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Advances in Mathematics 229 (2012) 3436–3467

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# Almost free modules and Mittag-Leffler conditions

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Received 16 October 2009; accepted 14 February 2012

Available online 28 February 2012

Communicated by the Managing Editors of AIM

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## Abstract

Drinfeld recently suggested to replace projective modules by the flat Mittag-Leffler ones in the definition of an infinite dimensional vector bundle on a scheme  $X$  (Drinfeld, 2006 [8]). Two questions arise: (1) What is the structure of the class  $\mathcal{D}$  of all flat Mittag-Leffler modules over a general ring? (2) Can flat Mittag-Leffler modules be used to build a Quillen model category structure on the category of all chain complexes of quasi-coherent sheaves on  $X$ ?

We answer (1) by showing that a module  $M$  is flat Mittag-Leffler, if and only if  $M$  is  $\aleph_1$ -projective in the sense of Eklof and Mekler (2002) [10]. We use this to characterize the rings such that  $\mathcal{D}$  is closed under products, and relate the classes of all Mittag-Leffler, strict Mittag-Leffler, and separable modules. Then we prove that the class  $\mathcal{D}$  is not deconstructible for any non-right perfect ring. So unlike the classes of all projective and flat modules, the class  $\mathcal{D}$  does not admit the homotopy theory tools developed recently by Hovey (2002) [26]. This gives a negative answer to (2).

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MSC: 16D40; 16E30; 14F05; 18F20; 03E75

Keywords: Mittag-Leffler module;  $\aleph_1$ -Projective module; Deconstructible class; Kaplansky class; Model category structure; Quasi-coherent sheaf

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<sup>1</sup> Supported by MEC-DGESIC (Spain) through Project MTM2005-00934, and by the Comissionat Per Universitats i Recerca de la Generalitat de Catalunya through Project 2009 SGR 1389.

<sup>2</sup> Supported by GAČR 201/09/0816 and MSM 0021620839.

## 1. Introduction

Mittag-Leffler modules were introduced by Raynaud and Gruson already in 1971 [31], but only recently Drinfeld suggested to employ them in infinite dimensional algebraic geometry. In [8, §2], he remarked that similarly as (infinitely generated) projective modules are used to define (infinite dimensional) vector bundles, the class  $\mathcal{D}$  of all flat Mittag-Leffler modules could yield a more general, but still tractable, subclass of the class of all flat quasi-coherent sheaves on a scheme. Two questions have thus arisen:

- (1) What is the structure of flat Mittag-Leffler modules over particular (notably commutative noetherian) rings?, and
- (2) Can one build a Quillen model category structure on the category  $\mathcal{U}$  of all unbounded chain complexes of quasi-coherent sheaves on a scheme  $X$  by applying the method of Hovey [26] to  $\mathcal{D}$ ?

Note that by [29], model category structures are essential for understanding the derived category  $\mathcal{C}$  of the category of all quasi-coherent sheaves on  $X$ . Namely, for each model category structure on  $\mathcal{U}$  whose weak equivalences are homotopy isomorphisms, the morphisms between two objects  $A$  and  $B$  of  $\mathcal{C}$  can be computed as the  $\mathcal{U}$ -morphisms between the cofibrant replacement of  $A$  and fibrant replacement of  $B$  modulo the homotopy equivalence.

In Corollary 2.10 below, we answer question (1) by proving that flat Mittag-Leffler modules coincide with the  $\aleph_1$ -projective modules in the sense of [10].

The study of  $\aleph_1$ -projective abelian groups goes back to a 1934 paper by Pontryagin [27], but it gathered momentum with the introduction of set-theoretic methods by Shelah, Eklof and Mekler in the 1970s. A new theory of almost free modules has emerged [10] which applies far beyond the original setting of abelian groups, to modules over arbitrary non-perfect rings. A surprising consequence of Corollary 2.10 is that  $\aleph_1$ -projective modules can be approached from a different perspective, via the tensor product functor. And conversely, the rich supply of set-theoretic tools, developed originally to study almost free modules, is now available for better understanding the class of all (flat) Mittag-Leffler modules. This is demonstrated in the second part of our paper dealing with question (2).

Recall that a positive answer to question (2) is known when  $\mathcal{D}$  is replaced by the class of all projective modules (in case  $X$  is the projective line), and by the class of all flat modules (in case  $X$  is quasi-compact and semi-separated), see [13] and [20], respectively.

In [20], the approach of Hovey [26] via small cotorsion pairs was used. This has recently been extended to classes of modules that are not necessarily closed under direct limits. Assuming that the scheme  $X$  is semi-separated, a positive answer to question (2) is given in [15] when  $\mathcal{D}$  is replaced by any class of modules of the form  ${}^\perp\mathcal{C}$  which is deconstructible (in the sense of Eklof [9], see Definition 6.3 below). However, since our setting for applying Hovey's approach is that of small cotorsion pairs over a Grothendieck category, deconstructibility is also a necessary condition here. So question (2) can be restated as follows:

- (2') Is the class of all flat Mittag-Leffler modules deconstructible?

Surprisingly, except for the trivial case when  $R$  is a perfect ring, the answer is always negative. We prove this in Corollary 7.3 below. Thus we obtain a negative answer to question (2).

We also show that the concept of a Kaplansky class employed in [20] coincides with the concept of a deconstructible class for all classes closed under direct limits, but it is weaker in general: for any non-artinian right self-injective von Neumann regular ring  $R$ , the class  $\mathcal{D}$  is a Kaplansky class, but as mentioned above,  $\mathcal{D}$  is not deconstructible (cf. Example 6.8).

Given a ring  $R$  and a class of (right  $R$ -) modules  $\mathcal{C}$ , we will denote by  ${}^\perp\mathcal{C}$  the class of all roots of Ext for  $\mathcal{C}$ , that is,  ${}^\perp\mathcal{C} = \text{Ker Ext}_R^1(-, \mathcal{C})$ . Similarly, we define  $\mathcal{C}^\perp = \text{Ker Ext}_R^1(\mathcal{C}, -)$ .

For example,  $\mathcal{P} = {}^\perp(\text{Mod-}R)$  is the class of all projective modules, and  $\mathcal{F} = {}^\perp\mathcal{I}$  the class of all flat modules, where  $\text{Mod-}R$  and  $\mathcal{I}$  denotes the class of all modules, and all pure-injective (= algebraically compact) modules, respectively. The structure of projective modules over many rings is known; in fact, by a classic theorem of Kaplansky, each projective module is a direct sum of countably generated projective modules. Flat modules, however, generally elude classification.

This is why Drinfeld suggested to consider the intermediate class  $\mathcal{D}$  of all flat Mittag-Leffler modules in [8]. Recall [31] that a module  $M$  is *Mittag-Leffler* if the canonical morphism

$$\rho : M \otimes_R \prod_{i \in I} Q_i \rightarrow \prod_{i \in I} M \otimes_R Q_i$$

is monic for each family of left  $R$ -modules  $(Q_i \mid i \in I)$ .

This is a particular instance of a more general relative notion introduced in [32] (and further studied in [33] and [1]): if  $\mathcal{Q}$  is a class of left  $R$ -modules, a module  $M$  is  *$\mathcal{Q}$ -Mittag-Leffler*, or *Mittag-Leffler relative to  $\mathcal{Q}$* , if  $\rho$  is monic for all families  $(Q_i \mid i \in I)$  consisting of modules from  $\mathcal{Q}$ .

We denote by  $\mathcal{M}$  the class of all Mittag-Leffler modules, by  $\mathcal{M}_{\mathcal{Q}}$  the class of all  $\mathcal{Q}$ -Mittag-Leffler modules, and by  $\mathcal{D}_{\mathcal{Q}}$  the class of all flat  $\mathcal{Q}$ -Mittag-Leffler modules. Clearly,

$$\mathcal{P} \subseteq \mathcal{D} \subseteq \mathcal{D}_{\mathcal{Q}} \subseteq \mathcal{F}.$$

Mittag-Leffler and relative Mittag-Leffler modules were studied in depth in [31] and [1], respectively.

Given a module  $M$ , an  $\aleph_1$ -dense system on  $M$  is a directed family  $\mathcal{C}$  consisting of submodules of  $M$  such that  $\mathcal{C}$  is closed under unions of countable ascending chains and such that any countable subset of  $M$  is contained in an element of  $\mathcal{C}$  (cf. Definition 2.5). In Theorem 2.6 we show that a module  $M$  is  $\mathcal{Q}$ -Mittag-Leffler if and only if it has an  $\aleph_1$ -dense system consisting of  $\mathcal{Q}$ -Mittag-Leffler modules.

This yields one of the main results of our paper: the modules in  $\mathcal{D}$  are exactly the ones having an  $\aleph_1$ -dense system consisting of projective modules, or equivalently, the  $\aleph_1$ -projective modules (Corollary 2.10). It is interesting to note that no cardinality conditions on the number of generators of the modules in the witnessing  $\aleph_1$ -dense system are needed.

We use the new approach via dense systems to study the (non-)deconstructibility of  $\mathcal{D}$  and, more generally, of classes of modules containing modules possessing  $\aleph_1$ -dense systems.

Given a module  $N$  which is a countable direct limit of a family of modules  $\mathcal{N} = \{F_n\}_{n \in \mathbb{N}}$ , we show in Section 5 that it is possible to construct arbitrarily large modules  $M$  with an  $\aleph_1$ -dense system of submodules that consist of countable direct sums of modules in  $\mathcal{N}$ , such that  $M$  has a filtration with many consecutive factors isomorphic to the initial module  $N$ . This implies that if  $\mathcal{N} \subseteq \mathcal{D}$ , then  $M \in \mathcal{D}$ , but if  $N \notin \mathcal{D}$  and  $M$  is large enough, then  $M$  cannot be filtered by smaller flat Mittag-Leffler modules. Since  $M$  can be taken to be arbitrarily large, this implies that  $\mathcal{D}$  is

not deconstructible unless it is closed under direct limits (which is well known to happen if and only if the underlying ring  $R$  is right perfect).

This idea is developed in a somewhat more general context in Theorem 6.10 and applied to relative flat Mittag-Leffler modules in Section 7. This type of proofs and constructions goes back to Eklof, Mekler, and Shelah [10], and the particular instance that we use here is based on [38].

If  $\mathcal{D}$  is deconstructible, then  $\mathcal{D} = {}^\perp(\mathcal{D}^\perp)$ . Since  $\mathcal{D}$  is deconstructible only for right perfect rings, a new challenge appears, namely to characterize the class  ${}^\perp(\mathcal{D}^\perp)$ .

As Theorem 6.13 indicates, the general problem of computing  ${}^\perp(\mathcal{A}^\perp)$  for a class  $\mathcal{A}$  of modules seems to be easier when  $\mathcal{A}$  is closed under products. Indeed, when  $\mathcal{D}$  is closed under arbitrary products (e.g. when  $R$  is left noetherian) we show in Corollary 7.8 that  ${}^\perp(\mathcal{D}^\perp)$  is closed under countable direct limits. This implies that if  $R$  is countable and  $\mathcal{D}$  is closed under products, then  ${}^\perp(\mathcal{D}^\perp)$  is the class of all flat modules.

In Section 4 we study systematically the closure under products of the classes  $\mathcal{D}_Q$ . In Theorem 4.7 we characterize the rings such that  $\mathcal{D}$  is closed under products. Finally, let us mention that in Section 3 we also pay some attention to the classes of all (flat) strict Mittag-Leffler modules, and of separable modules.

By a ring  $R$  we mean an associative ring with 1, all our modules are unital, and the unadorned term module means right  $R$ -module.

Throughout the paper we shall use freely that a finitely generated module  $M$  is finitely presented, if and only if  $M$  is  $\{R\}$ -Mittag-Leffler, if and only if the canonical map  $M \otimes R^M \rightarrow M^M$  is monic.

We also recall [31] that countably generated Mittag-Leffler modules are countably presented, and they coincide with the countably generated pure projective modules; the countably generated modules in  $\mathcal{D}$  are precisely the countably generated projective modules.

## 2. Relative Mittag-Leffler modules, dense systems, and $\aleph_1$ -projectivity

We start by characterizing the direct limits of  $\mathcal{Q}$ -Mittag-Leffler modules that are  $\mathcal{Q}$ -Mittag-Leffler. The argument for the proof follows the ideas from [1, Theorem 5.1] which in turn were inspired by Raynaud and Gruson’s original paper [31].

**Proposition 2.1.** *Let  $R$  be a ring and  $\mathcal{Q}$  be a class of left  $R$ -modules. Let  $(F_\alpha, u_{\beta\alpha} : F_\alpha \rightarrow F_\beta)_{\beta\alpha \in I}$  be a direct system of  $\mathcal{Q}$ -Mittag-Leffler modules such that the upward directed set  $I$  does not have a maximal element. Set  $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\alpha \leq \beta \in I}$  and, for each  $\alpha \in I$ , let  $u_\alpha : F_\alpha \rightarrow M$  denote the canonical map. Then the following statements are equivalent:*

- (1)  $M$  is  $\mathcal{Q}$ -Mittag-Leffler.
- (2) For any  $\alpha \in I$  and any finite subset  $x_1, \dots, x_n$  of  $F_\alpha$  there exists  $\beta > \alpha$  such that for any  $Q \in \mathcal{Q}$  and any family  $q_1, \dots, q_n$  of elements in  $Q$

$$\sum_{i=1}^n x_i \otimes q_i \in \text{Ker } u_\alpha \otimes Q \iff \sum_{i=1}^n x_i \otimes q_i \in \text{Ker } u_{\beta\alpha} \otimes Q.$$

- (3) For any family  $\{Q_k\}_{k \in K}$  of modules in  $\mathcal{Q}$  such that the cardinality of  $K$  is less or equal than  $\max(\aleph_0, |I|)$  the canonical morphism

$$\rho : M \otimes_R \prod_{k \in K} Q_k \rightarrow \prod_{k \in K} M \otimes_R Q_k$$

is injective.

**Proof.** It is clear that (1) implies (3).

(3)  $\Rightarrow$  (2). Assume, by the way of contradiction, there exist  $\alpha \in I$  and  $x_1, \dots, x_n$  in  $F_\alpha$ , satisfying that for any  $\beta > \alpha$  there exist  $Q_\beta \in \mathcal{Q}$  and elements  $q_\beta^1, \dots, q_\beta^n$  of  $Q_\beta$  such that

$$a_\beta = \sum_{i=1}^n x_i \otimes q_\beta^i \in \ker(u_\alpha \otimes Q_\beta) \setminus \ker(u_{\beta\alpha} \otimes Q_\beta).$$

Set  $x = \sum_{i=1}^n x_i \otimes (q_\beta^i)_{\beta > \alpha} \in F_\alpha \otimes_R \prod_{\beta > \alpha} Q_\beta$ . As

$$\left( \prod_{\beta > \alpha} (u_\alpha \otimes Q_\beta) \right) \rho'(x) = \left( \prod_{\beta > \alpha} (u_\alpha \otimes Q_\beta) \right) (a_\beta)_{\beta > \alpha} = 0$$

and, by hypothesis  $\rho$  is injective, the commutativity of the diagram,

$$\begin{array}{ccc} F_\alpha \otimes_R \prod_{\beta > \alpha} Q_\beta & \xrightarrow{u_\alpha \otimes \prod_{\beta > \alpha} Q_\beta} & M \otimes_R \prod_{\beta > \alpha} Q_\beta \\ \rho' \downarrow & & \rho \downarrow \\ \prod_{\beta > \alpha} (F_\alpha \otimes_R Q_\beta) & \xrightarrow{\prod_{\beta > \alpha} (u_\alpha \otimes Q_\beta)} & \prod_{\beta > \alpha} (M \otimes_R Q_\beta) \end{array}$$

implies that  $(u_\alpha \otimes \prod_{\beta > \alpha} Q_\beta)(x) = 0$ .

