JOURNAL OF DIFFERENTIAL EQUATIONS 104, 332–360 (1993)

Global Existence Theorem for Semilinear Wave Equations with Non-compact Data in Two Space Dimensions

KIMITOSHI TSUTAYA

Department of Mathematics, School of Science and Engineering, Waseda University, 3-4-1 Ohkubo, Shinjuku, Tokyo 169, Japan

Received May 7, 1991

1. INTRODUCTION

We consider the Cauchy problem for the semilinear wave equation

$$\begin{cases} u_{tt} - \Delta u = F(u) & (x, t) \in R^n \times (0, \infty) \\ u(x, 0) = f(x), \quad u_t(x, 0) = g(x) & x \in R^n \end{cases} \tag{1.1}$$

in low space dimensions ($n \leqq 3$) and for "small" initial data. It is well known that many global existence and blow-up theorems have been established under various conditions on $F(u)$, $f(x)$, $g(x)$. In particular, the following remarkable results are classical.

In three space dimensions ($n = 3$), John [5] has shown that global C^2-solutions exist if $F(u)$ satisfies

$$|F(u)| \leqq A|u|^p \qquad \text{for} \quad |u| \leqq 1 \tag{1.2}$$

with $p > 1 + \sqrt{2}$ and the initial data are sufficiently smooth, compactly supported, and also that non-trivial global solutions do not exist if $F(u)$ satisfies

$$F(u) \geqq Au^p \qquad \text{for} \quad u \geqq 0 \tag{1.3}$$

with $A > 0$, $1 < p < 1 + \sqrt{2}$ for the data satisfying $f = 0$, $g \geqq 0$. Recently, John [6] has proved that this blow-up result also holds for all data of compact support.

In two space dimensions, Glassey [3, 4] has also shown that (1.1) has global C^2-solutions when $F(u) = |u|^p$ for $p > (3 + \sqrt{17})/2$, provided the smooth data have compact support, and that for $F(u) = |u|^p$, $1 < p < (3 + \sqrt{17})/2$ the solution blows up in finite time, provided the data of compact support satisfy $\int g(x)\,dx > 0$.

0022-0396/93 $5.00

Glassey [3] and Schaeffer [11] have shown that the solutions blow up in finite time when p is the critical value, $1+\sqrt{2}$ in three space dimensions and $(3+\sqrt{17})/2$ in two space dimensions.

We note that in these results, the data are compactly supported. Since a global solution has the finite speed of propagation, one can expect that for the initial data which are not of compact support, by imposing a certain condition at infinity on the data, global solutions exist.

In fact, Asakura [2] has shown that if $F(u)$ satisfies (1.2) with $p>1+\sqrt{2}$, global solutions exist for small data $f(x)\in C^3(R^3)$, $g(x)\in C^2(R^3)$ satisfying

$$D_x^\alpha f(x),\ D_x^\beta g(x)=\mathcal{O}(|x|^{-1-k}) \text{ as } |x|\to\infty \quad |\alpha|\leqq 3,\ |\beta|\leqq 2 \tag{1.4}$$

provided $k>2/(p-1)$, and moreover that if $0<k<2/(p-1)$ and $F(u)$ satisfies (1.3), the solution blows up in finite time even with $p>1+\sqrt{2}$, for the data satisfying

$$f(x)=0, \qquad g(x)\geqq\frac{\varepsilon_0}{(1+|x|)^{1+k}}, \qquad \varepsilon_0>0. \tag{1.5}$$

In this paper, we shall prove that for $n=2$, the same results as Asakura's theorems hold. Our goal is to prove the following results.

THEOREM 1. *Consider the Cauchy problem*

$$\begin{cases} u_{tt}-\Delta u=F(u) & (x,t)\in R^2\times(0,\infty) \\ u(x,0)=f(x), \quad u_t(x,0)=g(x) & x\in R^2. \end{cases} \tag{1.6}$$

Assume that

(H1) *$F(u)\in C^2(R)$ satisfies the following condition: there exist $p>(3+\sqrt{17})/2$, $A>0$ such that*

$$|F^{(j)}(u)|\leqq A|u|^{p-j} \qquad \textit{for} \quad |u|\leqq 1, \qquad j=0,1,2$$

$$|F''(u)-F''(v)|\leqq A|\Phi|^{p-3}\,|u-v| \qquad \textit{for} \quad |u|,|v|\leqq 1,$$

$$\Phi=\max\{|u|,|v|\}$$

(H2) *$f(x)\in C^3(R^2)$, $g(x)\in C^2(R^2)$ satisfy*

$$\sum_{|\alpha|\leqq 3}|D_x^\alpha f(x)|+\sum_{|\beta|\leqq 2}|D_x^\beta g(x)|\leqslant\frac{G}{(1+|x|)^{1+k}}$$

with $k>0$, where G is a constant.

If $k > 2/(p-1)$ and G is sufficiently small, depending on p and k, then there exists a unique global C^2-solution of (1.6).

THEOREM 2. *Consider the problem*

$$\begin{cases} u_{tt} - \Delta u = F(u) & (x, t) \in R^2 \times (0, \infty) \\ u(x, 0) = 0, \quad u_t(x, 0) = g(x) & x \in R^2. \end{cases} \tag{1.7}$$

Assume that

(H3) $F(u) \in C^2(R)$ *satisfies*

$$F(u) \geqq A u^p, \qquad A > 0, \qquad p > \frac{3+\sqrt{17}}{2} \qquad \text{for} \quad u \geqq 0$$

(H4) $g(x) \in C^2(R^2)$ *satisfies*

$$g(x) \geqq \frac{\varepsilon_0}{(1+|x|)^{1+k}}, \qquad \varepsilon_0 > 0, \qquad k > 0.$$

Let $k < 2/(p-1)$. Then global solutions of (1.7) *do not exist.*

In Section 2, we derive the decay estimate of the solution for the homogeneous wave equation with the initial data satisfying (H2). Section 3 is devoted to establishing the basic estimate for the existence proof. These estimates can be obtained by using Kovalyov's method [8, 9] and Asakura's method [2]. In Section 4, we prove the global existence theorem, by using the basic estimate and the iteration. Finally, in Section 5 we prove the blow-up theorem following John's method [5]. Recently, R. Agemi and H. Takamura [1] and K. Kubota [10] have obtained independently similar results using different methods from ours.

We denote a constant in the estimate by C, which changes from step to step.

2. DECAY ESTIMATE FOR THE HOMOGENEOUS EQUATION

Consider the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & (x, t) \in R^2 \times (0, \infty) \\ u(x, 0) = f(x), \quad u_t(x, 0) = g(x) & x \in R^2. \end{cases} \tag{2.1}$$

We derive the decay estimate for the solution u of (2.1) under the hypothesis (H2).

LEMMA 2.1. *Let $f(x)$, $g(x)$ satisfy the hypothesis* (H2). *Then the solution u of* (2.1) *satisfies*

$$|D_x^\alpha u(x,t)| \leqq \begin{cases} \dfrac{C_k G}{\sqrt{(1+t+a)(1+|t-a|)}} & (k>1) \\[2ex] \dfrac{C_k G ln(2+|t-a|)}{\sqrt{1+t+a}\,(1+|t-a|)^{k-1/2}} & \left(\dfrac{1}{2}<k\leqq 1\right) \\[2ex] \dfrac{C_k G ln(2+t+a)\, ln(2+|t-a|)}{\sqrt{1+t+a}} & \left(k=\dfrac{1}{2}\right) \\[2ex] \dfrac{C_k G ln(2+|t-a|)}{(1+t+a)^k} & \left(0<k<\dfrac{1}{2}\right) \end{cases} \tag{2.2}$$

for $|\alpha|\leqq 2$, where $a=|x|$ and a constant C_k depends only on k.

