Vortex Dynamics of Ginzburg–Landau Equations in Inhomogeneous Superconductors

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We study a Ginzburg–Landau equation of parabolic type in inhomogeneous superconductors. It is proved that the vortices are attracted by impurities so that they are pinned by inhomogeneities in the superconducting materials. This fact was predicted recently by Chapman and Richardson using a method of formal matched asymptotics.

Key Words: Ginzburg–Landau system; vortex pinning; dynamics; parabolic estimate.

1. INTRODUCTION

Consider the following initial-boundary value problem of Ginzburg–Landau system for \( u = (u^1, u^2) : \Omega \to \mathbb{R}^2 \) with a smooth bounded domain \( \Omega \subset \mathbb{R}^2 \):

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= \frac{u}{\varepsilon^2} (a(x) - |u|^2) \quad \text{in} \quad \Omega \times (0, \infty) \\
\partial \Omega \times (0, \infty) \\
u_s(x, 0) &= u^0_s(x) \quad \text{in} \quad \Omega.
\end{aligned}
\]

(1.1)

The equation in (1.1) is a simple model which simulates inhomogeneous type-II superconducting materials. In this material, the equilibrium density of superconducting electrons is not a constant, but a positive and smooth function on \( \Omega \) which is characterised by \( a(x) \) in (1.1). We refer to [1–3] and the references there for the detailed physical background.

Physically, the points at which a solution to problem (1.1) equals zero are called vortices. In the case of \( a(x) \equiv 1 \), the vortex dynamics was studied first for the steady equations in [4] by Bethuel et al. and in [5] by Lin. (Also see [6] for the minimum solution.) Their results show that the

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vortices converge to a set of the critical points of the so-called renormalized energy functionals associated the steady problem. Furthermore, Lin [7], and independently Jerrard and Soner [8], studied the dynamical law for the vortices of $u(x, t)$ solving the initial-boundary value problem (1.1) with the log$rac{1}{t}$-time scaling factor for the case $a \equiv 1$. Their dynamical law is described by an ODE, $\frac{d}{dt} y(t) = -\nabla w(y(t))$. Here $w$ is the renormalized energy functional given by [4, p. 21]. The results in [7, 8] were generalized to the Neumann boundary condition by Lin [9].

In the case where $a(x)$ is not a constant, however, Chapman and Richardson [1] used a matched asymptotic method to predict a different phenomenon, i.e., the vortices for problem (1.1) (in fact, for a more complicated equation involving magnetic field and electric field), are attracted to the the minimum points of $a(x)$. In this paper we will prove this dynamical phenomenon rigorously.

The main results of this paper are Theorems 1.1, 1.2, and 1.3 below. The difference between these results and the results proved by Lin [7] and Jerrard and Soner [8] is this: first, our results are for the equation in (1.1) in nonhomogeneous case but their results are for its log$rac{1}{t}$-time scaled equation in homogeneous case; second, our vortex dynamics is described by Eq. (1.2) which depends only on the equilibrium density of superconducting electrons but theirs only on the renormalized energy functional; finally, the most important characteristic of our results shows that as $t \to \infty$ and $\varepsilon \to 0$, all vortices are pinned to the minimum points of $a(x)$.

We are now in the position to state the main results. we begin with the following assumptions:

\( (A_1) \quad g_1: \partial \Omega \to \mathbb{R}^2 \) is smooth, $|g_1(x)| = \sqrt{a(x)}$ on $\partial \Omega$ and $\deg(g_1, \partial \Omega) > 0$;
\( (A_2) \quad a \in C^{1, \alpha}(\overline{\Omega}) \ (\alpha > 0), \text{ and } a(x) > 0 \text{ for all } x \in \overline{\Omega}; \)
\( (A_3) \quad ) \text{ the initial data } u_0^\varepsilon \in C^2(\overline{\Omega}; \mathbb{R}^2) \ (\varepsilon > 0) \text{ satisfy } u_0^\varepsilon(x) = g_1(x) \text{ on } \partial \Omega \text{ and }\)
\[ ||u_0^\varepsilon||_{C(\overline{\Omega})} \leq K, \quad \int_{\Omega} \rho^2(x) [|\nabla u_0^\varepsilon|^2 + \frac{1}{2\varepsilon^2} (|u_0^\varepsilon|^2 - a(x))^2] \, dx \leq K \]
for a constant $K$ (independent of $\varepsilon$) and some $m$ distinct points $b_1, b_2, ..., b_m$ in $\Omega$, where $\rho(x) = \min\{|x - b_j|, j = 1, 2, ..., m\}$.