Since  $M \otimes_R \prod_{\beta > \alpha} Q_\beta = \varinjlim (F_\gamma \otimes_R \prod_{\beta > \alpha} Q_\beta)$ , there exists  $\beta_0 > \alpha$  such that  $x \in \ker(u_{\beta_0\alpha} \otimes \prod_{\beta > \alpha} Q_\beta)$ . The commutativity of the diagram

$$\begin{array}{ccc} F_\alpha \otimes_R \prod_{\beta > \alpha} Q_\beta & \xrightarrow{u_{\beta_0\alpha} \otimes \prod_{\beta > \alpha} Q_\beta} & F_{\beta_0} \otimes_R \prod_{\beta > \alpha} Q_\beta \\ \rho' \downarrow & & \rho' \downarrow \\ \prod_{\beta > \alpha} (F_\alpha \otimes_R Q_\beta) & \xrightarrow{\prod_{\beta > \alpha} (u_{\beta_0\alpha} \otimes Q_\beta)} & \prod_{\beta > \alpha} (F_{\beta_0} \otimes_R Q_\beta) \end{array}$$

and the fact that, by hypothesis,  $\rho'$  is injective imply that, for any  $\beta > \alpha$ ,  $a_\beta \in \ker(u_{\beta_0\alpha} \otimes Q_\beta)$ . In particular,  $a_{\beta_0} \in \ker(u_{\beta_0\alpha} \otimes Q_{\beta_0})$  which is a contradiction.

(2)  $\Rightarrow$  (1). Let  $\{Q_k\}_{k \in K}$  be a family of modules in  $\mathcal{Q}$ , and let  $x \in \text{Ker } \rho$  where  $\rho : M \otimes_R \prod_{k \in K} Q_k \rightarrow \prod_{k \in K} (M \otimes_R Q_k)$  denotes the canonical map. Since  $M \otimes_R \prod_{k \in K} Q_k = \varinjlim (F_\alpha \otimes_R \prod_{k \in K} Q_k)$  there exist  $\alpha \in I$  and  $x_\alpha = \sum_{i=1}^n x_i \otimes (q_k^i)_{k \in K} \in F_\alpha \otimes_R \prod_{k \in K} Q_k$  such that  $x = (u_\alpha \otimes \prod_{k \in K} Q_k)(x_\alpha)$ . The commutativity of the diagram

$$\begin{array}{ccc}
 F_\alpha \otimes_R \prod_{k \in K} Q_k & \xrightarrow{u_\alpha \otimes \prod_{k \in K} Q_k} & M \otimes_R \prod_{k \in K} Q_k \\
 \rho' \downarrow & & \rho \downarrow \\
 \prod_{k \in K} (F_\alpha \otimes_R Q_k) & \xrightarrow{\prod_{k \in K} (u_\alpha \otimes Q_k)} & \prod_{k \in K} (M \otimes_R Q_k)
 \end{array}$$

implies that, for each  $k \in K$ ,  $\sum_{i=1}^n x_i \otimes q_k^i \in \ker(u_\alpha \otimes Q_k)$ .

Let  $\beta > \alpha$  be such that, for any  $Q \in \mathcal{Q}$  and any family  $q_1, \dots, q_n$  of elements in  $Q$ ,

$$\sum_{i=1}^n x_i \otimes q_i \in \text{Ker } u_\alpha \otimes Q \iff \sum_{i=1}^n x_i \otimes q_i \in \text{Ker } u_{\beta\alpha} \otimes Q.$$

The commutativity of the diagram

$$\begin{array}{ccc}
 F_\alpha \otimes_R \prod_{k \in K} Q_k & \xrightarrow{u_{\beta\alpha} \otimes \prod_{k \in K} Q_k} & F_\beta \otimes_R \prod_{k \in K} Q_k \\
 \rho' \downarrow & & \rho' \downarrow \\
 \prod_{k \in K} (F_\alpha \otimes_R Q_k) & \xrightarrow{\prod_{k \in K} (u_{\beta\alpha} \otimes Q_k)} & \prod_{k \in K} (F_\beta \otimes_R Q_k)
 \end{array}$$

and the fact that, by hypothesis,  $\rho'$  is injective imply that  $(u_{\beta\alpha} \otimes \prod_{k \in K} Q_k)(x_\alpha) = 0$ . Hence  $x = (u_\beta u_{\beta\alpha} \otimes \prod_{k \in K} Q_k)(x_\alpha) = 0$ . This proves that  $\text{Ker } \rho = 0$ .  $\square$

**Proposition 2.2.** *Let  $R$  be a ring and  $\mathcal{Q}$  be a class of left  $R$ -modules. Let  $(F_\alpha, u_{\beta\alpha} : F_\alpha \rightarrow F_\beta)_{\alpha \leq \beta \in I}$  be a direct system of  $\mathcal{Q}$ -Mittag-Leffler modules with  $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ . Assume that for each increasing chain  $(\alpha_i \mid i < \omega)$  in  $I$ , the module  $\varinjlim F_{\alpha_i}$  is  $\mathcal{Q}$ -Mittag-Leffler. Then  $M$  is a  $\mathcal{Q}$ -Mittag-Leffler module.*

**Proof.** For each  $\alpha \in I$ , let  $u_\alpha : F_\alpha \rightarrow M$  denote the canonical map. Assume, by the way of contradiction, that  $M$  is not  $\mathcal{Q}$ -Mittag-Leffler. Therefore the upward directed set  $I$  does not have a maximal element.

By Proposition 2.1, there exist  $\alpha_0$  and a finite family  $x_1, \dots, x_n$  of elements of  $F_{\alpha_0}$  such that for any  $\beta > \alpha_0$  there exist  $Q_\beta \in \mathcal{Q}$  and a family of elements  $q_1^\beta, \dots, q_n^\beta$  in  $Q_\beta$  such that

$$a_\beta = \sum_{j=1}^n x_j \otimes q_j^\beta \in \text{Ker}(u_{\alpha_0} \otimes Q_\beta) \setminus \text{Ker}(u_{\beta\alpha_0} \otimes Q_\beta).$$

Note however that, for any  $\beta$ , there exists  $\beta' > \beta$  such that  $a_\beta \in \text{Ker}(u_{\beta'\alpha_0} \otimes Q_\beta)$ . These properties allow us to construct an increasing chain in  $I$

$$\alpha_0 < \dots < \alpha_i < \dots$$

such that for each  $i > 0$

$$a_{\alpha_i} = \sum_{j=1}^n x_j \otimes q_j^{\alpha_i} \in \text{Ker}(u_{\alpha_{i+1}\alpha_0} \otimes Q_{\alpha_i}) \setminus \text{Ker}(u_{\alpha_i\alpha_0} \otimes Q_{\alpha_i}).$$

But then the direct limit  $N = \varinjlim F_{\alpha_i}$  fails to satisfy Proposition 2.1(2), and hence  $N$  is not  $\mathcal{Q}$ -Mittag-Leffler which contradicts our hypothesis. Therefore  $M$  is  $\mathcal{Q}$ -Mittag-Leffler.  $\square$

**Remark 2.3.** As noted by the referee, Proposition 2.2 can also be obtained by model-theoretic means, using the fact that Mittag-Leffler modules are exactly the positively atomic modules (see [32, Theorem 2.2] or [28, Theorem 1.3.22]). The key point here is Herzog's model-theoretic criterion for  $a \otimes b = 0$  involving elementary duality, cf. [28, Theorem 1.3.7].

Let us record the following immediate corollary of Propositions 2.1 and 2.2 that provides a sufficient condition for a direct limit of  $\mathcal{Q}$ -Mittag-Leffler modules to be  $\mathcal{Q}$ -Mittag-Leffler involving only direct limits of chains of type  $\omega$  and countable subsets of  $\mathcal{Q}$ :

**Corollary 2.4.** *Let  $R$  be a ring and  $\mathcal{Q}$  be a class of left  $R$ -modules. Let  $(F_\alpha, u_{\beta\alpha} : F_\alpha \rightarrow F_\beta)_{\alpha \leq \beta \in I}$  be a direct system of  $\mathcal{Q}$ -Mittag-Leffler modules with  $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ . Assume that for each increasing chain  $(\alpha_i \mid i < \omega)$  in  $I$  and for each countable subset  $\mathcal{Q}'$  of  $\mathcal{Q}$ , the module  $\varinjlim F_{\alpha_i}$  is  $\mathcal{Q}'$ -Mittag-Leffler. Then  $M$  is a  $\mathcal{Q}$ -Mittag-Leffler module.*

Raynaud and Gruson characterized Mittag-Leffler modules as the ones satisfying that any countable subset is contained in a countably generated (presented) Mittag-Leffler pure submodule (see [31, seconde partie, Théorème 2.2.1]). By [1, Theorem 5.1], a version of this characterization for  $\mathcal{Q}$ -Mittag-Leffler modules is also available.

Proposition 2.2 allows us not only to substitute the purity condition in this characterization by one in the spirit of the almost freeness conditions that are the central topic of [10], but also to avoid the hypotheses on the number of generators. To this aim we find it useful to introduce the following terminology.

**Definition 2.5.** Let  $R$  be a ring, and  $M$  be a module. Let  $\kappa$  be a regular uncountable cardinal. A direct system,  $\mathcal{C}$ , of submodules of  $M$  is said to be a  $\kappa$ -dense system in  $M$  if

- (1)  $\mathcal{C}$  is closed under unions of well-ordered ascending chains of length  $< \kappa$ , and
- (2) every subset of  $M$  of cardinality  $< \kappa$  is contained in an element of  $\mathcal{C}$ .

Definition 2.5 follows [34, Definition 3.1], but notice that we are not making any assumption on the cardinality of a generating set of the modules in  $\mathcal{C}$ . In particular, if  $\kappa_1 < \kappa_2$  are two uncountable regular cardinals then a  $\kappa_2$ -dense system is also a  $\kappa_1$ -dense system.

From [31, seconde partie: Théorème 2.2.1, Lemme 2.2.2], it follows that a module  $M$  is Mittag-Leffler if and only if it has a dense system of consisting of countably generated pure submodules that are Mittag-Leffler. A relative version of this result can be deduced from [1, Corollary 5.2]. Using Proposition 2.2 we show in the next result that the cardinality and the (relative) purity hypotheses are redundant.

**Theorem 2.6.** *Let  $R$  be a ring,  $\mathcal{Q}$  be a class of left  $R$ -modules, and  $M$  be a module. The following statements are equivalent:*

- (i)  $M$  is  $\mathcal{Q}$ -Mittag-Leffler.

- (ii) For every countable subset  $X$  of  $M$  there is a countably generated  $\mathcal{Q}$ -Mittag-Leffler submodule  $N$  of  $M$  containing  $X$  such that  $\varepsilon \otimes_R \mathcal{Q} : N \otimes_R \mathcal{Q} \rightarrow M \otimes_R \mathcal{Q}$  is a monomorphism for all  $\mathcal{Q} \in \mathcal{Q}$ . Here  $\varepsilon : N \rightarrow M$  denotes the inclusion.
- (iii)  $M$  has an  $\aleph_1$ -dense system consisting of countably generated  $\mathcal{Q}$ -Mittag-Leffler modules.
- (iv)  $M$  has an  $\aleph_1$ -dense system consisting of  $\mathcal{Q}$ -Mittag-Leffler modules.

If, in addition,  $R \in \mathcal{Q}$  then the statements above are further equivalent to,

- (v)  $M$  has an  $\aleph_1$ -dense system consisting of countably presented  $\mathcal{Q}$ -Mittag-Leffler modules.

**Proof.** (i) implies (ii) by the implication (1)  $\Rightarrow$  (4) of [1, Theorem 5.1].

Assume (ii). Consider the set  $\mathcal{C}$  of all countably generated  $\mathcal{Q}$ -Mittag-Leffler submodules  $N$  of  $M$  satisfying that the canonical inclusion  $\varepsilon : N \rightarrow M$  remains injective when tensoring by any element  $\mathcal{Q} \in \mathcal{Q}$ . Then  $\mathcal{C}$  satisfies condition (1) of Definition 2.5 by [1, Corollary 5.2], and  $\mathcal{C}$  satisfies condition (2) by (ii). So  $\mathcal{C}$  is an  $\aleph_1$ -dense system in  $M$ .

That (iii) implies (iv) is clear. Now we prove that (iv) implies (i). Let  $\mathcal{C}$  be an  $\aleph_1$ -dense system consisting of  $\mathcal{Q}$ -Mittag-Leffler submodules of  $M$ . By condition (2) of Definition 2.5,  $M$  is a directed union of the elements of  $\mathcal{C}$ . By condition (1),  $\mathcal{C}$  is closed under unions of chains of type  $\omega$ , and Proposition 2.2 implies that  $M$  is  $\mathcal{Q}$ -Mittag-Leffler.

Finally, if  $R \in \mathcal{Q}$  then any countably generated  $\mathcal{Q}$ -Mittag-Leffler module is countably presented [1, Corollary 5.3], so that (iii) and (v) are equivalent statements.  $\square$

Since countably generated (presented) Mittag-Leffler modules are pure projective if we specialize Theorem 2.6 to them we obtain the following corollary.

**Corollary 2.7.** *Let  $R$  be a ring and  $M$  be a module. Then  $M$  is Mittag-Leffler if and only if  $M$  has an  $\aleph_1$ -dense system consisting of countably generated pure-projective modules.*

Let  $\kappa$  be a regular uncountable cardinal. An abelian group having a  $\kappa$ -dense system of  $< \kappa$ -generated free modules is called a  $\kappa$ -free abelian group. This class of groups as well as their module theoretic counterpart, the  $\kappa$ -free modules, have been studied in detail [10, Chapters IV and VII], see also [21]. A natural extension of these concepts to modules over non-hereditary rings is the following (cf. [10, p. 88, IV.1.1 Definition]):

**Definition 2.8.** Let  $R$  be a ring, and let  $\kappa$  be a regular uncountable cardinal. A module  $M$  is said to be  $\kappa$ -projective if  $M$  has a  $\kappa$ -dense system  $\mathcal{C}$  consisting of  $< \kappa$ -generated projective modules.

If  $R$  is a right hereditary ring then  $M$  is  $\kappa$ -projective, if and only if  $M$  has a family of  $< \kappa$ -generated projective submodules  $\mathcal{C}$  such that each  $< \kappa$ -generated submodule of  $M$  is contained in one of the family  $\mathcal{C}$ . Equivalently, if and only if each  $< \kappa$ -generated submodule of  $M$  is projective. Therefore, the condition that the family  $\mathcal{C}$  is closed under unions of well-ordered ascending chains of length  $< \kappa$  is redundant in this case.

If  $R$  is von Neumann regular then it is  $\aleph_0$ -hereditary. This implies that  $\aleph_1$ -projective modules coincide with the modules all of whose finitely (or countably) generated submodules are projective (see [22, corollary] or [38, Lemma 3.4]). So again, the closure under unions of countable chains in Definition 2.5 is redundant for  $\kappa = \aleph_1$ . This is not true for general rings as the following example shows.



**Example 2.9.** Let  $R$  be a commutative valuation domain with the quotient field  $Q$ . Assume that the  $R$ -module  $M = Q$  is not countably generated (that is, the projective dimension of  $M$  is bigger than 1, cf. [18, VI.3.3]). Then  $M$  is not  $\aleph_1$ -projective but

$$\mathcal{C} = \{r^{-1}R \mid r \in R \setminus \{0\}\}$$

is system of cyclic projective modules that satisfies condition (2) of Definition 2.5 for  $\kappa = \aleph_1$ .

To see that  $M$  is not  $\aleph_1$ -projective (or flat Mittag-Leffler, cf. Corollary 2.10) notice that  $R$  is not contained in any countably generated free pure submodule of  $M$  (cf. [31, seconde partie, Théorème 2.2.1] or just use Theorem 2.6).

In order to prove (2) of Definition 2.5 we first claim that for any sequence  $(r_n)_{n \in \mathbb{N}}$  of non-zero elements of  $R$ ,  $\bigcap_{n \in \mathbb{N}} r_n R \neq \{0\}$ . Indeed,  $R$  is a valuation domain, so if  $\bigcap_{n \in \mathbb{N}} r_n R = \{0\}$  then for each  $r \in R \setminus \{0\}$  there is  $n_0 \in \mathbb{N}$  such that  $rR \supseteq r_{n_0}R$ . This implies that  $r^{-1} \in \bigcup_{n \in \mathbb{N}} r_n^{-1}R$ , and  $M$  is be countably generated, a contradiction.

Consider a countable subset  $\mathcal{S} = \{s_n r_n^{-1}\}_{n \in \mathbb{N}}$  of  $M$ . By the previous claim there exists  $0 \neq r \in \bigcap_{n \in \mathbb{N}} r_n R$ . Then  $\mathcal{S} \subseteq r^{-1}R$ , and (2) holds.

The following surprising corollary makes it possible to describe  $\aleph_1$ -projectivity via the tensor product functor.