Proof. The solution of (2.1) is given by

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_0^t \frac{\rho}{\sqrt{t^2-\rho^2}}\int_{|\omega|=1} g(x+\rho\omega)\,d\omega\,d\rho \\ &\quad + \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_0^t \frac{\rho}{\sqrt{t^2-\rho^2}}\int_{|\omega|=1} f(x+\rho\omega)\,d\omega\,d\rho\right) \\ &= \frac{1}{2\pi}\int_0^t \frac{\rho}{\sqrt{t^2-\rho^2}}\int_{|\omega|=1} g(x+\rho\omega)\,d\omega\,d\rho \\ &\quad + \frac{1}{2\pi t}\int_0^t \left(\frac{\rho}{\sqrt{t^2-\rho^2}}\int_{|\omega|=1} f(x+\rho\omega)\,d\omega \right. \\ &\quad \left. + \frac{\rho^2}{\sqrt{t^2-\rho^2}}\int_{|\omega|=1} \omega\cdot\nabla f(x+\rho\omega)\,d\omega\right) d\rho \\ &\equiv u_g + u_f, \end{aligned} \tag{2.3}$$

where $d\omega$ denotes the surface measure on the unit circle in R^2. First, we estimate $|D_x^\alpha u_g|$ ($|\alpha|\leqq 2$). By (H2),

$$\begin{aligned} |D_x^\alpha u_g(x,t)| &\leqq \frac{1}{2\pi}\int_0^t \frac{\rho}{\sqrt{t^2-\rho^2}}\int_{|\omega|=1} |D_x^\alpha g(x+\rho\omega)|\,d\omega\,d\rho \\ &\leqq \frac{1}{2\pi}\int_0^t \frac{\rho}{\sqrt{t^2-\rho^2}}\int_{|\omega|=1} \frac{G}{(1+|x+\rho\omega|)^{1+k}}\,d\omega\,d\rho \\ &\leqq \frac{1}{2\pi}\int_{|x-y|\leqq t} \frac{G}{\sqrt{t^2-|x-y|^2}\,(1+|y|)^{1+k}}\,dy. \end{aligned}$$

Switching the last estimate to polar coordinates, we have

$$
\begin{aligned}
&|D_x^\alpha u_g(a, \theta, t)| \\
&\leqq \frac{G}{2\pi}\left(\int_{|t-a|}^{t+a} \frac{r}{(1+r)^{1+k}} \int_{-\varphi}^{\varphi} \frac{1}{\sqrt{t^2-a^2-r^2+2\,ar\cos\psi}}\,d\psi\,dr\right. \\
&\quad \left.+\chi(t-a)\int_0^{t-a} \frac{r}{(1+r)^{1+k}} \int_{-\pi}^{\pi} \frac{1}{\sqrt{t^2-a^2-r^2+2ar\cos\psi}}\,d\psi\,dr\right) \qquad (2.4)
\end{aligned}
$$

$$\varphi = \arccos\frac{a^2+r^2-t^2}{2ar},$$

where $x=(a\cos\theta, a\sin\theta)$, $y=(r\cos(\theta+\psi), r\sin(\theta+\psi))$, and χ is the characteristic function of positive numbers (see Kovalyov [8, 9]). The proof of this lemma is based essentially on the properties of the functional K defined as follows.

DEFINITION 2.2. Let $g(y)$ be a continuous function on R^2 and $y=(r\cos(\theta+\psi), r\sin(\theta+\psi))$. We define

$$
K(t, a, r, \theta, g)=\begin{cases} \displaystyle\int_{-\varphi}^{\varphi} \frac{g(r, \theta+\psi)}{\sqrt{t^2-a^2-r^2+2ar\cos\psi}}\,d\psi & \text{if } \left|\dfrac{a^2+r^2-t^2}{2ar}\right| \leqq 1 \\ \displaystyle\int_{-\pi}^{\pi} \frac{g(r, \theta+\psi)}{\sqrt{t^2-a^2-r^2+2ar\cos\psi}}\,d\psi & \text{if } \left|\dfrac{a^2+r^2-t^2}{2ar}\right| \geqq 1 \end{cases} \qquad (2.5)
$$

$$\varphi = \arccos\frac{a^2+r^2-t^2}{2ar}$$

and define

$$K(t, a, r) = K(t, a, r, \theta, 1). \qquad (2.6)$$

We use the following lemma which is proved in [8, 9].

LEMMA 2.3. (i) *If* $t \geqq a+r$ *and* $|(a^2+r^2-t^2)/2ar| \geqq 1$, *then* $K(t, a, r)$ *satisfies*

$$K(t, a, r) \leqq \frac{C\ln\left[2+\dfrac{ar}{t^2-(a+r)^2}\right]}{\sqrt{t^2-a^2-r^2}} \leqq \frac{C}{\sqrt{t^2-(a+r)^2}}. \qquad (2.7)$$

(ii) *If* $t \leqq a+r$ *and* $|(a^2+r^2-t^2)/2ar| \leqq 1$, *then*

$$K(t,a,r) \leqq \frac{C}{\sqrt{ar}} \ln\left[2+\frac{ar\chi(t-a)}{(a+r)^2-t^2}\right], \tag{2.8}$$

where χ *is the characteristic function of positive numbers.*

Now, we estimate the right-hand side of (2.4) by dividing into two cases.

Case 1. $a \geqq t$. (i) $k > 1/2$. Using (2.8), we have

$$\begin{aligned} |D_x^\alpha u_g(x,t)| &\leqq \frac{CG}{\sqrt{a}} \int_{a-t}^{t+a} \frac{1}{(1+r)^{1/2+k}}\, dr \\ &\leqq \frac{CG}{\sqrt{a}} \left\{ \frac{1}{(1+a-t)^{k-1/2}} - \frac{1}{(1+a+t)^{k-1/2}} \right\} \\ &= \frac{CG}{\sqrt{a}\,(1+a-t)^{k-1/2}} \left\{ 1 - \left(\frac{1+a-t}{1+a+t}\right)^{k-1/2} \right\}. \end{aligned} \tag{2.9}$$

Note that

$$1-s^{k-1/2} \leqq \max\left(1, k-\frac{1}{2}\right)(1-s) \qquad \text{for} \quad 0 \leqq s \leqq 1 \tag{2.10}$$

$$\frac{1+a-t}{1+a+t} = 1 - \frac{2t}{1+a+t}. \tag{2.11}$$

Thus

$$\begin{aligned} |D_x^\alpha u_g(x,t)| &\leqq \frac{CGt}{\sqrt{a}\,(1+a-t)^{k-1/2}\,(1+a+t)} \\ &\leqq \frac{CG}{\sqrt{1+t+a}\,(1+a-t)^{k-1/2}}. \end{aligned} \tag{2.12}$$

(ii) $0 < k < 1/2$. As above,

$$\begin{aligned} |D_x^\alpha u_g(x,t)| &\leqq \frac{CG(1+a+t)^{1/2-k}}{\sqrt{a}} \left\{ 1 - \left(\frac{1+a-t}{1+a+t}\right)^{1/2-k} \right\} \\ &\leqq \frac{CG}{(1+a+t)^k}. \end{aligned} \tag{2.13}$$

(iii) $k=1/2$. By (2.8),

$$|D_x^\alpha u_g(x,t)| \leqq \frac{CG}{\sqrt{a}} \int_{a-t}^{a+t} \frac{1}{1+r}\,dr$$
$$= \frac{CG}{\sqrt{a}} \ln \frac{1+a+t}{1+a-t}.$$

Since $a \geqq (1/2)(a+t)$, we have

$$|D_x^\alpha u_g(x,t)| \leqq \frac{CG \ln(1+a+t)}{\sqrt{a+t}}.$$

Moreover, since the right-hand side of the above inequality is bounded for $0 \leqq t+a \leqq 1$, hence we obtain

$$|D_x^\alpha u_g(x,t)| \leqq \frac{CG \ln(2+a+t)}{\sqrt{1+a+t}}. \tag{2.14}$$

Case 2. $t \geqq a$. We write

$$|D_x^\alpha u_g(x,t)| \leqq \mathbf{I} + \mathbf{II},$$

where

$$\mathbf{I} = \frac{G}{2\pi} \int_{t-a}^{t+a} \frac{K(t,a,r)\,r}{(1+r)^{1+k}}\,dr$$
$$\mathbf{II} = \frac{G}{2\pi} \int_0^{t-a} \frac{K(t,a,r)\,r}{(1+r)^{1+k}}\,dr.$$

We begin by estimating $\mathbf{I}$. From (2.8), we have

$$\mathbf{I} \leqq \frac{CG}{\sqrt{a}} \int_{t-a}^{t+a} \ln\left[2 + \frac{t-a}{a+r-t}\right] \frac{1}{(1+r)^{1/2+k}}\,dr \tag{2.15}$$

or

$$\mathbf{I} \leqq \frac{CG}{\sqrt{a}} \int_{t-a}^{t+a} \ln\left[2 + \frac{a}{a+r-t}\right] \frac{1}{(1+r)^{1/2+k}}\,dr. \tag{2.16}$$

We subdivide into two cases again.

(a) When $a \geqq 1$, we split the integral in the right-hand side of (2.15) into two parts:

$$\mathbf{I} \leqq \frac{CG}{\sqrt{a}} \int_{t-a+\varepsilon}^{t+a} \ln\left[2 + \frac{t-a}{a+r-t}\right] \frac{1}{(1+r)^{1/2+k}}\, dr$$

$$+ \frac{CG}{\sqrt{a}} \int_{t-a}^{t-a+\varepsilon} \ln\left[2 + \frac{t-a}{a+r-t}\right] \frac{1}{(1+r)^{1/2+k}}\, dr$$

$$\equiv \mathbf{I}' + \mathbf{I}'',$$

where $\varepsilon > 0$ is a sufficiently small constant.

To estimate $\mathbf{I}'$ we proceed as Case 1.