To describe the vortex dynamics, we need to consider the ODE system
\[
\begin{aligned}
\frac{d}{dt} y_j(t) &= -\nabla \ln a(y_j(t)), \\
y_j(0) &= b_j
\end{aligned}
\]
(1.2)
for \( j = 1, 2, \ldots, m \), where the \( b_j \)'s are the same as in \((A_3)\) and \( V \) is the gradient operator with respect to \( x = (x_1, x_2) \in \mathbb{R}^2 \).

In order to obtain the global solution to \((1.2)\), we suppose that for each \( j \) there is a Lipschitz domain \( G_j \) such that

\[
(A_4) \quad b_j \in G_j \subset \Omega, \min_{x \in G_j} a(x) > a(b_j), \quad j = 1, \ldots, m.
\]

Obviously, if \( V a(b_j) = 0 \) for some \( j = 1, 2, \ldots, m \), then \( y_j(t) \equiv b_j \) is the unique solution to \((1.2)\). More generally, we have the following result.

**Theorem 1.1.** (i) If hypotheses \((A_2)\) and \((A_4)\) are satisfied, then problem \((1.2)\) has a unique \( C^3 \)-solution \((y_1, y_2, \ldots, y_m) : [0, \infty) \rightarrow (\Omega)^m \). Moreover, the solution satisfies that for each \( j, \ l = 1, \ldots, m \), \( y_j(t) \neq y_l(t) \) for \( j \neq l \) and for all \( t \in [0, +\infty) \), and \( y_j(t) \in G_j \) for all \( t \in [0, +\infty) \).

(ii) Suppose that hypotheses \((A_2)\) and \((A_4)\) are satisfied. If the function \( a(x) \) is an analytic function in a neighborhood of any \( b \) in \( \Omega \) with \( V a(b) = 0 \), then for each \( j = 1, \ldots, m \), there exists a \( B_j \in G_j \) satisfying \( V a(B_j) = 0 \) such that \( y_j(t) \rightarrow B_j \) as \( t \to \infty \).

Essentially, Theorem 1.1 was proved by the first author in a short paper [10]. Its aim was to make the theorem 1.1 in [9, p. 390] more complete. Fortunately, we discover that our Theorem 1.1 can be used to describe the dynamical behaviour of the vortices of problem \((1.1)\). In fact, after cutting the domain \( \Omega \times (0, \infty) \) along the curves of the solutions to \((1.2)\), we will see that the energy of the solutions to \((1.1)\) is locally bounded (See Theorem 1.2 below). This suggests that the vortices of the solution to \((1.1)\) should evolve along the curves of solutions to \((1.2)\). We will prove this guess in the following Theorem 1.3.

**Theorem 1.2.** Suppose that hypotheses \((A_1), (A_2), (A_3), \text{ and } (A_4)\) are satisfied. Let \( y_j(t) (1 \leq j \leq m) \) be solutions to problem \((1.2)\) and denote

\[
\Omega(a) = \Omega \times (0, \infty) \setminus \bigcup_{j = 1}^{m} \{ (x, t) : x = y_j(t), \ 0 < t < \infty \}.
\]

Then the set \( \{ u : a > 0 \} \) of the classical solutions to problem \((1.1)\) is bounded in \( H^1_{0w}(\Omega(a)) \). Moreover, given any sequence \( \varepsilon_n \downarrow 0 \), there exists a subsequence (denoted still by itself \( u_n \)) such that \( u_n \rightharpoonup u \) weakly in \( H^1_{0w}(\Omega(a)) \), \( |u(x, t)| = \sqrt{a(x)} \ a.e. \text{ in } \Omega(a) \), \( a = g_1 \) on \( \partial \Omega \times (0, \infty) \) and \( u \) satisfies the equation

\[
\frac{\partial u}{\partial t} - Au = \frac{u}{2a} (2|Vu|^2 - Aa) \quad \text{in} \quad D'(\Omega(a)).
\]
Theorem 1.3. Suppose that the same hypotheses as in Theorem 1.2 are satisfied. Then for any $\delta > 0$ and any measurable set $I \subset (0, \infty)$ with $|I| > 0$, there exist a $t \in I$ and $a_\delta > 0$ such that

$$\{x \in \Omega: |u_j(x, t)| \leq \frac{1}{2} \min_{s \in \partial \Omega} \sqrt{a(x)} \} \subseteq \bigcup_{j=1}^{m} B_{\delta}(y_j(t)), \quad \forall \varepsilon \in (0, \varepsilon_0)$$

and

$$|u_j(x, t)| - \sqrt{a(x)} \leq C(\delta, I) \varepsilon, \quad \forall x \in \Omega \bigcup_{j=1}^{m} B_{\delta}(y_j(t)), \forall \varepsilon \in (0, \varepsilon_0).$$

The proofs of Theorems 1.1, 1.2, and 1.3 are arranged in the following three sections in order.

