Western Influence and Chinese Tradition in an Eighteenth Century Chinese Mathematical Work

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Ming Antu’s (?–1765?) Ge Yuan Mi Lü Jie Fa (Quick Methods for Trigonometry and for Determining the Precise Ratio of the Circle) illustrates nicely the situation of Chinese mathematics of the time, both by its history and by its contents. It contains the statement and proof of formulae of power series expansion of certain trigonometric functions, some of which had been introduced into China (without any proof) by a French Jesuit at the beginning of the 18th century. The author’s method consists of algebraicizing the traditional method of division of the circle (ge yuan). For this purpose he perfects an algebraic language constructed mostly by analogy with arithmetical operations. In one passage of the book, which appears as a defense of the Chinese tradition, he uses another type of analogy between arithmetic and algebra based on the geometric illustration of an algorithm. The way in which Western knowledge was introduced into China determined how it was assimilated by the Chinese: Euclidian geometry was adopted in its entirety, whereas power series, introduced divorced from their context (calculus), were reinterpreted so as to make sense within the Chinese system. At that time Chinese mathematics still maintained its independence: its specificity consisted of being a kind of synthesis between two mathematical traditions.

1.1. Historical Background

During the 17th and 18th centuries Jesuit missionaries introduced certain aspects of Western scientific knowledge into China. In the field of mathematics, the first Chinese translation of the first six books of Euclid’s *Elements* (the Chinese title being *Jihe Yuanben*) was made in 1670 from the Latin version of Clavius [Euclides 1574] by Matteo Ricci [1] and Xu Guangqi [2]. In addition to being the earliest, Euclidian geometry is probably the most significant branch of Western mathematics introduced into China during this period, since there was until then no deductive system based on axioms comparable to Euclidian geometry in the Chinese mathematical tradition; its geometry was instead based on the right-angled triangle (*gougu*). In arithmetic and algebra, areas usually considered more familiar to Chinese tradition, it seems that the Jesuits’ innovations had more to do with mathematical methods, such as written algorithms for elementary arithmetical operations which the Chinese at the time performed with the abacus [Needham 1959, 74–80], rather than with mathematical concepts. Symbolic algebra, on the other hand, was not widespread in China before the middle of the 19th century. This illustrates the fact that the Jesuits were not trying to promote modern science as it was developing in Europe at that time; they were instead using scientific knowledge as a means of arousing Chinese scholars’ interest in Christianity [3].

By the beginning of the 17th century an important part of China’s traditional mathematical knowledge had been lost or had become incomprehensible (this was most notable in the case of those advances in algebra made during the 13th century [Libbrecht 1973; Hoe 1977]). The Chinese, in learning Western science, not only rejuvenated their mathematics, but also rediscovered their own history, identifying some of the ancient methods with those they had learned from the Jesuits. The mathematics they created is a kind of synthesis between the two different types of mathematics.

1.2. The Ge Yuan Mi Lü Jie Fa

Ming Antu’s [4] *Ge Yuan Mi Lü Jie Fa* (Quick Methods for Trigonometry and for Determining the Precise Ratio of the Circle), completed in 1774, is characteris-
tic of this historical situation in both its contents and the history of its composition.

At the beginning of the 18th century, a French Jesuit, Pierre Jartoux [5] had introduced three formulae new to Chinese mathematicians: the power series expansions of the sine and versed sine functions and the series giving \( \pi \) that can be derived from the power series expansion of the arc sine function (first published in Europe by I. Newton [Newton 1967–1981, II, 237]). Jartoux provided no proof for them; calculus, on which they are based, remained unknown in China until the middle of the 19th century.

The Ge Yuan Mi Lü Jie Fu consists of the statement and proof of nine formulae: the “three formulae of Master Jartoux” (Du Shi san shu), the power series expansions of the chord and the sagitta, easily derived from those of the sine and versed sine, respectively, and those of the four inverse functions [6]. It is divided into four sections (juan). The first section contains the statement of the series and of a few elementary trigonometric formulae involving the difference between the given angle and those of 90°, 60°, 45°, and 30°. They are to be combined with the previous formulae so that the power series need be used only for “small angles,” the calculations thus involving only “simple numbers.” The second section gives some examples of utilizations of the formulae; these examples involve for the most part triangle problems and spherical trigonometry applied to astronomy. This gives an idea of the use of these power series in China at this time. The third and fourth sections contain the proofs of the formulae. Their structures are identical, the third section dealing with the chord and sine and the fourth with versed sine and sagitta.

It is possible that Ming Antu learned the three formulae from Jartoux himself, and the contents of his book as well as external evidence indicate that he derived the other six himself. After his death, his disciple Chen Jixin completed the book. Ming Antu was an astronomer of the Imperial Board of Astronomy (Qin Tian Jian), and his mathematical work seems to have been a hobby. This might be one of the reasons why his book, though completed in 1774, was not published before 1839 [Ming Antu 1839], when it was nevertheless not in the least out-of-date. In the meantime, the first section of the book, i.e., the statement of the nine formulae alone, had been circulated among Chinese mathematicians. Another proof of the formulae was provided at the beginning of the 19th century. To speak in more general terms, Ming’s book was the first of several similar works [Li 1955, 293–489].