(i) $k > 1/2$.

$$\mathbf{I}' \leqq \frac{CG \ln(2+t-a)}{\sqrt{a}} \int_{t-a+\varepsilon}^{t+a} \frac{1}{(1+r)^{1/2+k}}\, dr$$

$$\leqq \frac{CG \ln(2+t-a)}{\sqrt{1+t+a}\,(1+t-a)^{k-1/2}}. \tag{2.17}$$

(ii) $0 < k < 1/2$.

$$\mathbf{I}' \leqq \frac{CG \ln(2+t-a)}{(1+t+a)^k}. \tag{2.18}$$

(iii) $k = 1/2$.

$$\mathbf{I}' \leqq \frac{CG \ln(2+t-a)}{\sqrt{a}} \ln \frac{1+t+a}{1+t-a}.$$

If $t/2 \geqq a$, $t-a \geqq t/2 \geqq (1/4)(t+a)$. Note that $(1+t+a)/(1+t-a) = 1 + 2a/(1+t-a)$. Hence we have

$$\mathbf{I}' \leqq \frac{CG\sqrt{a}\ln(2+t-a)}{1+t-a}$$

$$\leqq \frac{CG \ln(2+t-a)}{\sqrt{1+t+a}}. \tag{2.19}$$

If $a \geqq t/2$, $a \geqq (1/4)(a+t)$. Hence

$$\mathbf{I}' \leqq \frac{CG \ln(2+t-a)\ln(1+t+a)}{\sqrt{t+a}}.$$

Since the right-hand side is bounded for $0 \leqq t+a \leqq 1$, thus we obtain

$$\mathbf{I}' \leqq \frac{CG \ln(2+t-a) \ln(2+t+a)}{\sqrt{1+t+a}}. \tag{2.20}$$

To estimate $\mathbf{I}''$, we make the change of variables $\xi = a+r-t$. Then,

$$\begin{aligned}
\mathbf{I}'' &= \frac{CG}{\sqrt{a}} \int_0^{\varepsilon} \ln\left[2+\frac{t-a}{\xi}\right] \frac{1}{(1+\xi+t-a)^{1/2+k}}\, d\xi \\
&\leqq \frac{CG}{\sqrt{a}\,(1+t-a)^{1/2+k}} \int_0^{\varepsilon} \ln \frac{C(2+t-a)}{\xi}\, d\xi \\
&\leqq \frac{CG}{\sqrt{a}\,(1+t-a)^{1/2+k}} \left\{\varepsilon \ln C + \varepsilon \ln(2+t-a) + \left|\int_0^{\varepsilon} \ln \xi \, d\xi\right|\right\} \\
&\leqq \frac{CG \ln(2+t-a)}{\sqrt{a+1}\,(1+t-a)^{1/2+k}}.
\end{aligned}$$

Note that

$$1+t+a \leqq 2(1+t-a)(1+a). \tag{2.21}$$

Hence we obtain

$$\mathbf{I}'' \leqq \frac{CG \ln(2+t-a)}{\sqrt{1+t+a}\,(1+t-a)^{k}}. \tag{2.22}$$

(b) When $a<1$, changing variables by $\xi = a+r-t$ in (2.16), we have

$$\begin{aligned}
\mathbf{I} &\leqq \frac{CG}{\sqrt{a}} \int_0^{2a} \ln\left[2+\frac{a}{\xi}\right] \frac{1}{(1+\xi+t-a)^{1/2+k}}\, d\xi \\
&\leqq \frac{CG}{\sqrt{a}\,(1+t-a)^{1/2+k}} \int_0^{2a} \ln \frac{Ca}{\xi}\, d\xi \\
&\leqq \frac{CG\sqrt{a}\,(1+|\ln a|)}{(1+t-a)^{1/2+k}} \\
&\leqq \frac{CG}{(1+t-a)^{1/2+k}}.
\end{aligned}$$

Since $3(1+t-a)>1+t+a$, we obtain

$$\mathbf{I} \leqq \frac{CG}{\sqrt{1+t+a}\,(1+t-a)^k}. \tag{2.23}$$

Now we turn to **II**. By (2.7),

$$\begin{aligned}\mathbf{II} &\leqq CG \int_0^{t-a} \frac{1}{\sqrt{(t+a+r)(t-a-r)}\,(1+r)^k}\,dr \\ &\leqq \frac{CG}{\sqrt{t+a}} \int_0^{t-a} \frac{1}{\sqrt{t-a-r}\,(1+r)^k}\,dr. \end{aligned} \tag{2.24}$$

To continue, we have to distinguish three cases.

(i) $k>1$. Changing variables by $\xi=\sqrt{t-a-r}$, we have

$$\begin{aligned}\mathbf{II} &\leqq \frac{CG}{\sqrt{t+a}} \int_0^{\sqrt{t-a}} \frac{2}{(1+t-a-\xi^2)^k}\,d\xi \\ &\leqq \frac{CG}{\sqrt{t+a}\,(1+t-a)^{k/2}} \int_0^{\sqrt{t-a}} \left\{\frac{1}{(\sqrt{1+t-a}+\xi)^k} \right. \\ &\quad \left. + \frac{1}{(\sqrt{1+t-a}-\xi)^k}\right\} d\xi \\ &\leqq \frac{CG}{\sqrt{t+a}\,(1+t-a)^{k/2}} \{(\sqrt{1+t-a}+\sqrt{t-a})^{k-1} \\ &\quad -(\sqrt{1+t-a}-\sqrt{t-a})^{k-1}\}. \end{aligned} \tag{2.25}$$

If $t \geqq a+1$, then

$$\mathbf{II} \leqq \frac{CG}{\sqrt{(1+t+a)(1+t-a)}}.$$

If $0 \leqq t-a<1$ and $a \geqq 1$, then $2(t+a) \geqq t+a+1$. Hence

$$\mathbf{II} \leqq \frac{CG}{\sqrt{1+t+a}\,(1+t-a)^{k/2}}.$$

If $0 \leqq t-a<1$ and $a<1$, the right-hand side of (2.25) is bounded. Therefore,

$$\mathbf{II} \leqq \frac{CG}{\sqrt{(1+t+a)(1+t-a)}}. \tag{2.26}$$

(ii) $1/2<k \leqq 1$. As above

$$\begin{aligned}
\mathbf{II} &\leqq \frac{CG(1+t-a)^{1-k}}{\sqrt{t+a}} \int_0^{t-a} \frac{1}{\sqrt{t-a-r}\,(1+r)}\,dr \\
&= \frac{CG(1+t-a)^{1-k}}{\sqrt{t+a}} \int_0^{\sqrt{t-a}} \frac{2}{1+t-a-\xi^2}\,d\xi \\
&\leqq \frac{CG \ln(\sqrt{1+t-a}+\sqrt{t-a})}{\sqrt{t+a}\,(1+t-a)^{k-1/2}} \\
&\leqq \frac{CG \ln(2+t-a)}{\sqrt{1+t+a}\,(1+t-a)^{k-1/2}}.
\end{aligned} \tag{2.27}$$

(iii) $0<k \leqq 1/2$. Similarly,

$$\begin{aligned}
\mathbf{II} &\leqq \frac{CG(1+t-a)^{1/2-k}}{\sqrt{t+a}} \int_0^{t-a} \frac{1}{\sqrt{(t-a-r)(1+r)}}\,dr \\
&\leqq \frac{CG(1+t-a)^{1/2-k}}{\sqrt{t+a}} \int_0^{\sqrt{t-a}} \frac{1}{\sqrt{1+t-a-\xi^2}}\,d\xi \\
&\leqq \frac{CG\sqrt{t-a}}{\sqrt{t+a}\,(1+t-a)^{k}} \\
&\leqq \frac{CG(1+t-a)^{1/2-k}}{\sqrt{1+t+a}}.
\end{aligned} \tag{2.28}$$

Next we estimate $|D_x^\alpha u_f|$. We can write

$$\begin{aligned}
|D_x^\alpha u_f(x,t)| &\leqq \frac{1}{2\pi t} \int_{|x-y| \leqq t} \frac{|D_y^\alpha f(y)|}{\sqrt{t^2-|x-y|^2}}\,dy \\
&\quad + \frac{1}{2\pi t} \int_{|x-y| \leqq t} \frac{|D_y^\alpha \nabla f(y)| \cdot |x-y|}{\sqrt{t^2-|x-y|^2}}\,dy \\
&\equiv \mathbf{III} + \mathbf{IV}.
\end{aligned}$$

Since $|x-y|/t \leqq 1$, by the hypothesis (H2), **IV** can be estimated in the same manner as $|D_x^\alpha u_g|$. Similarly, we see that for $t \geqq 1$, **III** has the same estimate as $|D_x^\alpha u_g|$. Thus it remains to estimate **III** for $0 < t < 1$.