**Corollary 2.10.** *Let  $R$  be a ring, and  $M$  be module. Then:*

- (i)  $M$  is  $\aleph_1$ -projective, if and only if it is a flat Mittag-Leffler module.
- (ii) If  $\kappa$  is a regular uncountable cardinal and  $M$  is  $\kappa$ -projective then  $M$  is a flat Mittag-Leffler module.

**Proof.** Statement (i) follows from Theorem 2.6 applied to  $\mathcal{Q} = R\text{-Mod}$ , and using the fact that a countably generated (presented) flat Mittag-Leffler module is projective.

To prove (ii), note that  $M$  is the directed union of the modules of the family  $\mathcal{C}$  witnessing the  $\kappa$ -projectivity of  $M$ . Since this directed union is closed under countable chains (as they have length  $< \kappa$ ) we deduce from Proposition 2.2 that  $M$  is a flat Mittag-Leffler module.  $\square$

We note that in the particular case of abelian groups, Corollary 2.10(i) follows from [4, Proposition 7].

Applying Proposition 2.1 to direct systems of finitely presented, hence Mittag-Leffler, modules we obtain the usual characterization of  $\mathcal{Q}$ -Mittag-Leffler modules.

**Corollary 2.11.** *Let  $R$  be a ring. Let  $\mathcal{Q}$  be a class of left  $R$ -modules, and  $M$  be a module. Let  $(F_\alpha, u_{\beta\alpha} : F_\alpha \rightarrow F_\beta)_{\beta, \alpha \in I}$  be a directed system of finitely presented modules with  $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ . For each  $\alpha \in I$ , let  $u_\alpha : F_\alpha \rightarrow M$  denote the canonical map. Then  $M$  is  $\mathcal{Q}$ -Mittag-Leffler if and only if for each  $\alpha \in I$  there exists  $\beta > \alpha$  such that for any  $Q \in \mathcal{Q}$*

$$\text{Ker}(u_\alpha \otimes Q) = \text{Ker}(u_{\beta\alpha} \otimes Q).$$

Specializing to left coherent rings and taking  $\mathcal{Q} = \mathcal{F}$ , the class of all flat modules, we obtain a characterization of  $\mathcal{F}$ -Mittag-Leffler modules due to Goodearl [22]. We state the result in

terms of  $\mathcal{F}$ -Mittag-Leffler left  $R$ -modules because this is the context we will use it later on (see Section 4).

**Corollary 2.12.** (See [22].) *Let  $R$  be a left coherent ring. A left  $R$ -module  $M$  is  $\mathcal{F}$ -Mittag-Leffler, if and only if any finitely generated left  $R$ -submodule of  $M$  is finitely presented.*

**Proof.** Assume  $M$  is  $\mathcal{F}$ -Mittag-Leffler, and let  $N$  be a finitely generated left  $R$ -submodule of  $M$ . Since  $R$  is left coherent,  $R^I$  is a flat module and then the injectivity of  $\rho : R^I \otimes M \rightarrow M^I$  implies the injectivity of  $\rho : R^I \otimes N \rightarrow N^I$ . Hence  $N$  is finitely presented.

Conversely, if  $M$  satisfies that each of its finitely generated left  $R$ -submodules is finitely presented, then write  $M$  as the directed union of its finitely generated (hence finitely presented) left  $R$ -submodules. This directed union clearly fulfills the left-hand version of Corollary 2.11 for  $\mathcal{Q} = \mathcal{F}$ , therefore  $M$  is  $\mathcal{F}$ -Mittag-Leffler.  $\square$

For further quoting, we note the following characterization of left Noetherian rings in terms of Mittag-Leffler conditions.

**Corollary 2.13.** (See [22], [32, Remark 2.8].) *A ring  $R$  is left Noetherian, if and only if each left  $R$ -module is  $\mathcal{F}$ -Mittag-Leffler.*

**Proof.** If  $R$  is left Noetherian then each left  $R$ -module satisfies Corollary 2.12, so each left  $R$ -module is  $\mathcal{F}$ -Mittag-Leffler.

Conversely, if any left  $R$ -module is  $\mathcal{F}$ -Mittag-Leffler then any finitely generated module is finitely presented so that  $R$  is left Noetherian.  $\square$

### 3. Strict Mittag-Leffler modules and separability

**Definition 3.1.** A module  $M$  is said to be *separable* if each finitely generated submodule of  $M$  is contained in a finitely presented direct summand of  $M$ .

Following Raynaud and Gruson [31, second partie, §2.3] (see also [1, Proposition 8.1]) a module  $M$  is said to be *strict Mittag-Leffler* if for any module homomorphism  $u : F \rightarrow M$ , with  $F$  a finitely presented module, there exist a finitely presented module  $S$  and a homomorphism  $v : F \rightarrow S$ , such that  $u$  factors through  $v$  (that is,  $u = v'v$  for a suitable  $v' : S \rightarrow M$ ), and such that for any module  $B$  and any module homomorphism  $f : S \rightarrow B$  there exists  $g : M \rightarrow B$  with  $gu = fv$ .

We say that a flat module  $M$  is *strongly  $\aleph_0$ -flat-Mittag-Leffler* if for each finitely generated submodule  $X$  of  $M$  there exists a finitely generated submodule  $N$  of  $M$  such that  $X \subseteq N$ , and both  $N$  and  $M/N$  are flat Mittag-Leffler modules.

We have borrowed the terminology of strongly  $\aleph_0$ -flat-Mittag-Leffler from Eklof and Mekler's book [10, p. 87 and p. 113] where the general notion of strongly  $\kappa$ -'free' is introduced for any infinite cardinal  $\kappa$ . This concept is in the heart of Shelah's Singular Compactness Theorem.

It is easy to see that each separable module is strict Mittag-Leffler, and by [31, §2.3], each strict Mittag-Leffler module is Mittag-Leffler.

Azumaya [3] realized that the class of strict Mittag-Leffler modules coincides with the class of locally pure projective modules. Flat strict Mittag-Leffler modules are also called locally pro-

jective modules and, in general, they are a particular type of pure submodules of products of copies of the ring (see [19] and [39]).

We will denote the class of all (flat) strict Mittag-Leffler modules by  $(\mathcal{SD}) \mathcal{SM}$ .

If a module is countably presented (or countably generated) then it is strict Mittag-Leffler if and only if it is Mittag-Leffler. Therefore a further variation of the results of the previous section allows us to describe the class of all Mittag-Leffler modules  $\mathcal{M}$  and the class  $\mathcal{D}$  of all flat Mittag-Leffler modules as the closure of  $\mathcal{SM}$  and  $\mathcal{SD}$ , respectively, under  $\aleph_1$ -dense systems.

**Corollary 3.2.** *Let  $R$  be a ring. Then  $\mathcal{M}$  is the class of all modules that have an  $\aleph_1$ -dense system of modules in  $\mathcal{SM}$ , and  $\mathcal{D}$  is the class of all modules having an  $\aleph_1$ -dense system of modules in  $\mathcal{SD}$ .*

Now we will show that the class of all (flat) strict Mittag-Leffler modules is the closure of the class of all (flat) separable modules under direct summands.

**Lemma 3.3.** *Let  $R$  be a ring. Let  $M$  be a strict Mittag-Leffler module, then any finitely generated pure submodule of  $M$  is a direct summand of  $M$ .*

**Proof.** Let  $N$  be a finitely generated pure submodule of  $M$ . As  $N$  is a pure submodule of a Mittag-Leffler module it is also Mittag-Leffler, and since  $N$  is finitely generated it must be finitely presented. Let  $\varepsilon : N \rightarrow M$  denote the canonical inclusion. By the definition of strict Mittag-Leffler module there exists a commutative diagram of module homomorphism

$$\begin{array}{ccc}
 N & \xrightarrow{v} & F \\
 \varepsilon \downarrow & \swarrow & \\
 M & & 
 \end{array}$$

with  $F$  a finitely presented module, such that for any module homomorphism  $h : F \rightarrow B$  there exists  $h' : M \rightarrow B$  such that  $h v = h' \varepsilon$ .

Since  $\varepsilon$  is a pure monomorphism, so is  $v : N \rightarrow F$ . Since  $N$  and  $F$  are finitely presented so is Coker  $v$ , and hence  $v$  splits. Therefore there exists  $h : F \rightarrow N$  such that  $h v = \text{id}_N$ . By the properties of the above diagram, there exists  $h' : M \rightarrow N$  such that  $\text{id}_N = h' \varepsilon$ , thus  $\varepsilon$  splits and therefore  $N$  is a direct summand of  $M$ .  $\square$

**Proposition 3.4.** *Let  $R$  be a ring such that all projective modules are direct sum of finitely generated ones. Let  $M$  be a flat module. Then:*

- (i)  $M_R$  is Mittag-Leffler, if and only if for any  $x_1, \dots, x_n \in M$  there exists a finitely generated projective and pure submodule  $N$  of  $M$  such that  $x_1 R + \dots + x_n R \subseteq N$ .
- (ii)  $M_R$  is Mittag-Leffler, if and only if  $M_R$  is strongly  $\aleph_0$ -flat-Mittag-Leffler.
- (iii)  $M_R$  is strict Mittag-Leffler, if and only if it is separable.

**Proof.** (i) is essentially due to Raynaud and Gruson [31]. We give a direct argument for completeness' sake.

If  $M_R$  is Mittag-Leffler and  $X$  is a finitely generated submodule of  $M$  then, by Corollary 2.10(i),  $X$  is contained in a countably generated projective pure submodule  $P$  of  $M$ . Since

$P$  is a direct sum of finitely generated modules,  $X$  is contained in a finitely generated direct summand  $N$  of  $P$ . Therefore  $N$  is the module we were looking for.

The converse follows by applying Theorem 2.6(ii).

(ii). If  $N$  is a finitely generated submodule of  $M$  then  $M/N$  is also Mittag-Leffler [1, Examples 1.6]. So by (i), if  $M$  is a flat Mittag-Leffler module then  $M$  is strongly  $\aleph_0$ -flat-Mittag-Leffler.

Conversely, it is easy to see that if  $M$  fits into an exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$$

with  $M/N \in \mathcal{D}$  and  $N$  a Mittag-Leffler module, then  $M$  is also a Mittag-Leffler module.

(iii). As remarked above, each separable module is strict Mittag-Leffler. For the converse implication combine (i) and Lemma 3.3.  $\square$

Particular instances of Proposition 3.4 are known: For example, if  $R$  is an Artin algebra then part (iii) was proved in [2, Lemma 20]; indeed, in this case separable modules coincide with the Mittag-Leffler ones.

It is interesting to note the following variation of the previous proposition that avoids the hypothesis on projective modules.

**Lemma 3.5.** *Let  $R$  be a ring and  $M$  be a flat module.*

- (i)  $M_R$  is Mittag-Leffler, if and only if for any  $x_1, \dots, x_n \in M$  there exists a finitely generated projective pure submodule  $N$  of  $M \oplus R^{(\aleph_0)}$  such that  $x_1R + \dots + x_nR \subseteq N$ , if and only if  $M_R \oplus R^{(\aleph_0)}$  is Mittag-Leffler.
- (ii)  $M_R$  is Mittag-Leffler, if and only if  $M_R \oplus R^{(\aleph_0)}$  is strongly  $\aleph_0$ -flat-Mittag-Leffler.
- (iii)  $M_R$  is strict Mittag-Leffler, if and only if  $M_R \oplus R^{(\aleph_0)}$  is separable if and only if  $M_R \oplus R^{(\aleph_0)}$  is strict Mittag-Leffler.

**Proof.** (i). If  $M_R$  is Mittag-Leffler and  $N$  is a finitely generated submodule of  $M$  then there exists a countably generated, projective, pure submodule  $N'$  of  $M$  that contains  $N$  (cf. Corollary 2.10(i)). Since  $N' \oplus R^{(\aleph_0)} \cong R^{(\aleph_0)}$ , there exists a finitely generated direct summand  $N''$  of  $N' \oplus R^{(\aleph_0)}$  containing  $N$ . Hence  $N''$  is the pure submodule of  $M \oplus R^{(\aleph_0)}$  we were looking for.

To prove the rest, notice that the second condition implies that  $M \oplus R^{(\aleph_0)}$  is Mittag-Leffler by Corollary 2.11, and that the property of being Mittag-Leffler is inherited by direct summands.

To prove (ii) proceed as in Proposition 3.4.

(iii). Assume that  $M$  is a flat strict Mittag-Leffler module, then so is  $M \oplus R^{(\aleph_0)}$  because a direct sum of two strict Mittag-Leffler modules is also strict Mittag-Leffler. By (i), any finitely generated submodule of  $M$  is contained in a finitely generated pure submodule  $N$  of  $M \oplus R^{(\aleph_0)}$ . By Lemma 3.3,  $N$  is a direct summand of  $M \oplus R^{(\aleph_0)}$ .

If  $X$  is a finitely generated submodule of  $M \oplus R^{(\aleph_0)}$  then  $X \subseteq X_1 \oplus X_2$  with  $X_1$  a finitely generated submodule of  $M$  and  $X_2$  a finitely generated submodule of  $R^{(\aleph_0)}$ . By the previous case it follows that  $X_1 \oplus X_2$ , and hence  $X$ , is contained in a direct summand of  $M \oplus R^{(\aleph_0)}$ . This shows that  $M \oplus R^{(\aleph_0)}$  is separable.

The remaining implications are clear.  $\square$

Finally we give a statement for general modules.

**Lemma 3.6.** *Let  $R$  be a ring, and let  $M$  be a module. Let  $L$  be the direct sum of a set of representatives, up to isomorphism, of the finitely presented modules. Then:*

- (i)  $M_R$  is Mittag-Leffler, if and only if for any  $x_1, \dots, x_n \in M$  there exists a finitely generated projective pure submodule  $N$  of  $M \oplus L^{(\aleph_0)}$  such that  $x_1R + \dots + x_nR \subseteq N$ .
- (ii)  $M_R$  is strict Mittag-Leffler, if and only if  $M_R \oplus L^{(\aleph_0)}$  is separable.

**Proof.** Since for any finitely presented module  $F$ ,  $F \oplus L^{(\aleph_0)} \cong L^{(\aleph_0)}$ , the proof of this result can be done in the same way as the one of Lemma 3.5.  $\square$

Now we can clarify the relation between strongly  $\aleph_0$ -Mittag-Leffler and Mittag-Leffler modules, and between separable modules and strict Mittag-Leffler modules, respectively.

**Corollary 3.7.** *Let  $R$  be a ring. Let  $\mathcal{S}$  denote the class of all separable modules. Then  $\text{Add}(\mathcal{S}) = \mathcal{SM}$ .*

*If  $R$  satisfies that all pure projective modules are direct sum of finitely generated ones, then  $\mathcal{S} = \mathcal{SM}$ .*

**Proof.** For the first part of the statement, apply Lemma 3.6, and use the fact that  $\mathcal{SM}$  is closed by arbitrary direct sums and by direct summands.

If all pure projective modules are direct sum of finitely generated ones, then it is easy to prove a result analogous to Proposition 3.4(iii) for  $\mathcal{SM}$  and then the conclusion follows.  $\square$

Specializing to the case of flat modules we obtain the following corollary.

**Corollary 3.8.** *Let  $R$  be a ring. Let  $\mathcal{T}$  denote the class of all strongly  $\aleph_0$ -flat-Mittag-Leffler modules, and  $\mathcal{SF}$  the class of all separable flat modules. Then:*

- (i)  $\text{Add}(\mathcal{T}) = \mathcal{D}$ .
- (ii)  $\text{Add}(\mathcal{SF}) = \mathcal{SD}$ .

*If all projective modules are direct sum of finitely generated ones then  $\mathcal{T} = \mathcal{D}$  and  $\mathcal{SD} = \mathcal{SF}$ .*

*If  $R_R$  is pure injective then  $\mathcal{T} = \mathcal{D} = \mathcal{SD} = \mathcal{SF}$ .*

**Proof.** For the first part of the statement proceed as in Corollary 3.7.

Assume that all projective modules are direct sum of cyclic modules. Then, by Proposition 3.4,  $\mathcal{T} = \mathcal{D}$  and  $\mathcal{SD} = \mathcal{SF}$ .

Assume  $R_R$  is pure injective, and recall that over a pure injective ring all projective modules are direct sum of cyclic ones. We only have to show that  $\mathcal{D} \subseteq \mathcal{SD}$ .