2. MING ANTU’S GENERAL METHOD

2.1. Elementary Tools

Ming’s proofs are based on Euclidian geometry and calculations with a certain type of algebraic expressions. The link between the two is the use of continued proportions (lian bili) (a sequence of numbers \( \beta_i \) such that \( \beta_{i-1}/\beta_i = \beta_i/\beta_{i+1} \)), first as a geometric object, then as an algebraic language. These continued proportions are always built on the same pattern (Fig. 1): taking any arc of a circle, the first
The terms of the continued proportions are called $lù$. This term is rooted in Chinese tradition: the concept of $lù$ is already found in the Jiǔ Zāng Suan Shū (Arithmetic in Nine Sections, first century A.D.), where $lù$ is a term referring to any number in a series of numbers which are defined in relation to one another [Li 1982]. Thus, in the second chapter, the exchange rates of cereals are expressed by $lù$, each $lù$ corresponding to one type of cereal. Likewise, in the history of circle measurement in China, what Greek tradition referred to as the "ratio" between the circumference and the diameter is characterized by a pair of numbers, the $lù$ of the diameter and the $lù$ of the circumference; dealing with the same problem, the two traditions do not name the same objects. In the context of Euclidian proportions the term $lù$ is naturally used to designate the terms of a proportion.

2.2. An Outline of the Proof

2.2.1. The iterative process. I will describe briefly only how the author proves the formula of the chord, which can be written as

$$\sum_{n=0}^{\infty} (-1)^n a^{2n+1} r^{2n+4} (2n + 1)!,$$

where $a$ is the length of the arc and $r$ is the radius; the case of the versed sine is treated in a similar way. I will attempt to use terms in English equivalent to those found in the book, referring to the given arc as "the whole arc" ($quān hu$) and to the part as "a fraction of the arc" ($yī fēn hu$, $fēn$ here being a classifier).
Starting from any arc, Ming Antu divides it into $n$ equal fractions and expresses the chord of the whole arc in terms of that of the fraction of arc; $n$ is taken successively as equal to 2, 3, 4, 5, 10, 100, 1000, and 10,000. The eight intermediate formulae obtained are expressed in terms of $\ell \ell$ of continued proportions (we only give the four first terms; in the text eight terms arc explicitly given for the formulae occurring in the proofs which contain more than eight terms):

- $n = 2$, $c = 2\beta_2 - \beta_4/4 - \beta_6/4.16 - 2\beta_8/4.16^2 - \ldots$
- $n = 3$, $c = 3\beta_2 - \beta_4$,
- $n = 4$, $c = 4\beta_2 + 10\beta_4/4 + 14\beta_6/4.16 + 12\beta_8/4.16^2 + \ldots$
- $n = 5$, $c = 5\beta_2 - 5\beta_4 + \beta_6$,
- $n = 10$, $c = 10\beta_2 - 165\beta_4/4 + 3003\beta_6/4.16 - 21,450\beta_8/4.16^2 + \ldots$
- $n = 100$, $c = 100\beta_2 - 166,650\beta_4/4 + 3,330,003\beta_6/4.16 - 316,350,028,500\beta_8/4.16^2 + \ldots$
- $n = 1000$, $c = 1000\beta_2 - 166,666,665\beta_4/4 + 33,333,000,000,300\beta_6/4.16 - 3,174,492,064,314,285,000\beta_8/4.16^2 + \ldots$
- $n = 10,000$, $c = 10,000\beta_2 - 166,666,666,500\beta_4/4 + 3,333,333,000,000,003,000\beta_6/4.16 - 31,746,020,634,921,457,142,850,000\beta_8/4.16^2 + \ldots$

Each of these steps, except the first one, is derived from one or two of the previous ones. The first four are based on geometric reasoning, whereas the four final steps only involve combining the formulae obtained in the previous steps (thus for $n = 10 = 2 \times 5$, the formula is derived by composing those corresponding to $n = 2$ and $n = 5$). In the first four steps, some geometric arguments lead first to the establishment of a relation between the chord of the whole arc and that of the fraction of arc. This relation is then expressed in terms of continued proportions, and some computations on them finally result in the expression of the former in terms of the latter (in the first section as well as in the intermediate formulae occurring in the proofs, only the first terms are stated; there is no expression of the generic term comparable to the one using symbols in modern mathematics). In this last stage, the continued proportions, by providing an algebraic language, permit operation on terms that do not correspond to any of the magnitudes represented on the geometric figures. Two or three different methods of taking the same step are sometimes given.

All the steps corresponding to the different values of $n$ are treated in parallel, which emphasizes the iterative aspect of the procedure and suggests its generality (which is pointed out explicitly at the beginning of the proof), thus making up, in some way, for the lack of algebraic symbols.

2.2.2. Description of an elementary step. Let us give a brief account of the first method given to derive the formula in the case $n = 2$ (using a language similar to Ming Antu’s, we will call “triangles in continued proportions” similar isosceles triangles the sides of which form a continued proportion) (Fig. 2).