Assume $|t-a| \leqq 1$. Then, by (H2) and changing variables by $\xi = |x-y|$, we have

$$\begin{aligned}\mathbf{III} &\leqq \frac{1}{2\pi t}\int_{|x-y|\leqq t} \frac{G}{\sqrt{t^2-|x-y|^2}\,(1+|y|)^{1+k}}\,dy \\ &\leqq \frac{G}{t}\int_0^t \frac{\xi}{\sqrt{t^2-\xi^2}}\,d\xi \\ &\leqq \frac{CG}{\sqrt{(1+t+a)(1+|t-a|)}}. \end{aligned} \tag{2.29}$$

On the other hand, for $|t-a|>1$ and $0<t<1$, we have $a>t+1$. Hence **III** can be estimated in the same manner as Case 1. Therefore for $|t-a|>1$ we obtain

$$\mathbf{III} \leqq \begin{cases} \dfrac{CG}{(1+a+t)^{3/2}\,(1+a-t)^{k-1/2}} & \left(k>\dfrac{1}{2}\right) \\ \dfrac{CG}{\sqrt{1+a+t}\,(1+a-t)} & \left(k=\dfrac{1}{2}\right) \\ \dfrac{CG}{(1+a+t)^{k+1}} & \left(0<k<\dfrac{1}{2}\right). \end{cases} \tag{2.30}$$

This completes the proof of Lemma 2.1.

3. The Basic Estimate

Consider the inhomogeneous Cauchy problem

$$\begin{cases} u_{tt}-\Delta u = w(x,t) & (x,t)\in R^2\times(0,\infty) \\ u(x,0)=u_t(x,0)=0 & x\in R^2. \end{cases} \tag{3.1}$$

Then the solution of (3.1) is given by

$$u(x,t)=\frac{1}{2\pi}\int_0^t\int_{|x-y|\leqq t-s}\frac{w(y,s)}{\sqrt{(t-s)^2-|x-y|^2}}\,dy\,ds. \tag{3.2}$$

We denote (3.2) by $u = Lw$. Note that $w \geqq 0$ implies $Lw \geqq 0$. As in Section 2, we switch (3.2) to polar coordinates. Then,

$$\begin{aligned} u(x, t) &= u(a, \theta, t) \\ &= \frac{1}{2\pi} \int_{\max(0, t-a)}^{t} \int_{a-t+s}^{a+t-s} r \\ &\quad \times \int_{-\varphi}^{\varphi} \frac{w(r, \theta+\psi, s)}{\sqrt{(t-s)^2 - a^2 - r^2 + 2ar\cos\psi}} \, d\psi \, dr \, ds \\ &\quad + \frac{\chi(t-a)}{2\pi} \int_0^{t-a} \int_{t-a-s}^{t+a-s} r \\ &\quad \times \int_{-\varphi}^{\varphi} \frac{w(r, \theta+\psi, s)}{\sqrt{(t-s)^2 - a^2 - r^2 + 2ar\cos\psi}} \, d\psi \, dr \, ds \\ &\quad + \frac{\chi(t-a)}{2\pi} \int_0^{t-a} \int_0^{t-a-s} r \\ &\quad \times \int_{-\pi}^{\pi} \frac{w(r, \theta+\psi, s)}{\sqrt{(t-s)^2 - a^2 - r^2 + 2ar\cos\psi}} \, d\psi \, dr \, ds \qquad (3.3) \\ \varphi &= \arccos \frac{a^2 + r^2 - (t-s)^2}{2ar}, \end{aligned}$$

where $x = (a\cos\theta, a\sin\theta)$, $y = (r\cos(\theta+\psi), r\sin(\theta+\psi))$ or

$$\begin{aligned} u(x, t) = \frac{1}{2\pi} \Big\{ &\int_{D'} K(t-s, a, r, \theta, w(s)) \, r \, dr \, ds \\ &+ \chi(t-a) \int_{D''} K(t-s, a, r, \theta, w(s)) \, r \, dr \, ds \Big\}, \end{aligned} \qquad (3.4)$$

where the domains D' and D'' denote the sets

$$D' = \{(s, r) : s + a - t \leqq r \leqq -s + t + a,\ 0 \leqq s \leqq t\} \qquad \text{if} \quad a \geqq t$$

$$\left. \begin{aligned} D' &= \{(s, r) : |s + a - t| \leqq r \leqq -s + t + a,\ 0 \leqq s \leqq t\} \\ D'' &= \{(s, r) : 0 \leqq r \leqq -s + t - a,\ 0 \leqq s \leqq t - a\} \end{aligned} \right\} \qquad \text{if} \quad t \geqq a$$

(see [8, 9]).

We prove the following lemma which is the basic estimate for the existence proof.

LEMMA 3.1. *Assume that* $u(x, t) \in C(R^2 \times [0, \infty))$ *satisfies*

$$|u(x, t)| \leqq \begin{cases} \dfrac{M \ln(2+|t-a|)}{\sqrt{1+t+a}\,(1+|t-a|)^m} & \left(k > \dfrac{1}{2}\right) \\ \dfrac{M \ln(2+t+a) \ln(2+|t-a|)}{\sqrt{1+t+a}} & \left(k = \dfrac{1}{2}\right) \\ \dfrac{M \ln(2+|t-a|)}{(1+t+a)^k} & \left(0 < k < \dfrac{1}{2}\right), \end{cases} \tag{3.5}$$

where $a = |x|$, $m = \min(1/2, (p-3)/2, k-1/2)$.
Let $k > 2/(p-1)$ *and* $p > (3+\sqrt{17})/2$. *Then,*

$$|L|u|^p(x, t)| \leqq \begin{cases} \dfrac{C_{p,k} M^p \ln(2+|t-a|)}{\sqrt{1+t+a}\,(1+|t-a|)^m} & \left(k > \dfrac{1}{2}\right) \\ \dfrac{C_{p,k} M^p \ln(2+|t-a|)}{(1+t+a)^k} & \left(0 < k \leqq \dfrac{1}{2}\right) \end{cases}, \tag{3.6}$$

where $C_{p,k}$ *depends on p and k, not on M.*

Proof. From (3.2) and (3.4), we have

$$|L|u|^p(x, t)| \leqq \frac{1}{2\pi}\left\{\int_{D'} K(t-s, a, r, \theta, |u|^p(s))\, r\, dr\, ds\right.$$

$$\left. + \chi(t-a) \int_{D''} K(t-s, a, r, \theta, |u|^p(s))\, r\, dr\, ds\right\}. \tag{3.7}$$

If $t \geqq a$, we split the domains of integration D', D'' into two sets respectively as follows:

$$D_1' = D' \cap \{(s, r) : r \geqq s\}$$

$$D_2' = D' \cap \{(s, r) : r \leqq s\}$$

$$D_1'' = D'' \cap \{(s, r) : r \geqq s\}$$

$$D_2'' = D'' \cap \{(s, r) : r \leqq s\}.$$

We denote the integral over the domains D_1', D_2', D_1'', D_2'' by $\mathbf{I}_{D_1'}$, $\mathbf{I}_{D_2'}$, $\mathbf{II}_{D_1''}$, and $\mathbf{II}_{D_2''}$, respectively.

(i) $k > 1/2$.

Case 1. $a \geqq t$. By (2.8)

$$|L|u|^p(x,t)| \leqq \frac{CM^p}{\sqrt{a}} \int_{D'} \frac{\sqrt{r}\,(\ln(2+r-s))^p}{(1+r+s)^{p/2}(1+r-s)^{pm}}\,dr\,ds$$
$$\leqq \frac{CM^p}{\sqrt{a}} \int_{D'} \frac{1}{(1+r+s)^{(p-1)/2}(1+r-s)^{pm-\varepsilon}}\,dr\,ds,$$

where $\varepsilon > 0$ is small and is determined later, and C depends on ε.
Changing variables by

$$\alpha = r+s, \qquad \beta = r-s \tag{3.8}$$

we have

$$|L|u|^p(x,t)| \leqq \frac{CM^p}{\sqrt{a}} \int_{a-t}^{a+t} \frac{1}{(1+\alpha)^{(p-1)/2}} \int_{a-t}^{\alpha} \frac{1}{(1+\beta)^{pm-\varepsilon}}\,d\beta\,d\alpha. \tag{3.9}$$

(a) $m = 1/2$. Since $p > 4$, we can choose ε such that $p/2 - \varepsilon > 1$. Hence as in (2.9)–(2.12),

$$|L|u|^p(x,t)| \leqq \frac{CM^p}{\sqrt{a}} \int_{a-t}^{a+t} \frac{1}{(1+\alpha)^{(p-1)/2}} \int_0^{\infty} \frac{1}{(1+\beta)^{p/2-\varepsilon}}\,d\beta\,d\alpha$$
$$\leqq \frac{CM^p}{\sqrt{a}} \left\{ \frac{1}{(1+a-t)^{(p-3)/2}} - \frac{1}{(1+a+t)^{(p-3)/2}} \right\}$$
$$\leqq \frac{CM^p}{\sqrt{1+a+t}\,(1+a-t)^{(p-3)/2}}$$
$$\leqq \frac{CM^p}{\sqrt{(1+a+t)(1+a-t)}}. \tag{3.10}$$

