Q_i \to M$, $1 \leqslant j \leqslant m_i$, be a set of generators for the image of the evaluation

$$Q \otimes_{\mathrm{End}_\Lambda(Q)} \mathrm{Hom}_\Lambda(Q, M) \to M.$$

Then $Q_0 = \sum_{i,j} \mathrm{Im}(\varphi_{ij})$ is the largest Γ-submodule of M, and by the above remark we have an extension

$$Q_0 \xrightarrow{\varphi} M \xrightarrow{\psi} Q_0^- \qquad \text{with } \varphi \cdot \psi = \iota,$$

where the composition ι is the natural injection $Q_o \to Q_0^-$. Obviously $Q_0^- = \Gamma \cdot M$, and since $\Gamma \cdot M/Q_0$ is semi-simple, we conclude $Q_0 = \mathrm{rad}(\Gamma) \cdot M$. Hence $\mathrm{rad}(\Gamma) \subset \Lambda$, and Λ is a subhereditary order. The remainining statements of the theorem are now easily verified.

Epiloque

The almost directed orders seem to be the analogue for integral representations to the simply connected artinian algebras. Theorem V shows that these give rise to simply connected socle-projective categories for $\mathscr{D}$. However, for $\mathscr{D}$ the Γ-lattices Q_i^- (in $\mathscr{K}$) become projective. If one wants to copy some of the results from the artinian situation to subhereditary orders, one has to develop a relative homological algebra for Λ-lattices (Λ is subhereditary for Γ), where also the Γ-lattices are made Λ-projective. This will be done in a subsequent paper.

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