The purpose is to express $BD$ in terms of $BC$. They are not in continued proportion. However, constructing $BG = BC$ and $CH = DC$ on $BD$, and drawing $CG$ and $CH$, one gets
\[ BD = 2BC - GH. \]  

Now \( BCG \) and \( CGH \), isosceles triangles, are similar to \( ABE \) (as their angles in \( B \) and \( C \) respectively are both equal to half that of \( ABC \) in \( A \)). Because \( CGH \) is in continued proportion with the former, and \( BEF \) (and, going on, \( EFJ \) and \( FJK \)) with the latter, \( BC : GH = AB : EF \) and \( AB : BE = BE : EF \). This gives \[ GH = BC \cdot EF/AB. \]  

Constructing \( BL = 2BE \), and then by successive symmetries \( BM \) and \( BN \) (\( BI = BC \) is built so that \( ABC \) and \( BCI \) are in continued proportion), and then \( P \) and \( O \) so that \( CMO \) and \( MOP \) are in continued proportion (these two isosceles triangles are equal respectively to \( EFJ \) and \( FJK \)), one gets \[ CI = CM + MN + NI - OP. \]  

Now, let us consider the two continued proportions occurring in this reasoning. The first one is \[ \beta_1 = AB, \beta_2 = BC, \text{ and } \beta_3 = CI, \] whereas the second one can be defined as \[ \beta'_1 = \beta_1 = AB, \beta'_2 = 2BE, \beta'_3 = 4EF, \beta'_4 = 8FJ, \text{ and } \beta'_5 = 16JK. \]  
The relation (3) now becomes \[ \beta_3 = \beta'_3 - \beta'_5/16. \]
The relation between geometric magnitudes once being expressed in terms of continued proportions, everything that follows consists of computations on the $\tilde{\ell}$ of these continued proportions. The next step is the inversion of this relation: having derived all the $\beta_{i+1} = \beta_{i-1} \cdot \beta_i / \beta_i$ in terms of the $\beta^i$ starting from $i = 1$ (formula (4)) up to $i = 8$, one obtains a triangular linear system which can be solved by successive subtractions. One then obtains the inverse of (4):

\[ \beta_i^i = \beta_3 + \beta_3/16 + 2\beta_3/16^2 + 5\beta_3/16^3 + 14\beta_3/16^4 + 42\beta_3/16^5 + 132\beta_3/16^6. \quad (5) \]

$\beta_3^i$ is equal to $4EF$. Writing (2) in terms of continued proportions simply means increasing the ranks of all the $\tilde{\ell}$ of $1$ in (5), and dividing all the terms by 4. Writing (1) in terms of continued proportions finally gives

\[ BD = 2\beta_2 - \beta_{14}/4 - \beta_{14}/4.16 - 2\beta_{14}/4.16^2 - 5\beta_{14}/4.16^3 - 14\beta_{14}/4.16^4 - 42\beta_{14}/4.16^5 - 132\beta_{14}/4.16^6. \quad (6) \]

(The subtraction corresponding to $2BC - GH$ is displayed in Fig. 3.)

2.2.3. Limits. After the eight formulae corresponding to the successive values of $n$ are obtained, the formula of the chord is finally derived by “taking the limit” of those obtained in the parallel steps described above. More precisely, Ming compares the coefficients of the last three formulae. For instance, in the case of the second $\tilde{\ell}$ of the continued proportion they are

<table>
<thead>
<tr>
<th></th>
<th>1/24.0024</th>
<th>for $n = 100$,</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1/24.000024</td>
<td>for $n = 1000$,</td>
</tr>
<tr>
<td></td>
<td>1/24.00000024</td>
<td>for $n = 10,000$.</td>
</tr>
</tbody>
</table>

The limit (which is known beforehand, since, as far as the direct formulae are concerned, Ming Antu’s proofs are essentially verifications of Jartouqu’s formulae) is taken to be $1/24$. The author justifies this as follows: straight lines ($\chi xian$) and circular lines ($yu an xian$) belong to different categories ($bu tong lei$); this corre-

\[
\begin{array}{cccccccc}
4 & 16 & 16 & 16 & 16 & 16 & 16 & 16 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]

Fig. 3. Transcription of a table displaying a subtraction which can be written as:

\[
2\beta_2 - \beta_{14}/4 - \beta_{14}/4.16 - 2\beta_{14}/4.16^2 - 5\beta_{14}/4.16^3 - 14\beta_{14}/4.16^4 - 42\beta_{14}/4.16^5 - 132\beta_{14}/4.16^6.
\]
sponds to the fact that the exact formula cannot be obtained in a finite number of steps; the difference of nature between the exact numbers and the approximate numbers reflects that between the circular line and the straight lines (by which the former is approached). The closer one approaches the arc, the more minute the difference between the numbers is; the two categories of lines can thus be unified (yi) by choosing the integer toward which the values obtained by finite procedure tend.

The power series thus obtained gives the chord (second lū of a continued proportion) in terms of the arc (second lū of another continued proportion, the first lū being the radius in both cases). To get the inverse formula, Ming then calculates the lū of the second continued proportion in terms of those of the first, deriving as many of them as are stated explicitly in the power series expansion of the chord. As in the derivation of the formula for \( n = 2 \) given above, he then has a triangular linear system, and by successive subtractions, finally obtains the arc in terms of the continued proportion built with the chord. The similarity between the two calculations is explicitly stated in the book. This suggests that the inversion method might have been perfected first to prove Jartoux's formulae, and then could have permitted the derivation of the inverse formulae that he had not given.