We should point out that Theorems 1.1 and 1.3 imply that for most sufficient large $t$, all the vortices of $u_j(x, t)$ are pinned together to the critical points of $a(x)$ in $\Omega$ as $\varepsilon \to 0$ if the assumptions of the conclusion (ii) in Theorem 1.1 hold true. In particular, if $a(x)$ has no other critical points than minimum points, all the vortices are pinned to the minimum points.

We conjecture that some results similar to Theorems 1.2 and 1.3 should be true for the problem (1.1)(1.3) in [9, p. 390].

Throughout this paper, we use the letter $C$ to denote various constants independent of $\varepsilon$ but maybe depending on $\Omega, a, g_1, K$ and other known constants.

2. PROOF OF THEOREM 1.1

Theorem 1.1 was essentially proved by the first author in [10]. For the sake of completeness, we outline the proof here.

Since $a(x)$ satisfies $(A_3)$, by a standard existence and uniqueness theorem for ODE we know that (1.2) has a unique solution defined on $[0, T)$ for some $T > 0$. If we assume $(A_4)$ and suppose that $y_j(t)$ satisfies (1.2) for all $t \in [0, T^*)$ with some $T^* > 0$, then

$$-\int_{0}^{t} \left| \frac{dy_j(s)}{ds} \right|^2 ds = \int_{0}^{t} \frac{d}{ds} \ln a(y_j(s)) ds = \ln a(y_j(t)) - \ln a(b_j),$$

which yields

$$a(y_j(t)) \leq a(b_j), \quad \forall t \in [0, T^*).$$

(2.1)

(2.2)

Using (2.2), $(A_4)$, and the continuity of $a(y_j(t))$, we see that $y_j(t)$ always stays in $G_j$ for all $t \in [0, T^*)$. Hence, we use the fact $G_j \subset \subset \Omega$ and apply the extension theorem (Theorem 2.3 of Chapter 1 in [11]) to obtain that
\( T = +\infty \). Furthermore, using a uniqueness theorem for Cauchy problem, we can easily obtain the conclusion (i).

Next, we are going to prove the conclusion (ii). Now that \( T = \infty \), by (2.1) we have

\[
\lim_{t \to \infty} \left( \frac{dy_j(t)}{dt} \right)^2 dt \leq \ln a(b) - \ln \min_{x \in \partial B} a(x) \leq C(a), \quad j = 1, \ldots, m. \tag{2.3}
\]

Since \( y_j(t) \) stays in \( G_j \) for all \( t > 0 \) and \( \Omega \) is bounded,

\[
|y_j(t)| \leq C(\Omega), \quad j = 1, \ldots, m, \quad \forall t \in [0, \infty). \tag{2.4}
\]

Now fix a \( j \) and write \( y(t) = y_j(t) \) for simplicity. Combining (2.3) and (2.4), one can find a sequence \( t_n \to \infty \) such that

\[
y(t_n) \to b \in \Omega \quad \text{and} \quad \frac{dy(t_n)}{dt} \to 0 \quad \text{as} \quad t_n \to \infty \tag{2.5}
\]

for some \( b \in \Omega \). This result, together with (1.2) yields \( \nabla \ln a(b) = 0 \). Furthermore, we have

\[
\frac{d}{dt} (\ln a(y(t)) - \ln a(b)) = - \left( \frac{dy(t)}{dt} \right)^2 \leq 0, \quad \forall t \in (0, \infty) \tag{2.6}
\]

and

\[
\ln a(y(t)) \geq \ln a(b) \quad \forall t \in (0, \infty), \quad \ln a(y(t)) \to \ln a(b) \quad \text{as} \quad t \to \infty. \tag{2.7}
\]

Since \( \ln a(x) \) is analytic at \( b \), we use the Lojasiewicz theorem \([12; 13, p. 538]\) concerning real analytic functions to obtain constants \( \theta_0 \) and \( \theta \) satisfying

\[
0 < \theta_0 < \text{dist}(b, \partial \Omega) \quad \text{and} \quad 0 < \theta < 2^{-1}
\]

such that

\[
|\nabla \ln a(x)| \geq ||\ln a(x) - \ln a(b)||^{1-\theta}, \quad \forall x \in B_{\theta_0}(b). \tag{2.8}
\]