(b) $m = (p-3)/2$. Since $p > (3+\sqrt{17})/2$ we can choose ε such that $p(p-3)/2 - \varepsilon > 1$. Hence as above

$$|L|u|^p(x,t)| \leqq \frac{CM^p}{\sqrt{1+a+t}\,(1+a-t)^{(p-3)/2}}. \tag{3.11}$$

(c) $m = k - 1/2$. If $p(k-1/2) > 1$, we can choose ε such that $p(k-1/2) - \varepsilon > 1$. Thus as above

$$|L|u|^p(x,t)| \leqq \frac{CM^p}{\sqrt{1+a+t}\,(1+a-t)^{(p-3)/2}}$$
$$\leqq \frac{CM^p}{\sqrt{1+a+t}\,(1+a-t)^{k-1/2}}. \tag{3.12}$$

If $p(k-1/2)\leqq 1$, we set $\delta=k-(5/2p)-(\varepsilon/p)-(\gamma/p)$ with small $\gamma>0$ which is determined later. We replace $(1+\alpha)^{p(1/2-\delta)-3/2}$ by $(1+\beta)^{p(1/2-\delta)-3/2}$ in (3.9). Note that

$$p\left(\frac{1}{2}-\delta\right)-\frac{3}{2}=p\left(\frac{1}{2}+\frac{1}{p}-k+\frac{\varepsilon}{p}+\frac{\gamma}{p}\right)>0.$$

Then,

$$\begin{aligned}|L|u|^p(x,t)| &\leqq \frac{CM^p}{\sqrt{a}}\int_{a-t}^{a+t}\frac{1}{(1+\alpha)^{1+p\delta}}\int_0^\infty \frac{1}{(1+\beta)^{1+\gamma}}\,d\beta\,d\alpha \\ &\leqq \frac{CM^p}{\sqrt{1+a+t}\,(1+a-t)^{p\delta}}.\end{aligned}$$

Since $k>2/(p-1)$, we can choose ε, γ such that

$$p\delta-\left(k-\frac{1}{2}\right)=(p-1)\left(k-\frac{2}{p-1}-\frac{\varepsilon}{p-1}-\frac{\gamma}{p-1}\right)>0.$$

Thus we obtain

$$|L|u|^p(x,t)|\leqq \frac{CM^p}{\sqrt{1+a+t}\,(1+a-t)^{k-1/2}}. \tag{3.13}$$

Case 2. $t\geqq a$. First, we estimate $\mathbf{I}_{D_1'}$. By (2.8) and changing variables by (3.8), we have

$$\begin{aligned}\mathbf{I}_{D_1'} &\leqq \frac{CM^p}{\sqrt{a}}\int_{D_1'}\ln\left[2+\frac{t-a}{a+r-t+s}\right]\frac{1}{(1+r+s)^{(p-1)/2}(1+r-s)^{pm-\varepsilon}}\,dr\,ds \\ &\leqq \frac{CM^p}{\sqrt{a}}\int_{t-a}^{t+a}\ln\left[2+\frac{t-a}{a-t+\alpha}\right]\frac{1}{(1+\alpha)^{(p-1)/2}}\int_0^\alpha\frac{1}{(1+\beta)^{pm-\varepsilon}}\,d\beta\,d\alpha.\end{aligned}$$

(a) $m=1/2$. We can estimate $\mathbf{I}_{D_1'}$ in the same manner as above Case 1(a) and $\mathbf{I}$ in Case 2 of the proof of Lemma 2.1. Hence we obtain

$$\mathbf{I}_{D_1'}\leqq \frac{CM^p\ln(2+t-a)}{\sqrt{(1+t+a)(1+t-a)}}. \tag{3.14}$$

(b) $m=(p-3)/2$. Similarly,

$$\mathbf{I}_{D_1'}\leqq \frac{CM^p\ln(2+t-a)}{\sqrt{1+t+a}\,(1+t-a)^{(p-3)/2}}. \tag{3.15}$$

(c) $m=k-(1/2)$. Similarly,

$$\mathbf{I}_{D_1'} \leqq \frac{CM^p \ln(2+t-a)}{\sqrt{1+t+a}\,(1+t-a)^{k-1/2}}. \tag{3.16}$$

$\mathbf{I}_{D_2'}$ can be estimated as above by changing variables by

$$\alpha = s+r, \qquad \beta = s-r. \tag{3.17}$$

To estimate $\mathbf{II}_{D_1''}$ we use (2.7) and make the change of variables as (3.8). Then,

$$\mathbf{II}_{D_1''} \leqq CM^p \int_{D_1''} \frac{1}{\begin{pmatrix}\sqrt{(t-s+a+r)(t-s-a-r)} \\ \times (1+r+s)^{(p-2)/2}\,(1+r-s)^{pm-\varepsilon}\end{pmatrix}}\,dr\,ds$$

$$\leqq \frac{CM^p}{\sqrt{t+a}} \int_0^{t-a} \frac{1}{\sqrt{t-a-\alpha}\,(1+\alpha)^{(p-2)/2}} \int_0^\infty \frac{1}{(1+\beta)^{pm-\varepsilon}}\,d\beta\,d\alpha.$$

(a) $m=1/2$. Since $p>4$, $\mathbf{II}_{D_1''}$ can be estimated in the same manner as in (2.24)–(2.26). Thus, we obtain

$$\mathbf{II}_{D_1''} \leqq \frac{CM^p}{\sqrt{(1+t+a)(1+t-a)}}. \tag{3.18}$$

(b) $m=(p-3)/2$. Since $1/2<(p-2)/2 \leqq 1$, as in (2.24), (2.27) we obtain

$$\mathbf{II}_{D_1''} \leqq \frac{CM^p \ln(2+t-a)}{\sqrt{1+t+a}\,(1+t-a)^{(p-3)/2}}. \tag{3.19}$$

(c) $m=k-1/2$. If $p(k-1/2)>1$, as above

$$\mathbf{II}_{D_1''} \leqq \frac{CM^p \ln(2+t-a)}{\sqrt{1+t+a}\,(1+t-a)^{m_0}} \qquad m_0 = \min\left(\frac{p-3}{2}, \frac{1}{2}\right)$$

$$\leqq \frac{CM^p \ln(2+t-a)}{\sqrt{1+t+a}\,(1+t-a)^{k-1/2}}. \tag{3.20}$$

If $p(k-1/2) \leqq 1$, we set $\delta = k-(2/p)-(\varepsilon/p)-(\gamma/p)$ with small $\varepsilon, \gamma>0$. $k>2/(p-1)>5/2p$ holds for $p<5$, and for $p \geqq 5$, since $k>1/2$, $k>5/2p$ holds. Hence we can choose ε, γ such that

$$p\delta = pk-2-\varepsilon-\gamma > \tfrac{1}{2}. \tag{3.21}$$

Thus, we have

$$\mathbf{II}_{D_1''} \leqq \frac{CM^p}{\sqrt{t+a}} \int_0^{t-a} \frac{1}{\sqrt{t-a-\alpha}\,(1+\alpha)^{p\delta}} \int_0^{\infty} \frac{1}{(1+\beta)^{1+\gamma}}\, d\beta\, d\alpha$$

$$\leqq \frac{CM^p \ln(2+t-a)}{\sqrt{1+t+a}\,(1+t-a)^{m_1}} \qquad m_1 = \min\left(p\delta - \frac{1}{2}, \frac{1}{2}\right).$$

Note that $1/2 \geqq k - 1/2$, since $m = k - 1/2$. Moreover, we can choose ε, γ such that

$$p\delta - \frac{1}{2} - \left(k - \frac{1}{2}\right) = (p-1)\left(k - \frac{2}{p-1} - \frac{\varepsilon}{p-1} - \frac{\gamma}{p-1}\right) > 0. \tag{3.22}$$

Therefore,

$$\mathbf{II}_{D_1''} \leqq \frac{CM^p \ln(2+t-a)}{\sqrt{1+t+a}\,(1+t-a)^{k-1/2}}. \tag{3.23}$$

To estimate $\mathbf{II}_{D_2''}$, using (2.7) and changing variables by (3.17), we have

$$\mathbf{II}_{D_2''} \leqq CM^p \int_0^{t-a} \frac{1}{\sqrt{t-a-\alpha}\,(1+\alpha)^{(p-2)/2}} \int_0^{\alpha} \frac{1}{\sqrt{t+a-\beta}\,(1+\beta)^{pm-\varepsilon}}\, d\beta\, d\alpha.$$

We can derive

$$\int_0^{\alpha} \frac{1}{\sqrt{t+a-\beta}\,(1+\beta)^{m_2}}\, d\beta$$

$$\leqq \frac{C}{\sqrt{1+t+a}} \qquad \text{for} \quad m_2 > 1 \text{ and } 0 < \alpha \leqq t-a \tag{3.24}$$

in the same manner as the estimate of the integral in (2.24) for $k > 1$. Therefore it follows that $\mathbf{II}_{D_2''}$ has the same estimate as $\mathbf{II}_{D_1''}$.