### 2.3. Sources of Inspiration

Ming Antu's general method appears as a generalization of a method occurring both in Chinese tradition and in the Western geometry introduced by the Jesuits, the method of division of the circle (ge yuan [Qian 1983]). In China, it is first found in Liu Hui's commentary (third century A.D.) on the Jiu Zhang Suan Shu [Qian 1963, 103–108]. The idea of this method is to approximate the circle by inscribed polygons, the number of sides of which is doubled at each step. The term ge yuan appears in the title of Ming's book. (I translate it there by "trigonometry" as it seems to have been used as a Chinese equivalent of this term in Ming's time: for instance, the trigonometric tables were called Ge Yuan Ba Xian Biao, literally "Tables of the eight ge yuan lines"; there was also a literal equivalent of the term "trigonometry," sanjiaodzu. The notion of angle, introduced by the Jesuits, is not found in Chinese tradition; however, in Ming's time, it had been assimilated by the Chinese and was used in circle measurement.) One of the book's prefaces (written in 1839 by Cen Jiangong, the editor of the book), recalling the history of circle measurement and of the method of division of the circle in China [Jami 1988], considers Ming's work a continuation of that tradition. Moreover, at the beginning of Section 3, Chen Jixin explains the origin of his master's idea as follows: "The ancient method includes a method for dividing an arc of a circle into two parts (er fen hu fa) [this refers to the traditional Chinese method of division of the circle, in which the operation of deriving the side of the 2n-sided regular polygon inscribed in the circle from that of the n-sided polygon is iterated]. The Western method includes a method for dividing an arc of a circle into three parts (san fen hu fa) too [this refers to the trisection of a 60° arc, to obtain the side of an
18-sided regular polygon; both this method and the ge yuan method are found in
the Yu Zhi Shu Li Jing Yun [7]. So it must also be possible to divide progressively
(di fen). . . .”

Ming generalizes the traditional method by using continued proportions as an
algebraic language, so that it extends to the measurement of any arc. Some of its
characteristics are discussed in the book, such as iteration, deduction of each step
from the previous one(s), and the very idea of dividing a whole arc into several
equal parts, the number of which increases at each step. Iteration and parallelism
play a very important part not only as a means of expression, but also as an
element in the structure of the book. Parallelism is used in the language of the
elementary steps of the proof to show that iteration is the backbone of the whole
procedure. It is also used in Section 1 in the stating of the formulae, providing a
uniform language fit for the expression of the recurrence formula by which each
term is deduced from the previous one. On the other hand, the exact parallelism
between Sections 3 and 4 is the result of a choice on the author’s part regarding
the structure of the book: the step $n = 4$ in the case of the chord is never used in
further steps, and seems to be there only as the symmetric step to the one that is
necessary in the case of the versed sine. This choice can be linked to the impor-
tance of parallelism as a characteristic of Chinese thought in general.

It is quite remarkable that both the form (parallelism and iteration) and the
substance (the ge yuan method) of Ming Antu’s proofs for “Jartoux’s formulae,”
which are typical of 17th century European mathematics, seem to be rooted in
Chinese tradition.

3. FROM ARITHMETIC TO ALGEBRA

There are two processes of algebraization in the book: One is that used to
generalize the ge yuan method, the language and notation defined for that purpose
being used throughout the book. The other one concerns only a short passage of
the book; it will here be analyzed further.

3.1. The Algebraic Language

Continued proportions are the basis of Ming Antu’s algebraic language. He
seems to have elaborated it himself, starting from notations and notions found in
the mathematics of his time. The way in which calculations on the lü of the
continued proportions are performed seems to be inspired both by the usual
notation of proportions and by the performing of arithmetical operations on paper
as they are found in the Yu Zhi Shu Li Jing Yun. In Ming’s book, they are shown
in tables (Fig. 3); the system of representation used is one of place-value. The
ranks of the lü occurring are written horizontally, increasing from right to left. The
value of each column is thus indicated once and for all in the table. This appears as
a generalization of the way proportions were written at the time: the four lü were
written in four columns, each bearing the rank of the corresponding lü. But, whereas this common notation is used only to express the lü of proportions, Ming
Antu uses it to perform computations. Since the coefficients assigned to the \( \text{lù} \) are usually fractions, the factors of the denominators are written above the rank; most of them are in geometric progression and are not affected by the operations. The numerators are written below, with their sign on their right (the characters \text{dùo} and \text{shào} corresponding respectively to the signs + and −). The whole expression is read from right to left, whereas numbers are written from left to right. This notation is similar both to the Western one for numbers and to the Chinese representation of numbers on the abacus, and before that with counting-rods (using a decimal place-value notation, read from left to right, in decreasing order of magnitude in both cases). However, numbers were not written down in Chinese texts in the same way with the usual characters used as figures until the Jesuits introduced written algorithms for elementary arithmetical operations. On the other hand, the algebraic notations found in the \text{Yu Zhi Shu Li Jing Yun} are read from left to right [Jami 1985, 43-49].

Addition and subtraction are performed on the \( \text{lù} \) column by column, which was the usual way to treat numbers in China at the time. The other type of operation is that of obtaining a \( \text{lù} \) of some continued proportion while knowing the previous ones (generally in terms of another continued proportion). The first table displaying such an operation which occurs is described as follows (Fig. 4): “First set out the coefficients above. Set them out again as in multiplication. Multiply the ‘above’ term with the ‘below’ term. There is no number for the first \( \text{lù} \), write a 0 underneath. Multiply the 1 of the second \( \text{lù} \) by the 0 of the first \( \text{lù} \), write a 0 underneath; multiply it by the 1 of the second \( \text{lù} \), this gives 1; write it under the third \( \text{lù} \). The lowering of one rank represents the division by the first \( \text{lù} \).”