Let  $M \in \mathcal{D}$ . By Proposition 3.4(i), any finitely generated submodule  $X$  of  $M$  is contained in a finitely generated projective pure submodule  $Y$  of  $M$ . The module  $Y$  is pure injective because  $R_R$  is, therefore  $Y$  is a direct summand of  $M$ . This shows that  $M$  is separable.  $\square$

**Remark 3.9.** As pointed out by the referee, some of the results of this section have recently been obtained in [24]: Lemma 3.3 is [24, Corollary 2.2], while [24, Theorem 4.2] covers the third part of Proposition 3.4 as well as the second part of Corollary 3.7.

#### 4. Closure under products

A well-known result by Chase says that the class of all flat right modules is closed under products, if and only if  $R^R$  is flat as a right module, and this happens if and only if  $R$  is a left coherent ring.

The rings such that the class  $\mathcal{SD}$  is closed under arbitrary products were characterized by Huisgen-Zimmermann in [39] and she called them left *strongly coherent rings*. Again, to test that a ring  $R$  is left strongly coherent it is enough to check that  $R^R$  is a strict Mittag-Leffler right module (see [39, Theorem 4.2]).

The class of all  $\mathcal{F}$ -Mittag-Leffler modules is closed under products, if and only if the ring is left  $\pi$ -coherent. This is to say that, for any set  $I$ , any finitely generated left  $R$ -submodule of  $R^I$  is finitely presented. As far as we know, this terminology is due to Camillo [7]. The first place that these rings were first considered is in a paper by Finkel Jones in [17, p. 103] in relation with closure under products of the class of  $\mathcal{F}$ -Mittag-Leffler modules.

In this section we study (coherent) rings such that  $\mathcal{D}_Q$  is closed under products. We start proving that this always happens when the ring  $R$  is left Noetherian.

We recall some closure properties of  $\mathcal{D}_Q$  and  $\mathcal{SD}$  that were already noticed in [31] (see also [32] or [1] for the relative version).

**Lemma 4.1.** *Let  $R$  be a ring.*

- (i) *For any class of left  $R$ -modules  $\mathcal{Q}$  the class  $\mathcal{D}_Q$  is closed under pure submodules and under (pure) extensions.*
- (ii) *[31, p. 75, Remarques 2.3.3] The class  $\mathcal{SD}$  is closed under pure submodules.*

In Section 7 we shall recall that  $\mathcal{D}$  is even closed under transfinite extensions. We stress the fact that, in general,  $\mathcal{SD}$  is not even closed under (pure) extensions. Next example, patterned on [31, p. 76], illustrates that.

**Example 4.2.** Let  $R$  be a left noetherian ring. By [39, Corollary 4.3],  $R^I \in \mathcal{SD}$  for any set  $I$ . So if there is  $I$  such that  $\text{Ext}_R^1(R^I, R) \neq 0$ , then any module  $M$  fitting in a non-split exact sequence

$$0 \rightarrow R \rightarrow M \rightarrow R^I \rightarrow 0$$

is a (pure) extension of modules in  $\mathcal{SD}$  but, by Lemma 3.3,  $M \in \mathcal{D} \setminus \mathcal{SD}$ .

For a concrete example take  $R = \mathbb{Z}$  and  $I = \mathbb{N}$ , cf. [1, Example 9.11] or [10, Exercise IV.16].

**Proposition 4.3.** *Let  $R$  be a left Noetherian ring, and let  $\mathcal{Q}$  be a class of left  $R$ -modules. Then  $\mathcal{D}_Q$  is closed under arbitrary products and pure submodules. In particular,  $\mathcal{D}_Q$  is a preenveloping class.*

**Proof.** Let  $\{M_i\}_{i \in I}$  be a family of modules in  $\mathcal{D}_Q$ , and let  $\{Q_k\}_{k \in K}$  be a family of modules of  $\mathcal{Q}$ . In the commutative diagram

$$\begin{array}{ccc}
 (\prod_{i \in I} M_i) \otimes_R \prod_{k \in K} Q_k & \xrightarrow{\rho} & \prod_{k \in K} ((\prod_{i \in I} M_i) \otimes_R Q_k) \\
 \rho_1 \downarrow & & \downarrow \rho_2 \\
 \prod_{i \in I} (M_i \otimes_R \prod_{k \in K} Q_k) & \xrightarrow{\prod_{i \in I} \rho'_i} & \prod_{i \in I} \prod_{k \in K} (M_i \otimes_R Q_k) \cong \prod_{k \in K} \prod_{i \in I} (M_i \otimes_R Q_k)
 \end{array}$$

$\rho_1$  and  $\rho_2$  are injective because each  $M_i$  is a flat module and, by Corollary 2.13, over any left Noetherian ring  $R$ , each left  $R$ -module is  $\mathcal{F}$ -Mittag-Leffler. By hypothesis, for each  $i \in I$ , the natural transformation  $\rho'_i : M_i \otimes_R \prod_{k \in K} Q_k \rightarrow \prod_{k \in K} M_i \otimes_R Q_k$  is injective. Therefore  $\prod_{i \in I} \rho'_i$  is also injective. The commutativity of the diagram implies that  $\rho$  is injective. Hence  $\prod_{i \in I} M_i$  is  $\mathcal{Q}$ -Mittag-Leffler. Since over a left Noetherian ring the product of flat modules is a flat module, we conclude that  $\prod_{i \in I} M_i$  is a flat  $\mathcal{Q}$ -Mittag-Leffler module.

By Lemma 4.1,  $\mathcal{D}_{\mathcal{Q}}$  is also closed under pure submodules. Then, by a result due to Rada and Saorín [30],  $\mathcal{D}_{\mathcal{Q}}$  is a preenveloping class.  $\square$

**Example 4.4.** Following with the notation of Proposition 4.3, we remark that if  $M$  is a module and  $f : M \rightarrow N$  is a  $\mathcal{D}_{\mathcal{Q}}$ -preenvelope. Then, in general,  $f$  is not injective.

For a simple example, consider any non-right perfect, but right hereditary, ring  $R$ . Let  $M$  be any flat non-projective countably generated module, and put  $\mathcal{Q} = R\text{-Mod}$ . Then  $f$  is not injective since in this case the class of all  $\aleph_1$ -projective modules is closed under submodules.

We can even have  $f = 0$ : Let  $R$  be a commutative Noetherian local domain, with maximal ideal  $I$  and ring of quotients  $Q$ . Take  $\mathcal{Q} = R\text{-Mod}$ , so that we are just considering flat Mittag-Leffler preenvelopes.

If  $N \in \mathcal{D}$  then the homomorphism

$$N \rightarrow N \otimes_R \prod_{n \in \mathbb{N}} R/I^n \xrightarrow{\rho} \prod_{n \in \mathbb{N}} N \otimes_R R/I^n \cong \prod_{n \in \mathbb{N}} N/N I^n$$

must be injective. Therefore the modules in  $\mathcal{D}$  are separated with respect to the  $I$ -adic topology, that is  $\bigcap_{n \in \mathbb{N}} N I^n = 0$ .

Assume that  $f : M \rightarrow N$  is a flat Mittag-Leffler preenvelope of a module  $M$ . Since submodules of separated modules are separated, we deduce that  $\bigcap_{n \in \mathbb{N}} M I^n \subseteq \text{Ker } f$ . In particular, the flat Mittag-Leffler preenvelope of the field of quotients  $Q$  is  $f : Q \rightarrow 0$ .

Now we will characterize the rings such that  $\mathcal{D}_{\mathcal{Q}}$  is closed under arbitrary products when  $\mathcal{Q} = \varinjlim \text{add } S$ , for a class  $S$  of finitely presented left  $R$ -modules. A key fact for that is the following result which can be seen as a consequence of [32, Theorem 2.2] and which is stated explicitly in [1, Theorem 1.3].

**Lemma 4.5.** *Let  $R$  be a ring, and let  $S$  be a class of left  $R$ -modules. A module  $M$  is  $S$ -Mittag-Leffler, if and only if it is  $\varinjlim \text{add } S$ -Mittag-Leffler.*

**Theorem 4.6.** *Let  $R$  be a ring. Let  $S$  be a class of finitely presented left  $R$ -modules, and let  $\mathcal{Q} = \varinjlim \text{add } S$ . Then the following statements are equivalent:*

- (i)  $\mathcal{D}_{\mathcal{Q}}$  is closed under arbitrary products.
- (ii) For any set  $I$ ,  $R^I \in \mathcal{D}_{\mathcal{Q}}$ .

- (iii)  $R^R \in \mathcal{D}_{\mathcal{Q}}$ .
- (iv)  $R$  is left coherent and, for any family of modules  $\{Q_\alpha\}_{\alpha \in \Lambda}$  in  $\mathcal{S}$ , any finitely generated submodule of  $\prod_{\alpha \in \Lambda} Q_\alpha$  is finitely presented.

**Proof.** It is clear that (i) implies (ii), and that (ii) implies (iii).

Assuming (iii), we will prove that (iv) holds. Let  $\{Q_\alpha\}_{\alpha \in \Lambda}$  be a family of modules in  $\mathcal{S}$ . Note that since, for any  $\alpha \in \Lambda$ ,  $Q_\alpha$  is a Mittag-Leffler left  $R$ -module then the composition of the two canonical maps

$$R^R \otimes \prod_{\alpha \in \Lambda} Q_\alpha \rightarrow \prod_{\alpha \in \Lambda} R^R \otimes Q_\alpha \rightarrow \prod_{\alpha \in \Lambda} Q_\alpha^R$$

is injective. Therefore, as in the proof of Proposition 4.3, we can deduce that the canonical map

$$\rho : R^R \otimes \prod_{\alpha \in \Lambda} Q_\alpha \rightarrow \left( R \otimes \prod_{\alpha \in \Lambda} Q_\alpha \right)^R$$

is also injective.

Let  $N$  be a finitely generated left  $R$ -submodule of  $\prod_{\alpha \in \Lambda} Q_\alpha$ . By assumption,  $R^R$  is a flat module, so we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & R^R \otimes N & \longrightarrow & R^R \otimes \prod_{\alpha \in \Lambda} Q_\alpha & & \\ & & \rho' \downarrow & & \rho \downarrow & & \\ 0 & \longrightarrow & (R \otimes N)^R & \longrightarrow & (R \otimes_R \prod_{\alpha \in \Lambda} Q_\alpha)^R & & \end{array}$$

As  $\rho$  is injective, so is  $\rho'$ . Therefore the finitely generated left  $R$ -module  $N$  is finitely presented.

To prove (iv)  $\Rightarrow$  (i) note that, by Corollary 2.12, condition (iv) implies that, for any family  $\{Q_\alpha\}_{\alpha \in \Lambda}$  of left  $R$ -modules in  $\mathcal{S}$ ,  $\prod_{\alpha \in \Lambda} Q_\alpha$  is Mittag-Leffler with respect to the class of all flat modules. As in the proof of Proposition 4.3, this implies that  $\mathcal{D}_{\mathcal{Q}}$  is closed under arbitrary products.  $\square$

In [32] there is a detailed analysis of whether there exists a cardinal  $\kappa$ , such that  $M^\kappa \mathcal{Q}$ -Mittag-Leffler implies  $M^I \mathcal{Q}$ -Mittag-Leffler for any set  $I$ . In particular, the equivalence of (ii) and (iii) in Theorem 4.6 is already contained in [32, Proposition 5.3, Remark 5.4].

In the next result we specialize to study the closure under products of  $\mathcal{D}$ . We remark that in this result, the equivalence of statements (ii), (iii) and (iv) is already proved in [33, Proposition 3.8], what seems to be new is that these statements are equivalent to the closure under products of the class  $\mathcal{D}$ .

**Theorem 4.7.** *Let  $R$  be a ring. Then the following statements are equivalent:*

- (i)  $\mathcal{D}$  is closed under arbitrary products.
- (ii) For any set  $I$ ,  $R^I \in \mathcal{D}$ .



(iii)  $R^R \in \mathcal{D}$ .

(iv)  $R$  is left coherent and, for any  $n \geq 1$ , intersections of arbitrary families of finitely generated left  $R$ -submodules of  $R^n$  are again finitely generated.

**Proof.** By applying Theorem 4.6 to the class  $\mathcal{S}$  of all finitely presented left  $R$ -modules, we deduce that (i)–(iii) are equivalent statements. The rest of the statement follows from [33, Proposition 3.8], for completeness' sake we show that statement (iv) is equivalent to Theorem 4.6(iv).

Assume Theorem 4.6(iv) holds for the class  $\mathcal{S}$  of all finitely presented left  $R$ -modules. Fix  $n \geq 1$ . Let  $\{N_\alpha\}_{\alpha \in \Lambda}$  be a family of finitely generated left  $R$ -submodules of  $R^n$ . For any  $\alpha \in \Lambda$ , set  $F_\alpha$  be the free left  $R$ -module of rank  $n$  and denote its canonical basis by  $(e_1^\alpha, \dots, e_n^\alpha)$ . Let  $Q_\alpha = F_\alpha/N_\alpha$ . For each  $i = 1, \dots, n$ , set  $q_i \in \prod_{\alpha \in \Lambda} Q_\alpha$  to be  $q_i = (e_i^\alpha + N_\alpha)_{\alpha \in \Lambda}$ . Since  $Rq_1 + \dots + Rq_n$  is a finitely generated submodule of  $\prod_{\alpha \in \Lambda} Q_\alpha$ , by assumption, it is finitely presented. Therefore the surjective morphism  $\pi : R^n \rightarrow Rq_1 + \dots + Rq_n$ , defined by  $\pi(e_i) = q_i$ , where  $e_1, \dots, e_n$  denotes the canonical basis of  $R^n$ , has finitely generated kernel. Since  $\text{Ker } \pi = \bigcap_{\alpha \in \Lambda} N_\alpha$  we deduce that  $\bigcap_{\alpha \in \Lambda} N_\alpha$  is finitely generated as wanted.

Assume (iv) holds. Let  $\{Q_\alpha\}_{\alpha \in \Lambda}$  be a family of finitely presented left  $R$ -modules, and let  $q_1, \dots, q_n$  be elements in  $\prod_{\alpha \in \Lambda} Q_\alpha$ . For any  $i = 1, \dots, n$ ,  $q_i = (q_\alpha^i)_{\alpha \in \Lambda}$  with  $q_\alpha^i \in Q_\alpha$ . As  $R$  is left coherent, for any  $\alpha \in \Lambda$ , there exists a finitely generated left  $R$ -submodule  $L_\alpha$  of  $R^n$  such that the sequence

$$0 \rightarrow L_\alpha \rightarrow R^n \xrightarrow{\pi_\alpha} \sum_{i=1}^n Rq_\alpha^i \rightarrow 0$$

is exact, where  $\pi_\alpha$  is the homomorphisms of left  $R$ -modules determined by  $\pi_\alpha(e_i) = q_\alpha^i$ , where  $(e_1, \dots, e_n)$  denotes the canonical basis of the free module  $R^n$ .

Let  $\pi : R^n \rightarrow \sum_{i=1}^n Rq_i$  be defined by  $\pi(e_i) = q_i$  for  $i = 1, \dots, n$ . As  $\text{Ker } \pi = \bigcap_{\alpha \in \Lambda} L_\alpha$ , our hypothesis implies that the finitely generated left  $R$ -submodule  $\sum_{i=1}^n Rq_i$  of  $\prod_{\alpha \in \Lambda} Q_\alpha$  is finitely presented.  $\square$

Following [32, §5], a ring is said to be left *pp-complete* provided that, for any  $n \geq 1$ , intersections of arbitrary families of finitely generated left  $R$ -submodules of  $R^n$  are again finitely generated.

**Examples 4.8.** If, in Theorem 4.6,  $\mathcal{Q} = \mathcal{F}$  the class of all flat left  $R$ -modules then, by Lemma 4.5,  $\mathcal{S}$  can be simply taken to be  $R$ . Therefore condition (iv) becomes: for any set  $I$ , any finitely generated left  $R$ -submodule of  $R^I$  is finitely presented, so that the rings obtained are exactly the left  $\pi$ -coherent rings.

Hence, the rings characterized by Theorem 4.7, that is, coherent and left *pp-complete* rings, are contained in the class of left  $\pi$ -coherent rings, but this inclusion is strict. For example, for any field  $k$ , the ring  $R = k[x_1, \dots, x_n, \dots]$  is  $\pi$ -coherent (cf. the work by Camillo [7, Theorem 6] for even a more general result) but, as observed by Garfinkel in [19, Example 5.2], it is not true that the intersection of an arbitrary family of finitely generated ideals of  $R$  is finitely generated. Hence  $R$  does not satisfy condition (iv) in Theorem 4.7.