For any \( \varepsilon \in (0, \frac{\theta_0}{4}) \), (2.5) and (2.7) imply that

\[
|y(t_n) - b| < \frac{\varepsilon}{4} \quad \text{and} \quad \theta^{-1}(\ln a(y(t_n)) - \ln a(b)) \leq \frac{\varepsilon}{4} \tag{2.9}
\]

for all \( n \geq N \) with some \( N > 0 \).
Let 
\[ i = \sup \{ s \geq t_N : |y(t) - b| < \theta_0 \text{ for all } t \in [t_N, s] \}. \]

We want to prove \( i = \infty \). Otherwise, \( |y(i) - b| = \theta_0 \). Moreover, by (1.2) and (2.8), we have, for all \( t \in (t_N, i) \), that
\[
\frac{d}{dt} (\ln a(y(t)) - \ln a(b))^{\frac{\theta}{\theta - 1}} \leq \theta \left| \frac{dy(t)}{dt} \right|.
\]

Thus,
\[
\int_{t_N}^{i} \left| \frac{dy(t)}{dt} \right| dt \leq \theta^{-1} [\ln a(y(t_N)) - \ln a(b)]^{\frac{\theta}{\theta - 1}} \leq \frac{e}{4} \quad \forall s \in (t_N, i). \tag{2.10}
\]

But
\[
|y(i) - b| \leq \int_{t_N}^{i} \left| \frac{dy(t)}{dt} \right| dt + |y(t_N) - b|.
\]

Hence, (2.9) and (2.10) give us \( |y(i) - b| \leq \frac{e}{2} \) contradicting the fact \( |y(i) - b| = \theta_0 > e \).

Now that \( i = \infty \), (2.10) reads as
\[
\int_{t_N}^{\infty} \left| \frac{dy(t)}{dt} \right| dt \leq \frac{e}{4}, \tag{2.11}
\]

Thus, for all \( T \geq t_N \)
\[
|y(T) - b| \leq \int_{t_N}^{T} \left| \frac{dy(t)}{dt} \right| dt + |y(t_N) - b| \leq 2 \frac{e}{2},
\]

which shows that \( y(t) \to b \) as \( t \to \infty \). In this way, we have proved Theorem 1.1.

3. PROOF OF THEOREM 1.2

Throughout this section, we assume \((A_1), (A_2), (A_3), \) and \((A_4)\), although some conclusions below need only part of these assumptions.

**Lemma 3.1.** Let \( u_t \) be classical solutions to (1.1). Then we have
\[
|u_t(x, t)|^2 \leq \max \{ \|a\|_{C(\Omega)} + 1, K \}, \quad \forall (x, t) \in \Omega \times [0, \infty) \tag{3.1}
\]
and
\[
|\nabla u(x, t)|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \leq \frac{C}{\varepsilon^2}, \quad \forall (x, t) \in \bar{\Omega} \times [\varepsilon^2, \infty).
\]  
(3.2)

**Proof.** Let \( w = |u| \). Dropping \( \varepsilon \), we see that the equation in (1.1) reads
\[
\partial_t w - \Delta w + 2|\nabla w|^2 = \frac{2w}{\varepsilon} (a - w).
\]  
(3.3)

If (3.1) were not true, we could use \((A_1)\) and \((A_3)\) and employ the usual arguments for maximum principle to find a point \((x_*, t_* \in \Omega \times (0, \infty))\) (for each \( \varepsilon \)) at which
\[
w > 1 + a(x_*, t_*), \quad \nabla w = 0, \quad \partial_t w \geq 0, \quad \text{and} \quad \Delta w \leq 0.
\]
Moreover, (3.3) gives us \( \partial_t w \leq -2w/\varepsilon^2 \) at \((x_*, t_*)\). This yields a contradiction.

By a scaling argument, considering the equation for \( U(x, t) = u(x, \varepsilon^2 t) \), and using (1.1) and standard local parabolic estimates, we immediately obtain (3.2).