The analogy with multiplication is explicit and evident from the table, whereas the “lowering of one rank” does not appear in it. A possible interpretation of its being mentioned in the text is that the rank of the \( \text{lù} \) obtained is the sum of those of the \( \text{lù} \) multiplied, minus one. Here we have \( \beta_2^2 / \beta_1 = \beta_3 \), corresponding to \( 2 + 2 - 1 = 3 \). Formally, multiplication and division of \( \text{lù} \) of the continued proportion would correspond to addition and subtraction of their ranks, respectively. Such a relation is stated for powers in the \text{Yu Zhi Shu Li Jing Yun}.

The notation Ming Antu uses to deal with continued proportions is derived from the place-value notation in the arithmetic of his time by replacing the powers of 10 (or successive figures) by the ranks of the \( \text{lù} \) in the tables which set out the calculations. Chinese algebra, which was based on the very same idea of place-value notation, was being rediscovered at the time Ming Antu was working on his
The notations of the 13th century algebra seem to be derived from the arrangement of counting-rods for root extraction [Chemla 1987], as well as from analogy between numbers and algebraic expressions. However, there is no evidence that they influenced Ming’s notations for continued proportions.

3.2. The “Third Method”

3.2.1. The metaphor of square root extraction. A passage of the text [9] presents another type of analogy between arithmetic and algebra. It consists of solving a right-angled triangle, the sides of which are expressed as algebraic terms. The base \(a\) (gou, i.e., the smaller side adjacent to the right angle) is the first lü of a continued proportion, the hypotenuse \(c\) (xian) is twice the second lü, and the purpose of the process is to express the altitude \(b\) (gu, i.e., the longer side adjacent to the right angle) in terms of that continued proportion. In modern terms, this would be the expansion of \(\sqrt{c^2 - a^2}\). The basic idea of Ming Antu’s method is to approach \(b\) by \(c\) in the right-angled triangle relation which is expressed as

\[
c - b = a^2|(c + b),
\]

and in the successive differences between the exact term and the approximation thus obtained. The terminology used in describing these computations is the same as that found in the square root extraction algorithm in the *Yu Zhi Shu Li Jing Yun*; it is very similar to the traditional one [Lam 1969], strongly connected with both geometric illustration and division terminology. I will refer hereafter to only some aspects of the algorithm, which can be summed as follows:

Let the number \(A\), the square root of which is to be extracted, be represented by a square. Its root is determined one decimal place at a time: these are the successive *shang* (the term refers to the quotient, in the context of division; they are called, respectively, *chushang*, *chishang*, *sanhang*, . . .). The factors involved in the calculation of the successive remainders (except the first one) are called the side-factors (*liangfa*) and corner factors (*yuafa*). If the square root of \(A\) is written as

\[
100a + 10b + c \quad \text{(with } 0 \leq a, b, c \leq 9 \text{ and } a \neq 0),
\]

then the first remainder is \(A - (100a)^2\). In the next two steps, the side-factors and corner-factors are respectively \(2 \times 100a\) and \(10b\) in the calculation of the second remainder, and \(2(100a + 10b)\) and \(c\) in the calculation of the third remainder. Each remainder is obtained by subtracting from the previous one the product of the corner-factor by the sum of the two factors.

The algebraic terms referred to as side-factors and corner-factors in Ming Antu’s text are processed in the same way as the numbers given that name in the *Yu Zhi Shu Li Jing Yun*’s root extraction algorithm: the operation (side-factor + corner-factor) · corner-factor is performed at each step of what appears as an iterative process. This suggests that a metaphor consisting of the use of terms traditionally denoting numbers to refer to algebraic expressions, in order to point
to the similarity of the processing of the two) is used to hint at the type of operation that is being performed, i.e., the square root extraction of an algebraic expression. This metaphor extends the language of arithmetic to algebra.

3.2.2. Step-by-step analysis. In order to understand the mathematical meaning of this analogy between arithmetic and algebra, it is necessary to analyze the procedure step by step. The successive steps give a sequence of quotients $q_i$, and the corresponding remainders $r_i$ (using the terminology of the square root extraction mentioned above) can be expressed as follows in terms of the base $a$ and the hypotenuse $c$:

$$
\begin{array}{l}
0 & c \\
1 & a^2/2c \\
2 & a^4/8c^3 \\
3 & a^6/16c^5 + a^8/128c^7 \\
4 & a^8/32c^9 + 3a^{10}/256c^9 + 3a^{12}/1024c^{11} + a^{14}/2048c^{13} \\
5 & a^{10}/64c^{19} + 5a^{12}/512c^{11} + 5a^{14}/1024c^{13} \\
6 & a^{12}/128c^{19} + 7a^{14}/1024c^{13} \\
7 & a^{14}/256c^{19} \\
\end{array}
$$