On the other hand, if  $R$  is left strongly coherent then, as strict Mittag-Leffler modules are Mittag-Leffler,  $R$  satisfies Theorem 4.7(ii). Hence  $\mathcal{D}$  is closed under products.

By [39, 4.3], each left noetherian ring is left strongly coherent. We believe that the class of coherent and left  $pp$ -complete ring is strictly bigger than the class of all left strongly coherent rings (cf. Proposition 3.4 and Lemma 3.5).

In the next result we note that for coherent rings being  $\pi$ -coherent is a symmetric condition.

**Lemma 4.9.** *Let  $R$  be a right and left coherent ring. Then  $R$  is right  $\pi$ -coherent, if and only if it is left  $\pi$ -coherent.*

**Proof.** For each pair of sets  $I$  and  $J$ , consider the following commutative diagram

$$\begin{array}{ccc}
 R^I \otimes_R R^J & \xrightarrow{\rho} & (R^I)^J \\
 \rho' \downarrow & & \downarrow \varphi \\
 (R^J)^I & \xrightarrow{\text{id}} & (R^J)^I.
 \end{array}$$

As  $\varphi$  is an isomorphism,  $\rho$  is injective, if and only if so is  $\rho'$ .

This shows that  $R^I$  is an  $R$ -Mittag-Leffler module if and only if  $R^J$  is an  $R$ -Mittag-Leffler left  $R$ -module. Now we conclude by Lemma 4.5 and Theorem 4.6.  $\square$

An interesting class of coherent rings are the von Neumann regular ones. In this setting, a module  $M$  is (flat) Mittag-Leffler, if and only if each finitely generated submodule of  $M$  is projective (cf. [22, Corollary] or Proposition 3.4). Also, again by Proposition 3.4, a module  $M$  is separable, if and only if  $M$  is strict Mittag-Leffler, if and only if each finitely generated submodule of  $M$  is a projective direct summand of  $M$ . If  $R$  is, in addition, right self-injective, then Mittag-Leffler modules coincide with the strict Mittag-Leffler ones by Corollary 3.8.

Since over a von Neumann regular ring all modules are flat,  $R$  is left  $\pi$ -coherent if and only if it is left complete, and by Lemma 4.9, if and only if  $R$  is right  $\pi$ -coherent if and only if it is right complete. See the work by Rothmaler [33] for these results and further developments in the von Neumann regular case.

### 5. Constructing large modules from countable patterns

In order to give an answer to question (2) from the Introduction, we will first develop a tool for constructing large modules using a pattern involving a countable direct limit.

Similar methods were employed in constructing almost free non-projective modules in [10]. However, since we aim at constructing  $\aleph_1$ -projective (and, more generally, flat  $\mathcal{Q}$ -Mittag-Leffler modules) rather than  $\kappa$ -projective modules, our construction will be performed in ZFC rather than in some of its forcing extensions (cf. Remark 6.11).

**Definition 5.1.** Let  $R$  be a ring, and let  $\kappa$  be an infinite cardinal. A module  $M$  is  $< \kappa$ -generated if it has a set of generators of cardinality  $< \kappa$ , and it is said to be  $< \kappa$ -presented if it has a presentation  $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$  with  $F$  free of rank  $< \kappa$ , and  $K$   $< \kappa$ -generated.

A *filtration* of  $M$  is an increasing chain  $\mathcal{M} = (M_\alpha \mid \alpha \leq \lambda)$  consisting of submodules of  $M$  such that  $M_0 = 0$ ,  $M_\alpha \subseteq M_{\alpha+1}$  for each  $\alpha < \lambda$ ,  $M = M_\lambda$ , and  $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$  for each limit ordinal  $\alpha \leq \lambda$ .

We recall the well-known fact that for each ring  $R$ , the class of all  $< \kappa$ -generated modules coincides with the class of  $< \kappa$ -presented modules for all large enough cardinals  $\kappa$ . More precisely:

**Lemma 5.2.** *Let  $R$  be a ring. Let  $\kappa$  be an infinite cardinal such that each right ideal of  $R$  is  $< \kappa$ -generated. Then if  $M$  is a  $< \kappa$ -generated module, then any submodule of  $M$  is also  $< \kappa$ -generated.*

*In particular, any  $< \kappa$ -generated module is  $< \kappa$ -presented.*

**Notation 5.3.** Let  $R$  be a ring. We fix

$$F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} \dots \xrightarrow{f_{i-1}} F_i \xrightarrow{f_i} F_{i+1} \xrightarrow{f_{i+1}} \dots$$

a countable direct system of modules with direct limit  $N = \varinjlim F_i \neq 0$ . Possibly replacing  $F_i$  by  $\bigoplus_{i < \omega} F_i$ , we can w.l.o.g. assume that  $F_i = F_j = F$  for all  $i, j < \omega$ . We will also canonically identify  $F_i$  with a submodule of  $F^{(\omega)}$  (namely with the one consisting of the sequences  $(x_j)_{j < \omega}$  such that  $x_j = 0$  for all  $j \neq i$ ). We have a pure exact sequence

$$0 \rightarrow F^{(\omega)} \xrightarrow{f} F^{(\omega)} \rightarrow N \rightarrow 0$$

where  $f$  is defined by  $f(x) = x - f_i(x)$  for all  $i < \omega$  and  $x \in F_i$ .

Let  $\kappa$  be an infinite cardinal and  $E = \{\alpha < \kappa^+ \mid \text{cf}(\alpha) = \aleph_0\}$ . Then  $E$  is a *stationary* subset of  $\kappa^+$ , that is,  $E$  has non-empty intersection with any closed and cofinal subset of  $\kappa^+$  (see [10, II.4.7]).

Let  $\nu$  be a limit ordinal of cofinality  $\aleph_0$ . A  $\nu$ -ladder is a strictly increasing sequence  $s_\nu = (s_\nu(i) \mid i < \omega)$  consisting of ordinals less than  $\nu$  such that  $\sup_{i < \omega} s_\nu(i) = \nu$ . A set  $\{s_\nu \mid \nu \in E\}$  is called a *ladder system* for  $E$  if  $s_\nu$  is a  $\nu$ -ladder for each  $\nu \in E$ .

If  $\text{cf}(\nu) = \aleph_0$  then a  $\nu$ -ladder always exists, and we can w.l.o.g. assume that  $s_\nu(i) = \tau_{\nu,i} + i + 1$  where  $\tau_{\nu,i}$  is a limit ordinal or 0. Thus we obtain a ladder system  $\{s_\nu \mid \nu \in E\}$  for  $E$  such that if  $\alpha = s_\mu(i) = s_\nu(j)$  for some  $\mu, \nu \in E$  and  $i, j < \omega$ , then  $i = j$ . This also guarantees that, for any  $\nu \in E$  and for any  $i < \omega$ ,  $s_\nu(i) \notin E$ .

Next, we use our ladder system to define a large module  $M$ , generalizing a construction in [38, §2]:

Let  $(F_\alpha \mid \alpha < \kappa^+)$  be a sequence of modules defined as follows:  $F_\alpha = F$  provided that  $\alpha \in \kappa^+ \setminus E$ , and  $F_\alpha = F^{(\omega)} = \bigoplus_{i < \omega} F_{\alpha,i}$  for  $\alpha \in E$ . Let  $P = \bigoplus_{\alpha < \kappa^+} F_\alpha$ . We will canonically identify the modules  $F_\alpha$  ( $\alpha < \kappa^+$ ) with submodules of  $P$ .

For  $\alpha \in \kappa^+ \setminus E$ , we denote by  $1_\alpha$  the endomorphism of  $P$  which is identity on  $F_\alpha$  and zero on  $F_\beta$  for  $\beta \neq \alpha$ . Similarly, for  $\alpha \in E$  and  $i < \omega$ , we let  $1_{\alpha,i}$  ( $f_{\alpha,i}$ ) denote the endomorphism of  $P$  which is identity on  $F_{\alpha,i}$  (resp., maps  $F_{\alpha,i}$  to  $F_{\alpha,i+1}$  by  $f_{\alpha,i}(x) = f_i(x)$ ) and is zero on  $F_{\alpha,j}$  and  $F_\beta$  for all  $\beta \neq \alpha$  and  $j \neq i$ . For each  $\alpha \in E$ , we define  $S_\alpha = \bigoplus_{i < \omega} \text{Im}(1_{\alpha,i} - f_{\alpha,i})$ . Then  $S_\alpha$  is a submodule of  $F_\alpha$  such that  $F_\alpha/S_\alpha \cong N$ .

For all  $\alpha \in E$  and  $i < \omega$ , we let  $g_{\alpha i} = 1_{s_\alpha(i)} - 1_{\alpha,i} + f_{\alpha,i} \in \text{End}_R(P)$ . It is easy to check that the images of endomorphisms  $\{g_{\alpha i} \mid \alpha \in E, i < \omega\}$  are  $R$ -independent submodules of  $P$ . We define  $G_\alpha = \bigoplus_{i < \omega} \text{Im}(g_{\alpha i})$  and  $G = \bigoplus_{\alpha \in E} G_\alpha$ . Finally, we define the module  $M = M_{\kappa^+} = P/G$ .

For each  $A \subseteq \{\beta < \kappa^+\}$  we define a submodule  $M_A$  of  $M$  by  $M_A = (\bigoplus_{\beta \in A} F_\beta + G)/G$ . In particular, since  $\alpha = \{\beta < \alpha\}$  for each ordinal  $\alpha \leq \kappa^+$ , we have  $M_\alpha = (\bigoplus_{\beta < \alpha} F_\beta + G)/G$ . Clearly  $M = M_{\kappa^+}$ .

Finally, we define  $Y = \bigcup_{\alpha \in E} \{s_{\alpha(i)} \mid i < \omega\}$  and  $X = \{\beta < \kappa^+ \mid \beta \notin E \cup Y\}$ . Note that  $\kappa^+$  is a disjoint union of the sets  $E$ ,  $X$ , and  $Y$ .

We notice the following simple facts:

**Lemma 5.4.**

- (i) For each  $A \subseteq E$ ,  $(\bigoplus_{\alpha \in A} F_{\alpha} + G) \cap (\bigoplus_{\beta \in X} F_{\beta}) = \{0\}$ ;
- (ii) For each  $B \subseteq \kappa^+$ , define  $\varepsilon_B : \bigoplus_{\alpha \in B} F_{\beta} \rightarrow M$  by  $\varepsilon_B(p) = p + G$ . Then the map  $\varepsilon_{B \cap X}$  is injective, and

$$M_B = \varepsilon_{B \cap X} \left( \bigoplus_{\beta \in B \cap X} F_{\beta} \right) \oplus \left( \left( \bigoplus_{\beta \in B \setminus X} F_{\beta} + G \right) / G \right);$$

- (iii) Let  $A, A'$  be subsets of  $E \cup X$ . Then  $A \subseteq A'$ , if and only if  $M_A \subseteq M_{A'}$ .

**Proof.** For each ordinal  $\beta < \kappa^+$ , let  $\pi_{\beta} : P \rightarrow F_{\beta}$  denote the canonical projection. Then statement (i) follows from the fact that  $\pi_{\beta}(G + \sum_{\alpha \in A} F_{\alpha}) = \{0\}$  for all  $\beta \in X$ .

(ii) is a consequence of (i).

(iii). Clearly,  $A \subseteq A'$  implies  $M_A \subseteq M_{A'}$ . Conversely, assume  $M_A \subseteq M_{A'}$ . If  $\alpha \in E \setminus A'$ , then by the definition of  $G$ ,  $(\sum_{\beta \in A'} F_{\beta} + G) \cap F_{\alpha} \subseteq S_{\alpha} \subsetneq F_{\alpha}$ , whence  $(F_{\alpha} + G)/G \not\subseteq M_{A'}$ , and  $\alpha \notin A$ . So  $A \cap E \subseteq A'$ . If  $\alpha \in A \cap X$  then  $(F_{\alpha} + G)/G \subseteq M_A \subseteq M_{A'}$  and the definitions of  $F$  and  $G$  yield  $\alpha \in A'$ .  $\square$

In the next result we single out a filtration of  $M = M_{\kappa^+}$ . In this filtration “many” consecutive factors are isomorphic to the initial module  $N$ .

**Proposition 5.5.**

- (i)  $\mathcal{M} = (M_{\alpha} \mid \alpha \leq \kappa^+)$  is a strictly increasing filtration of  $M$ .
- (ii) If  $\text{card}(F), \text{card}(R) \leq \kappa$ , then  $M_{\alpha}$  is a  $< \kappa^+$ -generated (equivalently,  $< \kappa^+$ -presented) module for each  $\alpha < \kappa^+$ . In particular,  $M$  is  $\kappa^+$ -generated.
- (iii) Let  $\nu < \mu \leq \kappa^+$  and assume that  $\nu \in E$ . Then there exists a module  $K \subseteq M_{\mu}/M_{\nu}$  such that  $M_{\mu}/M_{\nu} = M_{\nu+1}/M_{\nu} \oplus K$  and  $M_{\nu+1}/M_{\nu} \cong N$ .

**Proof.** Statements (i) and (ii) are clear from the definition of  $M$  and of the submodules  $M_{\alpha}$ . We shall prove (iii).

First, note that  $F_{\nu} \cap (\bigoplus_{\beta < \nu} F_{\beta} + G) = S_{\nu}$ . So

$$M_{\nu+1}/M_{\nu} \cong F_{\nu} / \left( F_{\nu} \cap \left( \bigoplus_{\beta < \nu} F_{\beta} + G \right) \right) \cong N.$$

We claim that  $(\bigoplus_{\beta \leq \nu} F_{\beta}) \cap (\bigoplus_{\nu < \gamma < \mu} F_{\gamma} + G) \subseteq \bigoplus_{\beta < \nu} F_{\beta} + G$ . Let  $x \in (\bigoplus_{\beta \leq \nu} F_{\beta}) \cap (\bigoplus_{\nu < \gamma < \mu} F_{\gamma} + G)$ . As  $x \in \bigoplus_{\beta \leq \nu} F_{\beta}$  there exist ordinals  $\beta_1 < \beta_2 < \dots < \beta_n = \nu$  such that  $x = \sum_{i=1}^n x_{\beta_i}$  and  $x_{\beta_i} \in F_{\beta_i}$ .

As  $x \in \bigoplus_{\nu < \gamma < \mu} F_\gamma + G$ ,  $x = y_1 + y_2$  where  $y_1 \in \bigoplus_{\nu < \gamma < \mu} F_\gamma$  and  $y_2 \in G$ . Let  $\pi_\nu : P \rightarrow F_\nu$  denote the canonical projection. Then  $x_\nu = \pi_\nu(x) = \pi_\nu(y_2)$ . Since  $\pi_\nu(y_2) \in S_\nu$ , we have  $x_\nu = -\sum_{r=1}^m (1_{\nu, j_r} - f_{\nu, j_r})(z_r)$  for some  $j_1, \dots, j_m \in \mathbb{N}$ , and some  $z_1, \dots, z_m$  in  $F_{\nu, j_1}, \dots, F_{\nu, j_m}$ , respectively. Hence

$$x_\nu - \sum_{r=1}^m g_{\nu, j_r}(z_r) \in \bigoplus_{i < \omega} F_{S_\nu(i)} \subseteq \bigoplus_{\beta < \nu} F_\beta.$$

This shows that  $x_\nu$  and, hence also  $x$  is an element of  $\bigoplus_{\beta < \nu} F_\beta + G$ , as we wanted to show.

Clearly  $M_\mu = M_{\nu+1} + H$  where  $H = (\bigoplus_{\nu < \gamma < \mu} F_\gamma + G)/G$ . The argument above shows that  $H \cap M_{\nu+1} \subseteq M_\nu$ . So

$$M_\mu/M_\nu \cong M_{\nu+1}/M_\nu \oplus (H + M_\nu/M_\nu).$$

This finishes the proof of (iii).  $\square$

Now we shall see that  $M = M_{\kappa^+}$  has an  $\aleph_1$ -dense system consisting of modules that are isomorphic to a countable direct sum of copies of  $F$ . Therefore, in contrast with the filtration given in Proposition 5.5, the dense system “does not see” the module  $N$ .