(ii) $k = 1/2$.

Case 1. $a \geqq t$. By (2.8) and changing variables by (3.8), we have

$$|L|u|^p(x,t)| \leqq \frac{CM^p}{\sqrt{a}} \int_{D'} \frac{\sqrt{r}\,(\ln(2+r+s)\ln(2+r-s))^p}{(1+r+s)^{p/2}}\,dr\,ds$$
$$\leqq \frac{CM^p}{\sqrt{a}} \int_{a-t}^{a+t} \frac{1}{(1+\alpha)^{(p-1)/2-\varepsilon}} \int_{a-t}^{\alpha} (\ln(2+\beta))^p\,d\beta\,d\alpha$$
$$\leqq \frac{CM^p}{\sqrt{a}} \int_{a-t}^{a+t} \frac{1}{(1+\alpha)^{(p-3)/2-\delta-\varepsilon}} \int_0^{\infty} \frac{1}{(1+\beta)^{1+\delta-\varepsilon}}\,d\beta\,d\alpha \quad (3.25)$$

with small $\delta > \varepsilon > 0$. $k = 1/2$ and $k > 2/(p-1)$ implies $p > 5$. Hence we can choose ε, δ such that

$$\frac{p-3}{2} - \delta - \varepsilon > 1, \qquad \delta > \varepsilon > 0. \quad (3.26)$$

Since the β-integral in the right-hand side of (3.25) is bounded, we see that the right-hand side is of exactly the same type as (2.9). Therefore,

$$|L|u|^p(x,t)| \leqq \frac{CM^p}{\sqrt{1+t+a}\,(1+a-t)^{(p-5)/2-\delta-\varepsilon}}$$
$$\leqq \frac{CM^p}{\sqrt{1+t+a}}. \quad (3.27)$$

Case 2. $t \geqq a$. As above

$$\mathbf{I}_{D_1'} \leqq \frac{CM^p}{\sqrt{a}} \int_{t-a}^{t+a} \ln\left[2 + \frac{t-a}{a-t+\alpha}\right] \frac{1}{(1+\alpha)^{(p-1)/2-\varepsilon}} \int_0^{\alpha} (\ln(2+\beta))^p\,d\beta\,d\alpha$$
$$\leqq \frac{CM^p}{\sqrt{a}} \int_{t-a}^{t+a} \ln\left[2 + \frac{t-a}{a-t+\alpha}\right] \frac{1}{(1+\alpha)^{(p-3)/2-\delta-\varepsilon}}\,d\alpha.$$

The last estimate is of exactly the same type as (2.15). Noting (3.26), therefore we obtain

$$\mathbf{I}_{D_1'} \leqq \frac{CM^p \ln(2+t-a)}{\sqrt{1+t+a}}. \quad (3.28)$$

Similarly,

$$\mathbf{I}_{D_2'} \leqq \frac{CM^p \ln(2+t-a)}{\sqrt{1+t+a}}. \quad (3.29)$$

Next we estimate $\mathrm{II}_{D_1''}$. By (2.7) and changing variables by (3.8), we have

$$\mathrm{II}_{D_1''} \leqq CM^p \int_{D_1''} \frac{r(\ln(2+r+s)\ln(2+r-s))^p}{\sqrt{(t-s+a+r)(t-s-a-r)}\,(1+r+s)^{p/2}}\,dr\,ds$$
$$\leqq \frac{CM^p}{\sqrt{t+a}} \int_0^{t-a} \frac{1}{\sqrt{t-a-\alpha}\,(1+\alpha)^{p/2-2-\delta-\varepsilon}}\,d\alpha$$

with small $\delta > \varepsilon > 0$.

This estimate is of exactly the same type as (2.24). Choosing ε, δ such that $(p-5)/2-\delta-\varepsilon>0$, therefore

$$\mathrm{II}_{D_1''} \leqq \frac{CM^p \ln(2+t-a)}{\sqrt{1+t+a}\,(1+t-a)^{m_3}} \qquad m_3 = \min\left(\frac{p-5}{2}-\delta-\varepsilon, \frac{1}{2}\right)$$
$$\leqq \frac{CM^p \ln(2+t-a)}{\sqrt{1+t+a}}. \tag{3.30}$$

As above, using (3.24) for $\mathrm{II}_{D_2''}$,

$$\mathrm{II}_{D_2''} \leqq CM^p \int_0^{t-a} \frac{1}{\sqrt{t-a-\alpha}\,(1+\alpha)^{p/2-2-\delta-\varepsilon}}$$
$$\times \int_0^{\alpha} \frac{1}{\sqrt{t+a-\beta}\,(1+\beta)^{1+\delta-\varepsilon}}\,d\beta\,d\alpha$$
$$\leqq \frac{CM^p \ln(2+t-a)}{\sqrt{1+t+a}}. \tag{3.31}$$

(iii) $0<k<1/2$.

Case 1. $a \geqq t$. It follows

$$|L|u|^p(x,t)| \leqq \frac{CM^p}{\sqrt{a}} \int_{D'} \frac{\sqrt{r}\,(\ln(2+r-s))^p}{(1+r+s)^{pk}}\,dr\,ds$$
$$\leqq \frac{CM^p}{\sqrt{a}} \int_{a-t}^{a+t} \frac{1}{(1+\alpha)^{pk-3/2-\delta}} \int_{a-t}^{\alpha} \frac{1}{(1+\beta)^{1+\delta-\varepsilon}}\,d\beta\,d\alpha.$$

Since $k>2/(p-1)>2/p$, we can choose small $\delta>0$ such that

$$pk-2-\delta>0, \qquad pk-\tfrac{3}{2}-\delta \neq 1. \tag{3.32}$$

Then we have in the same manner as Case 1 in the proof of Lemma 2.1:

$$|L|u|^p(x,t)| \leqq \begin{cases} \dfrac{CM^p}{\sqrt{1+t+a}\,(1+t-a)^{pk-5/2-\delta}} & \text{if } pk-\dfrac{3}{2}-\delta>1 \\ \dfrac{CM^p}{(1+t+a)^{pk-2-\delta}} & \text{if } 0<pk-\dfrac{3}{2}-\delta<1. \end{cases}$$

Choosing $\delta>0$ such that

$$pk-2-\delta-k=(p-1)\left(k-\frac{2}{p-1}-\frac{\delta}{p-1}\right)>0, \tag{3.33}$$

we obtain

$$|L|u|^p(x,t)| \leqq \frac{CM^p}{(1+a+t)^k}. \tag{3.34}$$

Case 2. $t \geqq a$. As above

$$\begin{aligned} \mathbf{I}_{D_1'} &\leqq \frac{CM^p}{\sqrt{a}} \int_{t-a}^{t+a} \ln\left[2+\frac{t-a}{a-t+\alpha}\right] \\ &\quad\times \frac{1}{(1+\alpha)^{pk-1/2}} \int_0^\alpha (\ln(2+\beta))^p\, d\beta\, d\alpha \\ &\leqq \frac{CM^p \ln(2+t-a)}{(1+t+a)^k}. \end{aligned} \tag{3.35}$$

Similarly, we see that $\mathbf{I}_{D_2'}$ has the same estimate as $\mathbf{I}_{D_1'}$. Next we estimate $\mathbf{II}_{D_1''}$. As before we have

$$\begin{aligned} \mathbf{II}_{D_1''} &\leqq CM^p \int_{D_1''} \frac{r(\ln(2+r-s))^p}{\sqrt{(t-s+a+r)(t-s-a-r)}\,(1+r+s)^{pk}}\, dr\, ds \\ &\leqq \frac{CM^p}{\sqrt{t+a}} \int_0^{t-a} \frac{1}{\sqrt{t-a-\alpha}\,(1+\alpha)^{pk-2-\delta}}\, d\alpha. \end{aligned}$$

This estimate is the same type as (2.24). Then it follows from (3.33) that

$$\mathbf{II}_{D_1''} \leqq \frac{CM^p}{(1+t+a)^k}. \tag{3.36}$$

Similarly, it follows from (3.24) that $\mathbf{II}_{D_2''}$ has the same estimate as $\mathbf{II}_{D_1''}$. This completes the proof of Lemma 3.1.

4. Proof of Theorem 1

Before proving Theorem 1, we have to make some preparations.