The quotients and remainders are not expressed in this way in the text, but are expressed instead as $lù$ of continued proportions (the first line is put between brackets since the term $q_0$ does not appear as a quotient in Ming Antu’s algorithm). However, this transcription enables us to compare Ming Antu’s algorithm to that given by Newton in *De Methodis Serierum et Fluxionum* [10] [Newton 1967–1981, III, 32–353], which would give, for the extraction of the square root of $c^2 - a^2$: 

$$
\begin{array}{l}
0 & c \\
1 & a^2 \\
2 & a^4/4c^2 \\
3 & a^6/8c^4 + a^8/64c^6 \\
4 & a^8/16c^6 + 3a^{10}/128c^6 + 3a^{12}/512c^{10} + a^{14}/1024c^{12} \\
5 & a^{12}/32c^{10} + 5a^{14}/256c^{10} + 5a^{16}/512c^{12} \\
6 & a^{16}/64c^{18} + 7a^{18}/512c^{12} \\
7 & a^{18}/128c^{18} \\
\end{array}
$$
Newton performs the operation strictly in increasing order of powers of $a$. He points to the analogy between arithmetic and algebra which enables him to perform algebraic operations: "To each single place in a decimal sequence decreasing continually to the right there corresponds a unique term in a variable array ordered according to the sequence of the dimensions of numerators or denominators continued in uniform progression to the infinity" [Newton 1967–1981, III, 35].

3.2.3. Geometric interpretation. Ming Antu proceeds quite differently. The parallel he draws between arithmetic and algebra is in fact based on the geometric interpretation of root extraction algorithms. This can be made clear by analyzing the geometric reasoning which is the basis of each step of his algorithm (Fig. 6).

$Bw$ is the hypotenuse $c$ of the right-angled triangle, and $zw$ is the unknown altitude $b$. The areas of squares $By$ and $zb$ are their respective squares, so that the area of the gnomon (qingzheji) $Buyx$ is the square of the base $a$.

The author wants to approximate $Bz$ which is equal to $c - b$. His first approximation being, as mentioned before, $a^2/2c$, the difference between the gnomon
Buyx and the gnomon Bgyx, corresponding respectively to the exact value and to
the approximation, is determined by a geometric reasoning: (i) The sum of the
areas of the rectangles ey and Bd is equal to the area of the gnomon Buyx. (ii) On
the other hand, the former is also equal to the sum of the areas of the square ed
and of the gnomon Bgyx. (iii) Thus the area of the gnomon cafg is equal to that of
the square ed. Because this gnomon is precisely the difference between the two
gnomons mentioned above, the value of the difference of areas, i.e., the first
remainder in our transcription, is $a^4/4c^2$.

This is the first step of the geometric reasoning, stated in the text before the
"algebraic algorithm." It displays an important characteristic of traditional Chi-
nese geometry, namely, the use of the "out–in complementary principle," defined
by Wu [1983] as "the assumption of the following obvious facts: 1) The area of a
planar figure remains the same when the figure is rigidly shifted to another place of
the plan. 2) If a planar figure is cut into several sections, the sum of the areas of the
sections is equal to the area of the original figure. It follows that the areas of the
various sections involved before and after the out–in procedures possess simple
arithmetical relations." He then analyzes one of the best-known examples of
utilization of this principle in the proof of the right-angled triangle theorem as it
has been reconstructed from ancient Chinese texts. The principle is used in
Ming’s reasoning, first to state the equality (ii) and then to derive (iii) from (i) and
(ii), by "subtracting" the gnomon Bgyx from the gnomon Buyx. On the other
hand, (i) is based on the relation existing between the algebraic expressions repre-
sented in the figure.

The second step is similar to the first one, except that the remainder is not
represented by a square but by a gnomon: (i) The sum of the areas of the rectan-
gles ei and dj is equal to the area of the gnomon cafg. (ii) On the other hand, the
former is also equal to the sum of the areas of the gnomons clf and hlkg. (iii) Thus
the area of the gnomon jail is equal to that of the gnomon hlkg. A third similar step
is described.

In terms of the algorithm, the procedure consisting of steps (i), (ii), and (iii) is
equivalent to the calculation of the remainder $q$, as expressed above.

On the other hand, from the geometrical point of view, the reasoning can be
summed up as follows: the problem of finding the "corner" of a gnomon, knowing
its area and its "side," is reduced to that of finding the side of a square, thanks to
the out–in procedure, which permits the expression of the gnomon in terms of a
part of the square corresponding to a step of the root extraction procedure. This
reduction is not done once and for all in the reasoning, but occurs at each step.

Ming Antu’s metaphor of root extraction is not applied to the expression $c^2 - a^2$,
which is represented by the square aw, but to $(c - b)^2$, represented by the
square xa; the latter is not actually drawn on the figure, while the "approached
squares" xg, xl, and xp are.

3.2.4. Chinese tradition in the face of Western innovation. The algorithm de-
scribed above constitutes the main point of the Third Method (San Fa) of obtaining
the expression of the chord of an arc in terms of that of the half arc. It uses
Chinese traditional methods, in strong contrast to all the other elementary steps of the same type (the first method has been outlined above); the former involves areas, while the latter involve only lines. The terminology of the Third Method is also specific, not only because of the occurrence of words characteristic of the traditional root extraction algorithms, but also because of the way right-angled triangles are referred to: the word used is gouguxing (gougu is the traditional term, gou and gu being respectively the smaller and longer side of the right angle; xing is the suffix equivalent to “figure” in the Euclidian geometry the Jesuits introduced). Moreover, the sums or differences of two or three of the right-angled triangle’s sides that occur in this passage are referred to using terms identical to those in Chinese works of the 13th century [Hoe 1977, 88–90]. In other parts of the book these terms do not occur, and gouguxing and zhijiao sanjiaoxing (literally “right-angled figure with three angles,” which is the term created by the Jesuits) are used interchangeably to refer to the right-angled triangle. Thus, the Third Method appears as the part of the book proposing a Chinese alternative to the Western methods: it is all the more striking because there is no necessity for it in the book, since two Euclidian-type reasonings leading to the same result have been given before (the first uses mostly parallelism, the second orthogonality). The purpose of the author in giving a Third Method might simply have been to assert the value of Chinese tradition in the face of Western innovations.