**Proposition 5.6.** *Let  $\mathcal{C}$  denote the class of all modules isomorphic to a countable (finite or infinite) direct sum of copies of  $F$ . Let  $\mathcal{S}$  be the set of all finite subsets of  $E \cup X$ .*

*Then:*

- (i)  $\{M_A\}_{A \in \mathcal{S}}$  is a direct system of submodules of  $M$ , and  $M = \bigcup_{A \in \mathcal{S}} M_A$ .
- (ii) For each  $A \in \mathcal{S}$ ,  $M_A \in \mathcal{C}$ .
- (iii) If  $A, A' \in \mathcal{S}$  are such that  $A \subseteq A'$ , then  $M_{A'} = M_A \oplus K_{(A, A')}$  for some  $K_{(A, A')} \in \mathcal{C}$ .
- (iv) If  $A_0 \subseteq A_1 \subseteq \dots \subseteq A_i \subseteq \dots$  is a countable ascending chain of elements of  $\mathcal{S}$ , then  $\bigcup_{i < \omega} M_{A_i} \in \mathcal{C}$ .

**Proof.** (i). By Lemma 5.4(iii),  $\{M_A\}_{A \in \mathcal{S}}$  is a direct system of submodules of  $M$ . That  $M = \bigcup_{A \in \mathcal{S}} M_A$  follows from the observation that  $F_{S_\beta(i)} \subseteq F_\beta + G$  for all  $\beta \in E$  and  $i < \omega$ .

Since for any  $A \in \mathcal{S}$ ,  $M_A \cong M_A/M_\emptyset$ , we see that (iii) implies (ii). Moreover, (iii) implies that  $\bigcup_{i < \omega} M_{A_i} \cong \bigoplus_{i < \omega} K_{(A_i, A_{i+1})} \in \mathcal{C}$ , so (iii) implies (iv).

(iii). Let  $A, A' \in \mathcal{S}$  be such that  $A \subseteq A'$ . In view of Lemma 5.4, it is enough to prove the statement for  $A \subseteq A' \subseteq E$ .

First, we define  $D = (\bigoplus_{\alpha \in (A' \setminus A)} F_\alpha) \cap (\bigoplus_{\alpha \in A} F_\alpha + G)$ . Then

$$M_{A'}/M_A \cong \left( \bigoplus_{\alpha \in A'} F_\alpha + G \right) / \left( \bigoplus_{\alpha \in A} F_\alpha + G \right) \cong \bigoplus_{\alpha \in A' \setminus A} F_\alpha / D.$$

We have  $A' \setminus A = \{\beta_0, \dots, \beta_{n-1}\}$  for some  $\beta_0 < \dots < \beta_{n-1}$ . For  $k < n$ , let  $I_k = \{i < \omega \mid (\exists k < j < n: s_{\beta_k}(i) = s_{\beta_j}(i)) \text{ or } (\exists \alpha \in A: s_{\beta_k}(i) = s_\alpha(i))\}$ . Since  $A$  is finite,  $I_k$  is finite for each  $k < n$ . Define  $C = \bigoplus_{k < n, i \notin I_k} F_{\beta_k, i} \in \mathcal{C}$ .

We will show that  $C \oplus D = \bigoplus_{\alpha \in A' \setminus A} F_\alpha$ .

In order to show that  $C + D = \bigoplus_{\alpha \in A' \setminus A} F_\alpha$ , we prove by reverse induction on  $k < n$  that  $\bigoplus_{i < \omega} F_{\beta_k, i} \subseteq C \oplus D$ . To this aim, for a fixed  $k < n$ , it suffices to show that  $F_{\beta_k, i} \subseteq C \oplus D$  for all  $i \in I_k$ . Since  $I_k$  is finite, we also make a reverse induction on  $I_k$ .

Let  $k = n - 1$  and  $i \in I_{n-1}$ . Then there exists  $\alpha \in A$  such that  $s_{\beta_k}(i) = s_\alpha(i)$ , and then  $h = 1_{\beta_k, i} - f_{\beta_k, i} = 1_{\alpha, i} - f_{\alpha, i} - g_{\beta_k, i} + g_{\alpha, i} \in \text{End}_R(P)$ . This implies that  $\text{Im}(h) \in D$ . As  $1_{\beta_k, i} = f_{\beta_k, i} + h$  and  $\text{Im}(f_{\beta_k, i}) \subseteq F_{\beta_k, i+1} \subseteq C + D$  by the inductive premise on  $I_{n-1}$ , so  $F_{\beta_k, i} = \text{Im}(1_{\beta_k, i}) \subseteq C + D$ .

If  $k < n - 1$  and  $i \in I_k$ , then either there exists  $\alpha \in A$  such that  $s_{\beta_k}(i) = s_\alpha(i)$  and we proceed as in the previous case, or there exists  $k < j < n$  such that  $s_{\beta_k}(i) = s_{\beta_j}(i)$ . Then the image of the map  $h = g_{\beta_k, i} - g_{\beta_j, i} = 1_{\beta_j, i} - f_{\beta_j, i} - 1_{\beta_k, i} + f_{\beta_k, i}$  is contained in  $D$ . However,  $1_{\beta_k, i} = f_{\beta_k, i} + 1_{\beta_j, i} - f_{\beta_j, i} - h$ , and  $F_{\beta_k, i+1} \oplus F_{\beta_j, i} \oplus F_{\beta_j, i+1} \subseteq C + D$  by the inductive premise. Therefore we can also conclude that  $F_{\beta_k, i} = \text{Im}(1_{\beta_k, i}) \subseteq C + D$ . This finishes the proof of  $C + D = \bigoplus_{\alpha \in A' \setminus A} F_\alpha$ .

Assume that  $0 \neq x \in C \cap (\bigoplus_{\alpha \in A} F_\alpha + G)$ . Since  $x \in C$ , there is  $k < n$  and a unique decomposition  $x = y + \sum_{k < j < n, i \notin I_k} x_{ij}$  where  $x_{ij} \in F_{\beta_j, i}$ , and  $0 \neq y = \sum_{i \notin I_k} x_i$  where  $x_i \in F_{\beta_k, i}$ . Let  $i' = \min\{i \notin I_k \mid x_i \neq 0\}$ .

Since also  $x \in \bigoplus_{\alpha \in A} F_\alpha + G$ ,  $x$  has a finite decomposition of the form

$$x = z + \sum_{k < j < n, i < \omega} z_{\beta_j, i} + \sum_{\alpha \in A} u_\alpha$$

where  $u_\alpha \in F_\alpha + G_\alpha$ ,  $0 \neq z = \sum_{i < \omega} z_{\beta_k, i}$ , and  $z_{\beta_j, i} \in \text{Im}(g_{\beta_j, i})$  for all  $k \leq j < n$  and  $i < \omega$ . Notice that  $i'$  must be also the least index  $i < \omega$  such that  $z_{\beta_k, i} \neq 0$ . But  $z_{\beta_k, i'}$  has a non-zero component in  $F_{s_{\beta_k}(i')}$ . This is only possible if either there exists  $k < j < n$  such that  $s_{\beta_k}(i') = s_{\beta_j}(i')$ , or there exists  $\alpha \in A$  such that  $s_{\beta_k}(i') = s_\alpha(i')$ . But in both cases it follows that  $i' \in I_k$ , which contradicts the fact that  $x \in C$ . This proves that  $C \cap (\bigoplus_{\alpha \in A} F_\alpha + G) = \{0\}$ . In particular,  $C \cap D = \{0\}$ .

Finally,  $M_{A'} = M_A + (\bigoplus_{\alpha \in A' \setminus A} F_\alpha + G)/G = M_A + K_{(A, A')}$  where  $K_{(A, A')} = (C + G)/G$ . But the previous argument implies that  $(C + G) \cap (\bigoplus_{\alpha \in A} F_\alpha + G) = G$ . Hence,  $M_{A'} = M_A \oplus K_{(A, A')}$ . Since  $C \cap G = \{0\}$ , we conclude that  $K_{(A, A')} \cong C \in \mathcal{C}$ .  $\square$

**Theorem 5.7.** Let  $\mathcal{C}$  denote the class of all modules that are isomorphic to a countable direct sum of copies of  $F$ . Let  $\mathcal{T}$  be the set of all countable subsets of  $E \cup X$ . Then  $\mathcal{U} = \{M_A\}_{A \in \mathcal{T}}$  is an  $\aleph_1$ -dense system in  $M$  consisting of modules from  $\mathcal{C}$ .

**Proof.** If  $A \in \mathcal{T}$  is finite, then  $M_A \in \mathcal{C}$  by Proposition 5.6(ii). If  $A \in \mathcal{T}$  is infinite, then  $A = \bigcup_{i < \omega} A_i$  for a strictly ascending chain  $A_0 \subset \dots \subset A_i \subset \dots$  of finite subsets of  $\mathcal{T}$ . By Proposition 5.6(iv),  $M_A = \bigcup_{i < \omega} M_{A_i} \in \mathcal{C}$ .

Clearly,  $\mathcal{U}$  is a direct system of submodules of  $M$ . By Proposition 5.6(i), its union is  $M$ , and each countable subset of  $M$  is contained in an element of  $\mathcal{U}$ . Finally, since  $\mathcal{T}$  is closed under unions of countable well-ordered ascending chains, so is  $\mathcal{U}$  by Lemma 5.4(iii). Therefore  $\mathcal{U}$  is an  $\aleph_1$ -dense system in  $M$ .  $\square$

### 6. Kaplansky classes and deconstructibility

Let  $R$  be a ring and let  $\mathcal{C}$  be a class of right (or left)  $R$ -modules. Recall that each class of the form  ${}^\perp \mathcal{C}$  is closed under extensions and arbitrary direct sums. These are particular instances of the more general notion of a transfinite extension:

**Definition 6.1.** Let  $R$  be a ring and  $\mathcal{A}$  a class of modules. A module  $M$  is a *transfinite extension* of modules in  $\mathcal{A}$  provided there exists a filtration  $\mathcal{M} = (M_\alpha \mid \alpha \leq \lambda)$  of  $M$  such that for each  $\alpha < \lambda$ ,  $M_{\alpha+1}/M_\alpha$  is isomorphic to an element of  $\mathcal{A}$ . In this case,  $\mathcal{M}$  is said to be a *witnessing chain* for  $M$ .

A class  $\mathcal{A}$  is *closed under transfinite extensions* provided that  $M \in \mathcal{A}$  whenever  $M$  is a transfinite extension of modules in  $\mathcal{A}$ . We will now see that this property is shared by the classes  $\mathcal{P}$ ,  $\mathcal{D}$ , and  $\mathcal{F}$ .

For the rest of the paper it is crucial to keep in mind the next result, known as the Eklof lemma, showing that Ext-orthogonal classes are closed under transfinite extensions.

**Lemma 6.2.** (See [10, XII.1.5].) *Let  $R$  be a ring. Let  $\mathcal{C}$  be any class of modules. Then the class  ${}^\perp\mathcal{C}$  is closed under transfinite extensions.*

Now we arrive at a key property of projective and flat modules that makes it possible to apply the homotopy theory tools developed in [26]. The term “deconstructible” is due to Eklof (see e.g. [9, Definition 5.1]).

**Definition 6.3.** Let  $R$  be a ring and  $\mathcal{A}$  a class of modules.

For an infinite cardinal  $\kappa$ , we define  $\mathcal{A}^{<\kappa}$  to be the class of all  $< \kappa$ -presented modules in  $\mathcal{A}$ . Then  $\mathcal{A}$  is called  $\kappa$ -*deconstructible* provided that each module  $M \in \mathcal{A}$  is a transfinite extension of modules in  $\mathcal{A}^{<\kappa}$ .

$\mathcal{A}$  is *deconstructible* in case there is a cardinal  $\kappa$  such that  $\mathcal{A}$  is  $\kappa$ -deconstructible.

#### Examples 6.4.

- (1) Let  $\mathcal{S}$  be a set of modules then  ${}^\perp(\mathcal{S}^\perp)$  is closed by transfinite extensions by Eklof Lemma 6.2, and it is deconstructible by [21, Theorem 4.2.11].
- (2) The classes  $\mathcal{P}$  and  $\mathcal{F}$  are particular instances of (1). Clearly  $\mathcal{P} = {}^\perp(\mathcal{P}^\perp)$ , and by Kaplansky theorem, the class  $\mathcal{P}$  is  $\aleph_1$ -deconstructible for any ring  $R$ . The class  $\mathcal{F}$  is  $\kappa^+$ -deconstructible where  $\kappa$  is the least infinite cardinal  $\geq \text{card } R$  [6], and also  $\mathcal{F} = {}^\perp\mathcal{C}$  where  $\mathcal{C}$  denotes the class of pure injective modules.
- (3) Let  $\mathcal{Q}$  be any class of left  $R$ -modules. Then the class  $\mathcal{D}_{\mathcal{Q}}$  is closed under transfinite extensions by [1, Proposition 1.9]. This is *not* a consequence of (1) – see Corollary 7.9(i) below.
- (4) If  $R$  is a right perfect ring and  $\mathcal{Q}$  is any class of left  $R$ -modules, then  $\mathcal{P} = \mathcal{D}_{\mathcal{Q}} = \mathcal{F}$ . Therefore,  $\mathcal{D}_{\mathcal{Q}} = {}^\perp(\mathcal{D}_{\mathcal{Q}}^\perp)$  is  $\aleph_0$ -deconstructible.

In order to study transfinite extensions and deconstructible classes, the following lemma, known as the Hill lemma, is very useful. It goes back to [25]; the general version needed here is [37, Theorem 6] (see also [21, 4.2.6]):

**Lemma 6.5.** *Let  $R$  be a ring,  $\kappa$  a regular infinite cardinal, and  $\mathcal{C}$  a class of  $< \kappa$ -presented modules. Let  $M$  be a transfinite extension of modules in  $\mathcal{C}$ , with a witnessing chain  $\mathcal{M} = (M_\alpha \mid \alpha \leq \lambda)$ . Then there is a family  $\mathcal{H}$  consisting of submodules of  $M$  such that*

- (i)  $\mathcal{M} \subseteq \mathcal{H}$ ,
- (ii)  $\mathcal{H}$  is closed under arbitrary sums and intersections,

- (iii)  $P/N$  is a transfinite extension of modules in  $\mathcal{C}$  for all  $N, P \in \mathcal{H}$  such that  $N \subseteq P$ , and
- (iv) if  $N \in \mathcal{H}$  and  $S$  is a subset of  $M$  of cardinality  $< \kappa$ , then there exists  $P \in \mathcal{H}$  such that  $N \cup S \subseteq P$  and  $P/N$  is  $< \kappa$ -presented.

If  $\kappa$  is a regular infinite cardinal and  $\mathcal{A}$  a  $\kappa$ -deconstructible class, then the Hill lemma implies that each  $M \in \mathcal{A}$  has a large family of chains witnessing that  $M$  is a transfinite extension of modules in  $\mathcal{A}^{<\kappa}$ . Thus we obtain a direct link between deconstructible classes and the Kaplansky classes in  $\text{Mod-}R$  in the sense of [20, Definition 4.9] (cf. [12, Definition 2.1]):

**Definition 6.6.** Let  $R$  be a ring,  $\kappa$  an infinite cardinal, and  $\mathcal{A}$  a class of modules.

$\mathcal{A}$  is said to be a  $\kappa$ -Kaplansky class provided that for each  $A \in \mathcal{A}$  and each  $\leq \kappa$ -generated submodule  $B \subseteq A$  there exists a  $\leq \kappa$ -presented submodule  $C \in \mathcal{A}$  such that  $B \subseteq C \subseteq A$  and  $A/C \in \mathcal{A}$ .

$\mathcal{A}$  is called a *Kaplansky class* in case there is an infinite cardinal  $\kappa$  such that  $\mathcal{A}$  is a  $\kappa$ -Kaplansky class.

**Lemma 6.7.** Let  $R$  be a ring,  $\kappa$  an infinite cardinal, and  $\mathcal{A}$  a  $\kappa^+$ -deconstructible class of modules closed under transfinite extensions. Then  $\mathcal{A}$  is a  $\kappa$ -Kaplansky class.

In particular, each deconstructible class closed under transfinite extensions is a Kaplansky class.