If there exists a solution $u \in C^2(R^2 \times [0, \infty))$ of (1.6) then u satisfies the integral equation

$$u = u_0 + LF(u), \tag{4.1}$$

where u_0 is the solution of (2.1) and L is given by (3.2). Conversely the following proposition holds (see [1, 3, 4]).

Proposition 4.1. *If u satisfies (4.1) and $D_x^\alpha u \in C(R^2 \times [0, \infty))$ for $|\alpha| \leqq 2$, u is the solution of (1.6).*

To prove Theorem 1, we define the norm for functions $u(x, t) \in C(R^2 \times [0, \infty))$

$$\|u\|_k = \begin{cases} \sup\limits_{\substack{x \in R^2 \\ t \geqq 0}} \dfrac{(1+t+|x|)^{1/2}(1+|t-|x||)^m}{\ln(2+|t-|x||)} |u(x,t)| & \left(k > \dfrac{1}{2}\right) \\ \sup\limits_{\substack{x \in R^2 \\ t \geqq 0}} \dfrac{(1+t+|x|)^{1/2}}{\ln(2+t+|x|)\ln(2+|t-|x||)} |u(x,t)| & \left(k = \dfrac{1}{2}\right) \\ \sup\limits_{\substack{x \in R^2 \\ t \geqq 0}} \dfrac{(1+t+|x|)^k}{\ln(2+|t-|x||)} |u(x,t)| & \left(0 < k < \dfrac{1}{2}\right), \end{cases} \tag{4.2}$$

where $m = \min(1/2, (p-3)/2, k-1/2)$.

Then, we have

$$|u(x,t)| \leqq C'_k \|u\|_k, \tag{4.3}$$

where $C'_k > 0$ is a constant depending only on k.

$$\||u|^\theta |v|^{1-\theta}\|_k \leqq \|u\|_k^\theta \|v\|_k^{1-\theta} \quad \text{for} \quad 0 \leqq \theta \leqq 1 \tag{4.4}$$

and by Lemma 3.1

$$\|L|u|^p\|_k \leqq C_{p,k} B_k \|u\|_k^p \tag{4.5}$$

$$\|L|u|^{\theta p} |v|^{(1-\theta)p}\|_k \leqq C_{p,k} B_k \|u\|_k^{\theta p} \|v\|_k^{(1-\theta)p} \quad \text{for} \quad 0 \leqq \theta \leqq 1, \tag{4.6}$$

where

$$B_k = \begin{cases} 1 & \left(k > 0, k \neq \frac{1}{2}\right) \\ \frac{1}{\ln 2} & \left(k = \frac{1}{2}\right) \end{cases}.$$

Let X_k be the linear space defined by

$$X_k = \{u(x, t) \colon D_x^\alpha u(x, t) \in C(R^2 \times [0, \infty)), \|D_x^\alpha u\|_k < \infty \text{ for } |\alpha| \leqq 2\} \tag{4.7}$$

with the norm

$$\|u\|_{X_k} = \sum_{|\alpha| \leqq 2} \|D_x^\alpha u\|_k.$$

We can verify easily that X_k is complete with respect to the norm $\|\cdot\|_{X_k}$. From now on to simplify the notation, we write X, $\|\cdot\|$, $\|\cdot\|_X$ instead of X_k, $\|\cdot\|_k$, $\|\cdot\|_{X_k}$, respectively.

Let $D_j = (\partial/\partial x_j)$ $(j = 1, 2)$. Let $u, v \in X$ with $\|u\| < 1$, $\|v\| < 1$.

We set

$$\Phi(x, t) = \max(|u(x, t)|, |v(x, t)|)$$

$$\|\Phi\| = \max(\|u\|, \|v\|).$$

It follows from (H1) and (4.6) that

$$\begin{aligned} \|L(F(u) - F(v))\| &\leqq A\|L(|\Phi|^{p-1}|u - v|)\| \\ &= A\|L(|\Phi|^{p\theta}|u - v|^{(1-\theta)p})\|, \qquad \theta = \frac{p-1}{p} \\ &\leqq C_{p,k} B_k A \|\Phi\|^{p-1} \|u - v\|. \end{aligned} \tag{4.8}$$

Similarly for the first derivatives, we have

$$\begin{aligned} &\|D_j L(F(u) - F(v))\| \\ &\quad = \|L\{(F'(u) - F'(v)) D_j u + F'(v)(D_j u - D_j v)\}\| \\ &\quad \leqq A\|L(|\Phi|^{p-2}|u - v|\,|D_j u|)\| + A\|L(|\Phi|^{p-1}|D_j u - D_j v|)\| \\ &\quad \leqq C_{p,k} B_k A \|\Phi\|^{p-2}(\|D_j u\|\,\|u - v\| + \|\Phi\|\,\|D_j u - D_j v\|). \end{aligned} \tag{4.9}$$

Finally for the second derivatives, we see that

$$\begin{aligned}
&\|D_iD_jL(F(u)-F(v))\| \\
&\quad=\|L\{(F'(u)-F'(v))\,D_i\,D_ju+F'(v)(D_i\,D_ju-D_i\,D_jv)\} \\
&\qquad+(F''(u)-F''(v))\,D_ju\,D_iu \\
&\qquad+F''(v)\,D_ju(D_iu-D_iv)+F''(v)\,D_iv(D_ju-D_jv)\}\| \\
&\quad\leqq A\|L(|\Phi|^{p-2}\,|u-v|\,|D_i\,D_ju| \\
&\qquad+|\Phi|^{p-1}\,|D_i\,D_ju-D_i\,D_jv|+|\Phi|^{p-3}\,|u-v|\,|D_ju\,D_iu| \\
&\qquad+|\Phi|^{p-2}\,|D_ju|\,|D_iu-D_iv|+|\Phi|^{p-2}\,|D_iv|\,|D_ju-D_jv|)\| \\
&\quad\leqq C_{p,k}B_kA(\|\Phi\|^{p-2}\,\|u-v\|\,\|D_i\,D_ju\| \\
&\qquad+\|\Phi\|^{p-1}\,\|D_i\,D_ju-D_i\,D_jv\|+\|\Phi\|^{p-3}\,\|u-v\|\,\|D_ju\|\,\|D_iu\| \\
&\qquad+\|\Phi\|^{p-2}\,\|D_ju\|\,\|D_iu-D_iv\|+\|\Phi\|^{p-2}\,\|D_iv\|\,\|D_ju-D_jv\|) \\
&\quad=C_{p,k}B_kA\{\|\Phi\|^{p-2}\,(\|D_i\,D_ju\|\,\|u-v\| \\
&\qquad+\|D_ju\|\,\|D_iu-D_iv\|+\|D_iv\|\,\|D_ju-D_jv\|) \\
&\qquad+\|\Phi\|^{p-1}\,\|D_i\,D_ju-D_i\,D_jv\|+\|\Phi\|^{p-3}\,\|D_iu\|\,\|D_ju\|\,\|u-v\|\}.
\end{aligned} \tag{4.10}$$

Now we define the closed subset of X

$$X_0=\{u\in X\colon \|u\|_X\leqq 12C_k\,D_kG\},$$

where C_k is a constant in (2.2) and

$$D_k=\begin{cases}\dfrac{1}{\ln 2} & (k>1)\\ 1 & (0<k\leqq 1).\end{cases}$$

It follows from Lemma 2.1 that

$$\|u_0\|_X\leqq 6C_k\,D_kG. \tag{4.11}$$

Hence, $u_0\in X_0$. We also define the map $\mathscr{F}$ by

$$\mathscr{F}u=u_0+LF(u).$$

Taking $u \in X_0$, $v=0$ in (4.8), (4.9), (4.10), we have

$$\|LF(u)\| \leqq C_{p,k} B_k A(12C_k D_k G)^p$$
$$\|D_j LF(u)\| \leqq 2C_{p,k} B_k A(12C_k D_k G)^p$$
$$\|D_i D_j LF(u)\| \leqq 5C_{p,k} B_k A(12C_k D_k G)^p.$$

Choosing G small enough such that

$$\tfrac{10}{3} C_{p,k} B_k A 12^p (C_k D_k G)^{p-1} \leqq 1, \tag{4.12}$$

we see that

$$\|\mathscr{F}u\|_X \leqq 6C_k D_k G + 20C_{p,k} B_k A(12C_k D_k G)^p$$
$$\leqq 12C_k D_k G,$$

which shows that $\mathscr{F}u \in X_0$ and $\mathscr{F}$ maps X_0 into itself.