The basis of the analogy between arithmetic and algebra is here the geometric interpretation of the problem. Ming Antu does not seem to consider the possibility of generalizing such a method: the converse of the versed sine formula obtained by the inversion method described above gives the square of the arc in terms of the versed sine (the situation is similar to that with the sagitta formula). This formula is given without any comment, and no attempt seems to have been made to “extract its square root.” This, as well as the fact that the Third Method is not based on the same place-value notation idea as the general calculations, suggests that there is no general and unified algebraic formalization which is used uniformly, as Newton uses the symbolic notation to perform on “literal numbers” any operation that is performed in arithmetic.

4. MATHEMATICAL OBJECTS AND THEIR CONTEXT

It is now possible to describe some characteristics of Ming Antu’s mathematics as representing a kind of synthesis between two different types of mathematics.

4.1. Power Series

Considering the modern expression which describes most accurately the formulae he deals with, “power series expansions of trigonometric functions,” it appears that the only term which had an equivalent in the 18th century Chinese mathematics is “trigonometry,” the Chinese term being sanjiaoshu or ge yuan (the latter refers to the Chinese traditional method of division of the circle, as mentioned above). There was no notion comparable to that of function (the term was used by Leibniz for the first time, and the concept in Europe at the time was
quite different from the modern one [Youschkevitch 1976]); in Chinese mathematics trigonometric lines had a geometric definition (found, for instance, in the *Yu Zhi Shu Li Jing Yun*), and their numerical values were expressed in tables [11] (the eight lines being the sine, cosine, versed sine, versed cosine, tangent, cotangent, secant, and cosecant). According to Cen Jiangong’s preface to the book, what is most interesting in Ming Antu’s formulae is that they make up for these tables.

The power series, as a mathematical object, was introduced into China divorced from its European context, i.e., calculus. Ming Antu’s way of dealing with it shows that the object had been reinterpreted in order to make sense within the framework of the Chinese mathematical system. In that context, it appears as a finite sum of the terms of a sequence, expressed as *lù* of a continued proportion with certain coefficients, each term of the sequence being obtained from the previous one by a recurrence formula. The number of terms to be taken into account for the sum is chosen according to the precision needed. The most appropriate way to characterize the object is once more to use a parallel between arithmetic and algebra: the series Ming Antu deals with is an expansion of the same type as the decimal expansion of numbers, applied to variable quantities. There is no explicit notion of expansion in the book, and no specific name is given to the object just described. However, the fact that Ming Antu derived six “power series” by himself and that more were derived by his successors suggest that there was some awareness of a new specific object among Chinese mathematicians.

The “power series” being thus defined, some of the basic questions they aroused (or answered) in 18th century European mathematics were completely irrelevant to the Chinese framework. There was no reason to consider the convergence or the meaning of infinite sums in it: the Chinese approach to the power series is essentially different.

4.2. The Status of Proof

In the same way, what is the status of Ming Antu’s derivation of the formulae? In more general terms, what constitutes a proof in Chinese mathematics of this time? According to Cen Jiangong’s preface to the book, Ming Antu’s proof of Jarioux’s three formulae (and derivation of the six others) was acknowledged as a valid justification for them; the preface mentions that “Wang Lai, at first, disparaged these formulae, saying they had been obtained by chance. Having seen this book, he completely changed his mind.” Wang Lai (1768–1813) was one of the greatest mathematicians of his time. This passage shows that he was not content with the formulae, although they were efficient, as long as they were not justified: they could not be accepted as mathematical knowledge without justification. On the other hand, he found the one given by Ming Antu sufficient. Thus, the nature of the “proofs” in the book is relevant to understanding their status in Chinese mathematics at the time.

The title of Sections 3 and 4 in the table of contents is *Fa Jie*, literally “Explanation of the Formulae.” The term *jie* does not seem to occur in Chinese traditional texts. In the propositions of Ricci’s *Jihe Yuanben*, the *jie* is placed between the
statement of the proposition in general terms and its proof: it consists of the translation of the proposition’s hypotheses into the description of a geometric figure having those required properties, or, in other words, in a language fit for the proof (the proof itself is called lun). However, the title given at the beginning of the section is different: Tu Jie, which means “illustration.” It must be pointed out that there seems to be no general equivalent for the term “proof” in the terminology of the time.