**Proof.** Assume that  $\mathcal{A}$  is  $\kappa^+$ -deconstructible. Let  $0 \neq A \in \mathcal{A}$  and let  $\mathcal{M} = (M_\alpha \mid \alpha \leq \lambda)$  be a witnessing chain for  $A$ . Consider the corresponding family  $\mathcal{H}$  from Lemma 6.5 (for the infinite regular cardinal  $\kappa^+$ , and for  $\mathcal{C} = \mathcal{A}^{<\kappa^+}$ ). Let  $B$  be a  $\leq \kappa$ -generated submodule of  $A$ . By condition (iv) of Lemma 6.5 (for  $N = 0$  and  $S$  a generating subset of  $B$  of cardinality  $\leq \kappa$ ), there exists  $C \in \mathcal{H}$  such that  $C$  is  $\leq \kappa$ -presented and  $B \subseteq C$ . By condition (iii), both  $C$  and  $A/C$  are transfinite extensions of modules in  $\mathcal{C}$ . Since  $\mathcal{C} \subseteq \mathcal{A}$ , we conclude that  $C, A/C \in \mathcal{A}$ .  $\square$

The converse of Lemma 6.7 fails in general, as shown by the following example:

**Example 6.8.** Let  $R$  be a non-artinian von Neumann regular right self-injective ring (for example, let  $R$  be the endomorphism ring of an infinite dimensional right linear space). Let  $\mathcal{A} = \mathcal{D}$  be the class of all  $\aleph_1$ -projective modules. As observed in Section 4, since  $R$  is von Neumann regular,  $\mathcal{A}$  is the class of all modules  $M$  such that each finitely generated submodule of  $M$  is projective. In particular, since  $R$  is right non-singular, so is each  $M \in \mathcal{A}$ . Conversely, if  $M$  is non-singular and  $N$  is a finitely generated submodule of  $M$ , then  $N$  is projective by [23, 9.2]. So  $\mathcal{A}$  also coincides with the class of all non-singular modules. By Corollary 2.10 and Example 6.4(3),  $\mathcal{A}$  is closed under transfinite extensions.

We will show that  $\mathcal{A}$  is a Kaplansky class.<sup>3</sup> Let  $\lambda (\geq \aleph_0)$  be the cardinality of  $R$  and let  $\kappa = 2^\lambda$ . In order to prove that  $\mathcal{A}$  is a  $\kappa$ -Kaplansky class, it suffices to show that if  $A$  is an  $\aleph_1$ -projective module,  $B$  is its submodule of cardinality  $\leq \kappa$ , and  $B \subseteq C \subseteq A$  is such that  $C/B$  is the singular submodule of  $A/B$ , then  $C$  has cardinality  $\leq \kappa$  (then also  $A/C \in \mathcal{A}$ , because  $R$  is non-singular).

Consider the set of all pairs  $(I, \{b_i \mid i \in I\})$  where  $I$  is an essential right ideal of  $R$  and  $b_i \in B$  for each  $i \in I$ . Notice that for each pair  $(I, \{b_i \mid i \in I\})$ , there is at most one  $x \in A$

<sup>3</sup> Added in proof: In [35], it has recently been proved that  $\mathcal{D}_{\mathcal{Q}}$  is a Kaplansky class for any ring  $R$  and any class  $\mathcal{Q}$ .



such that  $I$  is the annihilator of  $x + B$ , and  $x \cdot i = b_i$  for each  $i \in I$  (if  $x' \in A$  is another such element, then  $x - x'$  is annihilated by  $I$ , so  $x = x'$  because  $A$  is non-singular). The number of essential ideals of  $R$  is at most  $\kappa = 2^\lambda$ , and since  $I$  has cardinality  $\leq \lambda$ , the number of the sequences of the form  $\{b_i \mid i \in I\}$  is again at most  $\kappa = \kappa^\lambda$ . It follows that  $C$  has cardinality  $\leq \kappa$ .

Finally, by Corollary 2.10, the fact that  $\mathcal{A}$  is not deconstructible is a particular instance of Corollary 7.3 below.

However, the converse of Lemma 6.7 does hold in the particular case of classes closed under extensions and direct limits (which is the setting where Kaplansky classes were employed in [14] and [20]):

**Lemma 6.9.** *Let  $R$  be a ring,  $\kappa$  an infinite cardinal, and  $\mathcal{A}$  a class of modules closed under extensions and direct limits. Then  $\mathcal{A}$  is  $\kappa^+$ -deconstructible iff  $\mathcal{A}$  is a  $\kappa$ -Kaplansky class.*

*In particular,  $\mathcal{A}$  is deconstructible iff  $\mathcal{A}$  is a Kaplansky class.*

**Proof.** It is easy to see that our assumptions on  $\mathcal{A}$  imply that  $\mathcal{A}$  is closed under transfinite extensions. So the only if part follows by Lemma 6.7.

Conversely, assume  $\mathcal{A}$  is a  $\kappa$ -Kaplansky class and let  $M \in \mathcal{A}$ . Taking a generating set  $L = \{g_\alpha \mid \alpha < \lambda\}$  of  $M$ , we construct a witnessing chain  $\mathcal{M} = (M_\alpha \mid \alpha \leq \lambda)$  for  $M$  as follows:  $M_0 = 0$ ; if  $M_\alpha$  is defined so that  $M_\alpha, M/M_\alpha \in \mathcal{A}$ , we use Definition 6.6 for  $A = M/M_\alpha$  and  $B = (g_\alpha + M_\alpha)R$  in order to obtain  $M_{\alpha+1}$  such that  $M_\alpha \cup \{g_\alpha\} \subseteq M_{\alpha+1}$  and  $C = M_{\alpha+1}/M_\alpha \in \mathcal{A}$ . Then  $M/M_{\alpha+1} \cong A/C \in \mathcal{A}$ , and  $M_{\alpha+1} \in \mathcal{A}$  because  $\mathcal{A}$  is closed under extensions. If  $\alpha \leq \lambda$  is a limit ordinal, we let  $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ . Then  $M_\alpha \in \mathcal{A}$  by Eklof Lemma 6.2. Moreover,  $M/M_\alpha \cong \varinjlim_{\beta < \alpha} M/M_\beta$ , so  $M/M_\alpha \in \mathcal{A}$  by assumption. We conclude that  $L \subseteq M_\lambda$ , so  $M_\lambda = M$ .  $\square$

The following result, based on the constructions in Section 5, gives a useful criterion for deconstructibility of classes of modules.

**Theorem 6.10.** *Let  $R$  be a ring, and let  $\mathcal{A}' \subseteq \mathcal{A}$  be classes of modules closed under isomorphisms. Assume also that  $\mathcal{A}'$  is closed under countable direct sums, and that  $\mathcal{A}$  is a deconstructible class closed under direct summands such that  $\mathcal{A}$  contains all modules possessing an  $\aleph_1$ -dense system of modules in  $\mathcal{A}'$ . Then  $\mathcal{A}$  is closed under countable direct limits of modules from  $\mathcal{A}'$ .*

**Proof.** Let  $\kappa$  be an infinite cardinal such that  $\mathcal{A}$  is  $\kappa^+$ -deconstructible. Assume, by the way of contradiction, that there is a module  $N \notin \mathcal{A}$  that is a countable direct limit of the modules  $F_i = F \in \mathcal{A}'$  ( $i < \omega$ ). We may suppose that  $\kappa \geq \text{card}(F), \text{card}(R)$ . Then, using this data, the module  $M = M_{\kappa^+}$  constructed in Notation 5.3 has an  $\aleph_1$ -dense system of modules in  $\mathcal{A}'$  by Theorem 5.7. Therefore,  $M \in \mathcal{A}$  by assumption. Moreover,  $M$  is  $\kappa^+$ -generated by Proposition 5.5.

By assumption, there is a witnessing chain  $\mathcal{N}$  for  $M$  being a transfinite extension of  $\leq \kappa$ -generated modules in  $\mathcal{A}$ . Using Lemma 6.5 (with  $\mathcal{M}$  replaced by  $\mathcal{N}$ ,  $\kappa$  by  $\kappa^+$ , and  $\mathcal{C} = \mathcal{A}^{<\kappa^+}$ ) and the fact that  $M$  is  $\kappa^+$ -generated, we can select from the family  $\mathcal{H}$  a new witnessing chain  $\mathcal{M}'$  for  $M$  of length  $\kappa^+$ , so  $\mathcal{M}' = (M'_\alpha \mid \alpha \leq \kappa^+)$ , such that  $M'_\alpha$  is  $\leq \kappa$ -generated for each  $\alpha < \kappa^+$ . Then  $C = \{\alpha < \kappa^+ \mid M_\alpha = M'_\alpha\}$  is a closed and unbounded subset of  $\kappa^+$ . Since  $E$  is stationary, there exists  $\nu \in C \cap E$ , and also  $\nu < \mu \in C \cap E$ . Then  $N_\mu/N_\nu \in \mathcal{A}$  because  $N_\mu, N_\nu \in \mathcal{H}$ , but  $N_\mu/N_\nu = M_\mu/M_\nu \notin \mathcal{A}$  because, by Proposition 5.5,  $M_\mu/M_\nu$  has a direct summand isomorphic

to  $N \notin \mathcal{A}$ . This contradicts the initial assumption of  $\mathcal{A}$  being  $\kappa^+$ -deconstructible. Therefore we conclude that  $N \in \mathcal{A}$ .  $\square$

**Remark 6.11.** The properties of the module  $M$  proved in Section 5 still hold if we replace the set  $E$  in the construction by any of its stationary subsets. This makes it possible to prove the stronger claim that  $M$  has a  $\kappa^+$ -dense system of  $\leq \kappa$ -generated submodules (so not just the  $\aleph_1$ -density, cf. Theorem 5.7) under the extra set-theoretic hypothesis of the Axiom of Constructibility ( $V = L$ ). The point is that by [10, VI.3.1],  $V = L$  implies that for each infinite cardinal  $\kappa$  there is a non-reflecting stationary subset  $\tilde{E}$  of  $\kappa^+$  consisting of ordinals of cofinality  $\aleph_0$ . As in [10, VII.1.4], we then infer that the module  $M$  defined for  $E = \tilde{E}$  has  $\kappa^+$ -dense system of submodules. That  $M$  is not a transfinite extension of modules in  $\mathcal{A}^{<\kappa^+}$  then follows exactly as in the proof of Theorem 6.10.

Now we prove another general result that ensures the closure under countable direct limits, this time for classes of modules of the form  ${}^\perp\mathcal{C}$ . We substitute the hypothesis of deconstructibility from Theorem 6.10 by closure under products and pure submodules.

If  $\kappa$  is an ordinal and  $(M_\alpha, \alpha < \kappa)$  is a family of modules over a ring  $R$  we denote by  $\prod_{\alpha < \kappa}^b M_\alpha$  the submodule of  $\prod_{\alpha < \kappa} M_\alpha$  formed by the elements with bounded support in  $\kappa$ . If, for any  $\alpha, \beta, M_\alpha = M_\beta$  we simply write  $\prod_{\alpha < \kappa} M_\alpha = M^\kappa$  and  $\prod_{\alpha < \kappa}^b M_\alpha = M^{<\kappa}$ .

The following result is just a variation of [36, Lemmas 7 and 8] (see also [21, Lemma 4.3.17, Lemma 4.3.18]). The proof is just a straightforward adaptation of the original one.

**Lemma 6.12.** (See [21, Lemma 4.3.17, Lemma 4.3.18].) *Let  $R$  be a ring and  $C$  be a module. Then:*

- (i) *Let  $M$  be a module such that, for any set  $I$ , any pure submodule of  $M^I$  is in  ${}^\perp C$ . Then for any regular cardinal  $\kappa, M^\kappa / M^{<\kappa} \in {}^\perp C$ .*
- (ii) *Let  $\mathcal{A}'$  be a class of modules closed under products and such that all pure submodules of elements of  $\mathcal{A}'$  are in  ${}^\perp C$ . Let  $\kappa$  be a regular cardinal, then  $\prod_{\alpha < \kappa} M_\alpha / \prod_{\alpha < \kappa}^b M_\alpha \in {}^\perp C$  for any family  $(M_\alpha, \alpha < \kappa)$  of modules in  $\mathcal{A}'$ .*

**Theorem 6.13.** *Let  $R$  be a ring. Let  $\mathcal{A}'$  be a class of modules that is closed under products. Assume that  $\mathcal{A}' \subseteq \mathcal{A} = {}^\perp C$  for a suitable class of modules  $C$ , and that  $\mathcal{A}$  is closed under pure submodules. Then  $\mathcal{A}$  contains all countable direct limits of modules in  $\mathcal{A}'$ .*

**Proof.** Let  $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} \dots$  be a countable direct system of modules in  $\mathcal{A}'$ . Let  $M = \varinjlim M_n$ . Then  $M \cong \bigoplus_{n \in \mathbb{N}} M_n / \Phi(\bigoplus_{n \in \mathbb{N}} M_n)$  where  $\Phi : \bigoplus_{n \in \mathbb{N}} M_n \rightarrow \bigoplus_{n \in \mathbb{N}} M_n$  is the map defined by  $\Phi(0, \dots, 0, m_n, 0, \dots) = (0, \dots, m_n, -f_n(m_n), 0, \dots)$  for any  $m_n \in M_n$ . Notice that  $\Phi$  can be extended to an isomorphism  $\Phi' : \prod_{n \in \mathbb{N}} M_n \rightarrow \prod_{n \in \mathbb{N}} M_n$  by setting  $\Phi'(m_1, m_2, \dots, m_n, \dots) = (m_1, m_2 - f_1(m_1), \dots, m_n - f_{n-1}(m_{n-1}), \dots)$ .

By Lemma 6.12,

$$\left( \prod_{n \in \mathbb{N}} M_n \right) / \bigoplus_{n \in \mathbb{N}} M_n \cong \left( \prod_{n \in \mathbb{N}} M_n \right) / \Phi \left( \bigoplus_{n \in \mathbb{N}} M_n \right) \in \mathcal{A}.$$

Since the inclusion  $\bigoplus_{n \in \mathbb{N}} M_n \subseteq \prod_{n \in \mathbb{N}} M_n$  is a pure embedding,  $\bigoplus_{n \in \mathbb{N}} M_n / \Phi(\bigoplus_{n \in \mathbb{N}} M_n)$  is a pure submodule of  $\prod_{n \in \mathbb{N}} M_n / \Phi(\prod_{n \in \mathbb{N}} M_n)$ . Since  $\mathcal{A}$  is closed under pure submodules, we conclude that  $M \in \mathcal{A}$ .  $\square$

### 7. Non-deconstructibility of flat Mittag-Leffler modules and cotorsion pairs

We recall that a pair of classes of modules  $(\mathcal{A}, \mathcal{B})$  is a *cotorsion pair* if  $\mathcal{A} = {}^\perp \mathcal{B}$  and  $\mathcal{A}^\perp = \mathcal{B}$ . If  $\mathcal{S}$  is a class of modules then the cotorsion pair *generated by*  $\mathcal{S}$  is  $({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$ .

Cotorsion pairs can also be considered in more general categories. In [20, §§4–5], Gillespie employed Kaplansky classes closed under direct limits in constructing Quillen model category structures on the category of all unbounded chain complexes over a Grothendieck category  $\mathcal{G}$ , using the approach via small cotorsion pairs from [26].

In the particular case when  $\mathcal{G}$  is the category of all quasi-coherent sheaves on a scheme  $X$ , and  $V$  denotes the set of all affine open subsets of  $X$ ,  $\mathcal{G}$  can be identified with the category of ‘quasi-coherent modules’  $\mathcal{M} = \{M(v) \mid v \in V\}$  over a representation  $\mathcal{R} = \{R(v) \mid v \in V\}$  of a particular quiver [12, §2]. The generalized infinite dimensional vector bundles suggested by Drinfeld in [8] (see Introduction) then correspond to the ‘quasi-coherent modules’  $\mathcal{M}$  such that  $M(v)$  is a flat Mittag-Leffler  $R(v)$ -module for each  $v \in V$  [15].

In [15], Gillespie’s result was extended further, to deconstructible classes, for quasi-coherent sheaves on a semi-separated scheme  $X$ . However, as mentioned in the Introduction, deconstructibility is also a necessary condition for making Hovey’s approach from [26] applicable in this setting.

Therefore, in this section, we study deconstructibility of the classes  $\mathcal{D}_Q$  and of  ${}^\perp(\mathcal{D}_Q^\perp)$ . We answer question (2’), and hence also question (2) from the Introduction, in the negative for each non-right perfect ring  $R$ .