Moreover, taking $u, v \in X_0$ in (4.8), (4.9), (4.10), we have

$$\|\mathscr{F}u - \mathscr{F}v\| \leqq C_{p,k} B_k A(12C_k D_k G)^{p-1} \|u-v\|_X$$
$$\|D_j(\mathscr{F}u - \mathscr{F}v)\| \leqq 2C_{p,k} B_k A(12C_k D_k G)^{p-1} \|u-v\|_X$$
$$\|D_i D_j(\mathscr{F}u - \mathscr{F}v)\| \leqq 5C_{p,k} B_k A(12C_k D_k G)^{p-1} \|u-v\|_X.$$

Thus from (4.12) we obtain

$$\|\mathscr{F}u - \mathscr{F}v\|_X \leqq 20C_{p,k} B_k A(12C_k D_k G)^{p-1} \|u-v\|_X$$
$$\leqq \tfrac{1}{2}\|u-v\|_X,$$

which shows that $\mathscr{F}$ is a contraction on X_0. Hence $\mathscr{F}$ admits a unique fixed point $u \in X_0$, which satisfies (4.1) and $D_x^\alpha u \in C(R^2 \times [0, \infty))$ for $|\alpha| \leqq 2$. Therefore, it follows from Proposition 4.1 that u is the solution of (1.6). This completes the proof of Theorem 1.

5. Proof of Theorem 2

To prove Theorem 2, we use the following proposition and lemma.

PROPOSITION 5.1. *Let $f(x)=0$ and $g(x) \in C^2(R^2)$ satisfy the hypothesis* (H4) *in* (2.1). *Then the solution u of* (2.1) *satisfies*

$$u(x, t) \geqq \frac{\varepsilon_0 t}{(1+t+|x|)^{1+k}} \qquad \textit{for} \quad t>0. \tag{5.1}$$

Proof. From (2.3), we have

$$u(x,t)=\frac{1}{2\pi}\int_{|x-y|\leqq t}\frac{g(y)}{\sqrt{t^2-|x-y|^2}}\,dy$$

$$=\frac{1}{2\pi}\int_0^t\frac{\rho}{\sqrt{t^2-\rho^2}}\int_{|\omega|=1}g(x+\rho\omega)\,d\omega\,d\rho$$

$$\geqq\frac{1}{2\pi}\int_0^t\frac{\rho}{\sqrt{t^2-\rho^2}}\int_{|\omega|=1}\frac{\varepsilon_0}{(1+|x+\rho\omega|)^{1+k}}\,d\omega\,d\rho$$

$$\geqq\frac{\varepsilon_0 t}{(1+t+|x|)^{1+k}}.$$

Hence we obtain the result.

Remark. From (2.3), $f(x)=0$ and $g(x)\geqq 0$ implies $u\geqq 0$.

LEMMA 5.2. *Assume that $F\in C^2(R)$ satisfies $F(u)\geqq 0$ for $u\geqq 0$. Let u be a C^2-solution of* (1.7) *with $g(x)>0$. Then $u>0$.*

Proof. This lemma is a special case of the Keller comparison theorem (see Keller [7]).

Now, we prove Theorem 2 following John [5]. We assume that a global C^2-solution u of (1.7) exists.

By Lemma 5.2, $u>0$. By (H3), we have

$$u\geqq u_0+ALu^p. \tag{5.2}$$

Since $u\geqq u_0$, it follows from Proposition 5.1 that

$$u(x,t)\geqq\frac{\varepsilon_0 t}{(1+t+|x|)^{1+k}}. \tag{5.3}$$

We assume that the following estimate holds:

$$u(x,t)\geqq\frac{Ct^a}{(1+t+|x|)^b}. \tag{5.4}$$

Here the constants C, a, b satisfy $C>0$, $a\geqq 1$, $b\geqq 0$. We observe that (5.4) with $a=1$, $b=1+k$, $C=\varepsilon_0$ corresponds to (5.3). Then from (5.2)

$$\begin{aligned}
u(x,t) &\geqq \frac{A}{2\pi}\int_0^t \int_{|x-y|\leqq t-s} \frac{u(y,s)^p}{\sqrt{(t-s)^2-|x-y|^2}}\,dy\,ds \\
&= \frac{A}{2\pi}\int_0^t \int_0^{t-s} \frac{\rho}{\sqrt{(t-s)^2-\rho^2}} \int_{|\omega|=1} u(x+\rho\omega, s)^p\,d\omega\,d\rho\,ds \\
&\geqq \frac{AC^p}{2\pi}\int_0^t \int_0^{t-s} \frac{\rho}{\sqrt{(t-s)^2-\rho^2}} \\
&\quad \times \int_{|\omega|=1} \frac{s^{ap}}{(1+s+|x+\rho\omega|)^{bp}}\,d\omega\,d\rho\,ds \\
&\geqq \frac{AC^p}{(1+t+|x|)^{bp}}\int_0^t s^{ap}\int_0^{t-s} \frac{\rho}{\sqrt{(t-s)^2-\rho^2}}\,d\rho\,ds \\
&= \frac{AC^p t^{ap+2}}{(ap+1)(ap+2)(1+t+|x|)^{bp}} \\
&\geqq \frac{AC^p t^{ap+2}}{(ap+2)^2\,(1+t+|x|)^{bp}}. \qquad (5.5)
\end{aligned}$$

We define the sequences a_n, b_n, c_n for $n=0, 1, 2, \ldots$ by

$$a_{n+1}=pa_n+2, \qquad b_{n+1}=pb_n, \qquad C_{n+1}=\frac{AC_n^p}{(a_n p+2)^2} \qquad (5.6)$$

$$a_0=1, \qquad b_0=1+k, \qquad C_0=\varepsilon_0. \qquad (5.7)$$

Solving (5.6), (5.7), we obtain

$$a_n=\frac{p^n(p+1)-2}{p-1}, \qquad b_n=p^n(1+k),$$

and thus

$$C_{n+1}=\frac{AC_n^p}{a_{n+1}^2}\geqq\frac{AC_n^p}{4p^{2n+2}}.$$

Then

$$\ln C_n \geqq \frac{\ln(A/4)}{p-1}(p^n-1)-2p^n\sum_{j=1}^{n}\frac{j}{p^j}\ln p+p^n\ln C_0.$$

For sufficiently large n, we have

$$C_n\geqq \exp(Ep^n),$$

where

$$E = \frac{1}{p-1} \min\left(0, \ln \frac{A}{4}\right) - 2 \sum_{j=1}^{\infty} \frac{j}{p^j} \ln p + \ln C_0.$$

Replacing a, b, C by a_n, b_n, C_n respectively in (5.4), we obtain

$$u(x, t) \geqq t^{-2/(p-1)} \exp\left[\left\{E + \frac{p+1}{p-1} \ln t - (1+k) \ln(1+t+|x|)\right\} p^n\right]. \quad (5.8)$$

Since $k < 2/(p-1)$, by choosing t large enough, we can find a positive δ such that

$$E + \frac{p+1}{p-1} \ln t - (1+k) \ln(1+t+|x|) \geqq \delta > 0.$$

Then, it follows from (5.8) for $n \to \infty$ that $u(x, t) = \infty$ for sufficiently large t. This is a contradiction. Therefore, we conclude that global C^2-solutions do not exist.

Acknowledgments

The author thanks Professor Takao Kakita and Professor Kenji Nishihara for their helpful suggestions and constant encouragement. It is also a pleasure to acknowledge Professor Mikhail Kovalyov for his very helpful advice.

References

1. R. Agemi and H. Takamura, The lifespan of classical solutions to nonlinear wave equations in two space dimensions, *Hokkaido Math. Journal* **21** (1992), 517–542.
2. F. Asakura, Existence of a global solution to a semi-linear wave equation with slowly decreasing initial data in three space dimensions, *Comm. Partial Differential Equations* **11**, No. 13 (1986), 1459–1487.
3. R. T. Glassey, Finite-time blow-up for solutions of nonlinear wave equations, *Math. Z.* **177** (1981), 323–340.
4. R. T. Glassey, Existence in the large for $\Box u = F(u)$ in two space dimensions, *Math. Z.* **178** (1981), 233–261.
5. F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.* **28** (1979), 235–268.
6. F. John, "Nonlinear Wave Equations, Formation of Singularities," Lehigh University, Lecture Series, Amer. Math. Soc., Providence, RI, 1990.
7. J. B. Keller, On solutions of nonlinear wave equations, *Comm. Pure Appl. Math.* **10** (1957), 523–530.

8. M. Kovalyov, Long-time behaviour of solutions of a system of nonlinear wave equations, *Comm. Partial Differential Equations* **12**, No. 5 (1987), 471–501.
9. M. Kovalyov, "Long-Time Existence of Solutions of Nonlinear Wave Equations," Ph.D. Thesis, Courant Institute of Mathematical Sciences, New York University, 1986.
10. K. Kubota, Existence of a global solution to a semi-linear wave equation with non compact supported initial data in two space dimensions, preprint.
11. J. Schaeffer, The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of p, *Proc. Roy. Soc. Edinburgh Sect. A* **101** (1985), 31–44.