Ming Antu’s geometric reasonings are quite similar to “usual” Euclidian geometry. This is a sign of his mastery of the subject. Both his “power series” and the way he deals with limits reveal a practice of mathematics closely concerned with the nonmathematical world. The power series appear as partial sums of sequences; the number of terms actually used in them is determined by an external criterion, namely, the precision required in the calculations involving trigonometric lines that are being performed. On the other hand, the “criterion of convergence” that can be deduced from the text is an “experimental” one. The observation of the value of the coefficients of the lü (until the rank 16) of the continued proportion, in three approximation formulae, is sufficient for verifying that they tend to the coefficient of the final power series (and that the sequences of coefficients of the lü of all ranks converge likewise). This procedure is reminiscent of the verification of a scientific law by experimentation in which margins of error are taken into account. It suggests that Ming Antu’s practice as an astronomer might have influenced his conception of proof. To speak in more general terms, one might wonder about the effect of the subordination of mathematics to astronomy on the methods used in the mathematics of the time.

5. CONCLUSION

The way mathematical knowledge was introduced into China determined its assimilation by the Chinese and its integration in their mathematical system. In this respect, one can only contrast Euclid’s Elements and “Jartoux’s formulae.” Whereas the former, presented as a complete system, was (to some extent) accepted by the Chinese as a whole (even if they tried to interpret it as a part of their own tradition [Martzloff 1981]), the latter, given without any hint of their meaning within their original context, were understood and used consistently in the Chinese system, the Chinese mathematicians having thus defined a different object. On the other hand, the story of “Jartoux’s formulae” is typical of the scientific innovations of the Jesuits at the end of the 17th and at the beginning of the 18th century: a system of astronomical and mathematical knowledge having been given in the Xi Yang Xin Fa Li Shu [12], there never was any organized attempt to modify its contents in accordance with the evolution of science in Europe. Thus, only isolated elements were introduced. Western and Chinese methods sometimes merely appear juxtaposed, or opposed in Ming Antu’s book. However, his work as a whole appears as a synthesis of both, at least because the structure of the proofs is similar to that of the Chinese traditional ge yuan method, and of the
importance of parallelism in the mathematical style of the author: it provides a language to introduce imported concepts into the Chinese mathematical system. Through the *Ge Yuan Mi Lü Jie Fa*, Chinese mathematics during the 18th century and the first part of the 19th century appears to have been a lively and productive discipline in which the Chinese did much more than merely assimilate what they had been taught by the Jesuits; they continued in the directions determined by that teaching. It maintains a separate identity, an independence characterized by its being a synthesis between Western and traditional Chinese elements.

GLOSSARY

bu tong lei 不同類
Cen Jiangong 岑建功
Chen Jixin 陳際新
Chi Shui Yi Zhen 赤水逸珍
Chongzhen Li Shu 崇禎曆書
chushang 初商
cishang 次商
di fen 邁分
Du Shi san shu 杜氏三術
duo 多
er fen hu fa 二分弧法
Fa jie 注解
fen 分
gou 句
gougu 旬設
gouguxing 句設形
gu 股
Jihe Yuanben 幾何原本
jie 解
Jiu Zhang Suan Shu 九章算術
Jiu Zhang Suan Shu de bilu lilun 九章算術的比率理論
Jiu Zhang Suan Shu yu Liu Hui 九章算術與劉徽
juan 卷
Li Jimin 李繼閔
Li Yan 李演
lian bili 達比例
lianta 理法
Liu Hui 演繹
lun 論
lu 質
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NOTES

1. Matteo Ricci (1552–1610) was the founder of the Jesuit mission in China. Clavius had been his master in Rome [Pfister 1932–1934, 22–42].

2. Xu Guangqi (1562–1633) was one of the first high-ranking Chinese officials to convert to Christianity. He studied Western science with the Jesuits.
3. Ricci was the first to promote such a policy, choosing to gain the friendship and esteem of the scholars, and, through them, the Court [Gernet 1982, 25–38; Bernard-Maître 1935].

4. Ming Antu (?–1765?) was an astronomer. Educated at the time of the Kangxi Emperor, he worked with the Jesuits in cartography, and later, in modernizing some aspects of the astronomical system. At the time of his death, he was director of the Imperial Board of Astronomy (Qin Tian Jian).

5. Pierre Jartoux (1669–1720) arrived in China in 1701 and belonged to the French mission; he was one of the teachers of Kangxi in mathematics and astronomy.

6. It is interesting to note that in his Tetsujutsu Sankei (1722), the Japanese mathematician Takebe Katahiro (1664–1739) gives one of the inverse formulae, which he derived independently and by a different method [Murata 1980].

7. The Yu Zhi Shu Li Jing Yun (Basic Principles of Mathematics Collected by Imperial Order) is a compendium issued in 1723; it includes both traditional and imported mathematics. I often refer to it hereafter, as it represents the basic culture of the 18th century Chinese mathematicians.

8. The first book that analyzes the 13th century positional algebra, the Chi Shui Yi Zhen (Pearls Lost in the Red River), by Mei Juecheng, was published in 1759.

9. The “Third Method” for deriving the chord of the whole arc in terms of that of the half arc (Section 3, pp. 18a–23b).

10. It seems very unlikely that the Chinese mathematicians of the 18th century ever became acquainted with Newton’s work. My purpose here is to compare two algorithmic procedures; this comparison brings out the difference between the two.

11. These tables were introduced by the Jesuits during the 17th century.

12. The Xi Yang Xin Fa Li Shu (Treatise on the Calendar According to the New Western Method) was first written under the name of Chongzhen Li Shu (Chongzhen Reign Period Treatise on the Calendar), between 1629 and 1635, by the Jesuits. It was a compendium of mathematics and astronomy, based on the Tychonic theory.

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