We start by observing the following closure properties of any cotorsion pair generated by  $\mathcal{D}_Q$ -Mittag-Leffler modules. They will allow us to apply the results from Section 6.

A class  $\mathcal{C}$  is called *resolving* if  $\mathcal{C}$  is closed under extensions,  $\mathcal{P} \subseteq \mathcal{C}$ , and  $A \in \mathcal{C}$  whenever  $B, C \in \mathcal{C}$  fit into an exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ . The classes  $\mathcal{P}$  and  $\mathcal{F}$  are resolving and, as we recall in the following lemma, so is the class  $\mathcal{D}_Q$  for any class  $Q$  of left  $R$ -modules.

**Lemma 7.1.** *Let  $R$  be a ring,  $F$  be a flat module, and  $Q$  be a class of left  $R$ -modules. Then:*

- (i)  ${}^\perp(\mathcal{D}^\perp) \subseteq {}^\perp(\mathcal{D}_Q^\perp) \subseteq \mathcal{F}$ .
- (ii) For each  $n \geq 1$ ,  $\Omega_n(F) \in \mathcal{SD} \subseteq \mathcal{D} \subseteq \mathcal{D}_Q$ . Here  $\Omega_n(F)$  denotes any  $n$ -th syzygy of  $F$ .
- (iii) For all  $C \in \mathcal{D}_Q^\perp$  and  $n \geq 2$ ,  $\text{Ext}_R^n(F, C) = 0$ .
- (iv)  ${}^\perp(\mathcal{D}_Q^\perp)$  is closed under pure submodules.
- (v) The cotorsion pair  $({}^\perp(\mathcal{D}_Q^\perp), \mathcal{D}_Q^\perp)$  is hereditary, that is, for any  $n \geq 1$  each  $n$ -th syzygy of a module in  ${}^\perp(\mathcal{D}_Q^\perp)$  is also in  ${}^\perp(\mathcal{D}_Q^\perp)$ .

**Proof.** (i). The class  $\mathcal{F}$  of all flat modules coincides with  ${}^\perp(\mathcal{F}^\perp)$ . Therefore if  $\mathcal{C}$  is any class of flat modules  ${}^\perp(\mathcal{C}^\perp) \subseteq \mathcal{F}$ .

To finish the proof of (i), note that  $\mathcal{D} = \mathcal{D}_Q$  where  $Q = R\text{-Mod}$ .

(ii). Fix  $n \geq 1$  and consider an exact sequence

$$0 \rightarrow \Omega_n(F) \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow F = \Omega_0(F) \rightarrow 0,$$

where  $P_i$  are projective modules. Since any syzygy of a flat module is flat, for any  $n \geq 1$ , the exact sequence

$$0 \rightarrow \Omega_n(F) \rightarrow P_{n-1} \rightarrow \Omega_{n-1}(F) \rightarrow 0$$

is pure. So  $\Omega_n(F) \in \mathcal{SD}$  because it is a flat pure submodule of the projective, hence strict Mittag-Leffler, module  $P_{n-1}$ .

Statement (ii) allows us to use a dimension shifting argument to prove (iii).

(iv). Let

$$0 \rightarrow X \rightarrow A \rightarrow A/X \rightarrow 0 \tag{*}$$

be a pure exact sequence such that  $A \in \mathcal{A} = {}^\perp(\mathcal{D}_Q^\perp)$ . Note that since by (i),  $A$  is flat, so are  $X$  and  $A/X$ .

Let  $C \in \mathcal{D}_Q^\perp$ . Applying the contravariant functor  $\text{Hom}_R(-, C)$  to (\*) we get the exact sequence

$$0 = \text{Ext}_R^1(A, C) \rightarrow \text{Ext}_R^1(X, C) \rightarrow \text{Ext}_R^2(A/X, C).$$

By (iii),  $\text{Ext}_R^2(A/X, C) = 0$  so that  $\text{Ext}_R^1(X, C) = 0$ .

Statement (v) follows from (ii) (or (iii)).  $\square$

**Corollary 7.2.** *Let  $R$  be a ring, and let  $\mathcal{Q}$  be a class of left  $R$ -modules. Then:*

- (i) *For each cardinal  $\kappa$ ,  ${}^\perp((\mathcal{D}_Q^{\leq \kappa})^\perp) \subseteq \mathcal{D}_Q$ .*
- (ii) *There exists a cardinal  $\kappa$  such that  ${}^\perp((\mathcal{D}_Q^{\leq \kappa})^\perp) = \mathcal{D}_Q$  if and only if  $\mathcal{D}_Q = \mathcal{F}$ .*

**Proof.** Statement (i) follows from [21, 4.2.11] and Example 6.4(3).

To prove (ii) assume first that  ${}^\perp((\mathcal{D}_Q^{\leq \kappa})^\perp) = \mathcal{D}_Q$ . By Theorem 2.6, we can apply Theorem 6.10 with  $\mathcal{A} = \mathcal{A}' = \mathcal{D}_Q$  to conclude that  $\mathcal{D}_Q$  must be closed under countable direct limits. In particular, it follows that any countable direct limit of projective modules is in  $\mathcal{D}_Q$ . By Corollary 2.4, we can deduce that  $\mathcal{D}_Q$  is closed under arbitrary direct limits. Hence  $\mathcal{D}_Q = \mathcal{F}$ . The converse follows from the fact that the class of flat modules is deconstructible [6].  $\square$

Notice that if  $R$  is right Noetherian and  $\mathcal{Q}$  is the class of all flat left  $R$ -modules then, by Corollary 2.13,  $\mathcal{D}_Q$  is the class of all flat modules. But this is no longer true in general (it fails for all non-artinian von Neumann regular rings, for example).

Specializing Corollary 7.2 to the class of flat Mittag-Leffler modules (i.e., to  $\mathcal{Q} = R\text{-Mod}$ ) we obtain the announced negative answer to question (2).

**Corollary 7.3.** *Let  $R$  be a ring. Then  $\mathcal{D}$  is deconstructible if and only if  $R$  is a right perfect ring.*

**Proof.** By Corollary 7.2(ii),  $\mathcal{D}$  is deconstructible if and only if  $\mathcal{D} = \mathcal{F}$ . In particular, all countably presented flat modules must be projective. It is a classical result of Bass that this holds if and only if  $R$  is a right perfect ring.  $\square$

As we have seen in Corollary 7.2, the problem of non-deconstructibility of  $\mathcal{D}$  can be avoided on the account of taking smaller subclasses of  $\mathcal{D}$ : for each uncountable cardinal  $\kappa$ , we can replace  $\mathcal{D}$  by the deconstructible subclass  $\mathcal{D}_\kappa = {}^\perp((\mathcal{D}^{<\kappa})^\perp)$ .

By (the proof of) Theorem 6.10, the module  $M_{\kappa^+}$  constructed in Notation 5.3 satisfies  $M_{\kappa^+} \in \mathcal{D}_{\kappa^+} \setminus \mathcal{D}_\kappa$ , so for each non-right perfect ring, there is a strictly increasing chain

$$\mathcal{P} = \mathcal{D}_{\aleph_1} \subset \mathcal{D}_{\aleph_2} \subset \cdots \subset \mathcal{D}_\kappa \subset \mathcal{D}_{\kappa^+} \subset \cdots \subset \mathcal{D} = \bigcup_{\kappa} \mathcal{D}_\kappa$$

indexed by the (proper) class of all uncountable cardinals. Of course,  $\mathcal{D}_{\aleph_1} = \mathcal{P}$ , but also the other classes  $\mathcal{D}_\kappa$  are of interest:

**Example 7.4.** A module  $M$  is called *Whitehead* provided that  $\text{Ext}_R^1(M, R) = 0$ . For each uncountable cardinal  $\kappa$ , Eklof and Shelah [11] have constructed a local PID  $R_\kappa$  of cardinality  $2^\kappa$  such that for each  $\leq \kappa$ -generated module  $M$ ,  $M$  is Whitehead, if and only if  $M$  is flat Mittag-Leffler. Moreover, under the assumption of  $V = L$ , the class  $\mathcal{D}_\kappa$  coincides with the class of all Whitehead  $R_\kappa$ -modules.

**Remark 7.5.** Each of the classes  $\mathcal{D}_\kappa$  ( $\kappa \geq \aleph_1$ ), as well as the class  $\mathcal{D}$ , determine the corresponding notion of a generalized vector bundle on a scheme  $X$  in the sense of [8]. All these notions are *local*, that is, do not depend on the particular choice of an open affine covering of  $X$ . In the case of  $\mathcal{D}$  and  $\mathcal{P} = \mathcal{D}_{\aleph_1}$ , this is a classic fact proved by Raynaud and Gruson [31], but the case of  $\mathcal{D}_\kappa$  for  $\kappa > \aleph_1$  is quite recent [16].

We conjecture that, in general,  ${}^\perp(\mathcal{D}^\perp) = \mathcal{F}$  and, hence, also  ${}^\perp(\mathcal{D}_Q^\perp) = \mathcal{F}$  for any class of left  $R$ -modules  $\mathcal{Q}$ . We explain some first criteria for this to happen in the following

**Proposition 7.6.** *Let  $R$  be a ring. Then the following statements are equivalent.*

- (i)  ${}^\perp(\mathcal{D}^\perp) = \mathcal{F}$ .
- (ii) For each class of left  $R$ -modules  $\mathcal{Q}$ ,  ${}^\perp(\mathcal{D}_Q^\perp) = \mathcal{F}$ .
- (iii)  ${}^\perp(\mathcal{D}^\perp)$  is closed under pure epimorphic images of modules in  $\mathcal{SD}$  (that is,  $Z \in {}^\perp(\mathcal{D}^\perp)$  whenever there exists an exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  with  $X, Y \in \mathcal{SD}$ ).
- (iv)  $\varinjlim \mathcal{D} \subseteq {}^\perp(\mathcal{D}^\perp)$ .

*If, in addition,  $R$  is left coherent then the statements above are also equivalent to*

- (v)  ${}^\perp(\mathcal{D}^\perp)$  is closed under products.

*If any of the statements above holds, then the class  ${}^\perp(\mathcal{D}^\perp)$  is deconstructible.*

**Proof.** As all projective modules are in  $\mathcal{D}$ , it is clear that (i) and (iv) are equivalent, and Lemma 7.1 easily yields that (i) and (ii) are also equivalent. To prove that (i) and (iii) are equivalent observe that if  $F$  is a flat module with a presentation

$$0 \rightarrow \Omega_1(M) \rightarrow P_0 \rightarrow M \rightarrow 0,$$

where  $P_0$  is projective, hence in  $\mathcal{SD}$ , then the exact sequence is pure and  $\Omega_1(M) \in \mathcal{SD}$  as  $\mathcal{SD}$  is closed under pure submodules (cf. Lemma 4.1).

If  $R$  is left coherent and  ${}^\perp(\mathcal{D}^\perp)$  is closed under products then, as  ${}^\perp(\mathcal{D}^\perp)$  is closed by pure submodules by Lemma 7.1, we can apply [21, Theorem 4.3.21] to deduce that it is also closed under pure epimorphic images. So (iii) holds. The converse implication is clear because the class  $\mathcal{F}$  is closed under direct products for each left coherent ring  $R$ .  $\square$

Using the results from Section 5 we will now deduce that the deconstructibility of  ${}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$  implies closure under countable direct limits of modules in  $\mathcal{D}_\mathcal{Q}$ .

**Corollary 7.7.** *Let  $R$  be a ring, and let  $\mathcal{Q}$  be a class of left  $R$ -modules. If the class  ${}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$  is deconstructible, then it contains all countable direct limits of modules in  $\mathcal{D}_\mathcal{Q}$ . In particular,  ${}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$  contains all countably presented flat modules.*

*If, in addition,  $R$  is countable then  ${}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$  is deconstructible if and only if  ${}^\perp(\mathcal{D}_\mathcal{Q}^\perp) = \mathcal{F}$ .*

**Proof.** By Theorem 2.6, we can apply Theorem 6.10 with  $\mathcal{A}' = \mathcal{D}_\mathcal{Q}$  and  $\mathcal{A} = {}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$  to conclude that  ${}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$  must be closed under countable direct limits. Since any projective module is Mittag-Leffler, we deduce that all countably presented flat modules must be in  ${}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$ .

If  $R$  is countable then a flat module is a transfinite extension of countably presented flat modules. Hence any flat module is a transfinite extension of a module in  ${}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$ . Since the latter class is closed under transfinite extensions and it is contained in  $\mathcal{F}$ , we conclude that it must coincide with  $\mathcal{F}$ .  $\square$

We conjecture that the deconstructibility of  ${}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$  is equivalent to the fact that  $\mathcal{F} = \varinjlim \mathcal{D}_\mathcal{Q} = {}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$ .

We finish by showing that if  $\mathcal{D}$  is closed under products (see Theorem 4.7), then  ${}^\perp(\mathcal{D}^\perp)$  is in fact closed under countable direct limits of modules in  $\mathcal{D}$ .

As a consequence we prove that if  $R$  is a countable ring such that  $\mathcal{D}$  is closed under products then  $\mathcal{F} = {}^\perp(\mathcal{D}^\perp)$ . (In the particular case of  $R = \mathbb{Z}$ , the latter result was proved using specific methods of abelian group theory in [15, §5].)<sup>4</sup>

**Corollary 7.8.** *Let  $R$  be a ring and let  $\mathcal{Q}$  be a class of left  $R$ -module such that  $\mathcal{D}_\mathcal{Q}$  is closed under products (e.g., let  $R$  be a left Noetherian ring). Then  ${}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$  contains all countable direct limits of modules in  $\mathcal{D}_\mathcal{Q}$ . In particular, any countably presented flat module is in  ${}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$ .*

**Proof.** Our hypothesis and Lemma 7.1 made it possible to apply Theorem 6.13 with  $\mathcal{A}' = \mathcal{D}_\mathcal{Q}$  and  $\mathcal{A} = {}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$  to deduce that countable direct limits of modules in  $\mathcal{D}_\mathcal{Q}$  are in  ${}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$ .

Since any countably presented flat module  $M$  is a countable direct limit of finitely generated free modules we deduce that  $M \in {}^\perp(\mathcal{D}_\mathcal{Q}^\perp)$ .  $\square$

<sup>4</sup> *Added in proof:* Recently, the assumption of  $\mathcal{D}$  being closed under products has been removed, so Corollary 7.9(ii) holds for arbitrary countable rings  $R$  – see [5] and [35]. However, the question of whether  $\mathcal{F} = {}^\perp(\mathcal{D}^\perp)$  remains open for all uncountable non-perfect rings.

**Corollary 7.9.**

- (i) Let  $R$  be a non-right perfect ring such that  $\mathcal{D}$  is closed under products (e.g., let  $R$  be a left Noetherian ring which is not artinian). Then  $\mathcal{D}$  is closed under transfinite extensions, but it is not of the form  ${}^{\perp}\mathcal{C}$  for any class of modules  $\mathcal{C}$ .
- (ii) Let  $R$  be a countable ring such that  $\mathcal{D}$  is closed under products (e.g., let  $R$  be a countable left Noetherian ring). Then  ${}^{\perp}(\mathcal{D}^{\perp}) = \mathcal{F}$ .

**Proof.** (i).  $\mathcal{D}$  is closed under transfinite extensions by Example 6.4(3). If  $\mathcal{D} = {}^{\perp}\mathcal{C}$  for a class of modules  $\mathcal{C}$ , then  $\mathcal{D} = {}^{\perp}(\mathcal{D}^{\perp})$ , so  $\mathcal{D}$  contains the class  $\mathcal{B}$  all countable direct limits of modules in  $\mathcal{D}$  by Corollary 7.8. However, since  $R$  is non-right perfect,  $\mathcal{B}$  contains a countably presented flat non-projective module  $F$ , by a classic result of Bass. So  $F \in \mathcal{D}$ , a contradiction.

(ii). Since  $R$  is countable, any flat module has a filtration of countably generated (hence, countably presented) flat modules [6]. Hence, by Corollary 7.8, any flat module is filtered by modules in  ${}^{\perp}(\mathcal{D}^{\perp})$ . By Eklof Lemma 6.2,  $\mathcal{F} \subseteq {}^{\perp}(\mathcal{D}^{\perp})$ . Hence,  ${}^{\perp}(\mathcal{D}^{\perp}) = \mathcal{F}$ .  $\square$

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