For classical solutions \( u \) to problem (1.1), define
\[
A(x) = \sqrt{a(x)}, \quad V(x) = \frac{u(x)}{A(x)}, \quad g = \frac{g_1}{A}, \quad V^0 = \frac{u^0}{A}.
\]  
(3.4)

Then \( V \) satisfies
\[
\begin{cases}
\partial_t V - A^{-1} A V = \frac{A^2}{\varepsilon^2} V(1 - |V|^2), & \text{in } \Omega \times (0, \infty) \\
V = g, & \text{on } \partial \Omega \times (0, \infty) \\
V(x, 0) = V^0(x), & \text{in } \Omega.
\end{cases}
\]  
(3.5)

Set
\[
\epsilon(V) = \frac{1}{2} \left[ |\nabla(A V)|^2 + \frac{A^4}{2\varepsilon^2} (1 - |V|^2)^2 \right].
\]  
(3.6)

**Lemma 3.2.** For any \( T > 0 \), there exist two positive constants \( C(T) \) and \( \sigma(T) \) (both depending on \( T \)) such that for all \( \varepsilon > 0 \), all \( \delta \in (0, \sigma(T)) \) and all \( t \in [0, T] \), one has
\[
B_{\delta}(y(t)) \subset \Omega, \quad B_{\delta}(y(t)) \cap B_{\delta}(y_j(t)) = \emptyset \quad \text{for} \quad l \neq j
\]
and
\[ \int_0^T \int_{\Omega(t)} \left| \frac{\partial V}{\partial t} \right|^2 dx \, dt + \sup_{0 \leq t \leq T} \int_{\partial\Omega(t)} e(V) \, dx \leq \delta^{-2} C(T). \]

**Proof.** For each \( T > 0 \), by Theorem 1.1 we can find a \( \sigma = \sigma(T) > 0 \) such that
\[ \sigma \leq (2 \sup_{x \in \Omega} |V\ln a(x)|)^{-1} \]  
(3.7)
and for all \( t \in [0, T] \),
\[ \min_{1 \leq j \leq m} \left\{ \text{dist}(y_j(t), \partial\Omega), \ |y_j(t) - y_j(t)| \text{ for } l \neq j \right\} \geq 4\sigma. \]  
(3.8)
Basing on the method used first by Jerrard and Soner [8], we choose a smooth monotone function \( \phi : [0, \infty) \to [0, \infty) \) such that
\[ \phi(r) = \begin{cases} r^2, & \text{if } r \leq \sigma \\ \sigma^2, & \text{if } r \geq 2\sigma. \end{cases} \]  
(3.9)
Let
\[ \rho(x, t) = \min_{1 \leq j \leq m} |x - y_j(t)|. \]
It follows easily from (3.8) that \( \phi(\rho(x, t)) \) is smooth in \( x \) as well as in \( t \) for all \( (x, t) \in \Omega \times [0, T] \). Dropping \( \epsilon \), applying integration by parts, noting \( \partial_t V = \partial_t g = 0 \) on \( \partial\Omega \times (0, \infty) \), and using (3.5) and (3.6), we obtain
\[
\frac{d}{dt} \int_\Omega \phi(\rho(x, t)) e(V) \, dx \\
= \int_\Omega \frac{d\phi(\rho)}{dt} e(V) + \int_\Omega \phi(\rho) \left[ \nabla(AV) \nabla(AV_i) - \epsilon^{-2} A^4 V (1 - |V|^2) \right] V_i \\
= \int_\Omega \frac{d\phi(\rho)}{dt} e(V) - \int_\Omega A V_i \nabla(AV) \nabla(\phi(\rho)) \\
- \int_\Omega \phi(\rho) \left[ A \partial_t(AV) + \epsilon^{-2} A^4 V (1 - |V|^2) \right] V_i \\
= \int_\Omega \frac{d\phi(\rho)}{dt} e(V) - \int_\Omega A^2 \nabla(V_i)^2 \phi(\rho) - \int_\Omega A V_i V \nabla \phi(\rho) \\
- \int_\Omega A^2 V_i \nabla \nabla \phi(\rho) \\
\]
\[
\begin{align*}
\leq & \int_\Omega \frac{d\phi(\rho)}{dt} e(V) - \frac{1}{2} \int_\Omega \phi(\rho) A^2 |V_t|^2 \\
&+ 2 \int_\Omega |V|^2 |\nabla AV\sqrt{\phi(\rho)}|^2 - \int_\Omega A^2 V \nabla V \nabla \phi(\rho) \\
&\leq C(K, \sigma) - \frac{1}{2} \int_\Omega \phi(\rho) A^2 |V_t|^2 + \int_\Omega \left[ \frac{d\phi(\rho)}{dt} e(V) - A^2 V \nabla V \nabla \phi(\rho) \right] dx,
\end{align*}
\]

(3.10)

where we have used (3.1) and (3.9).

One can use the notation \( \omega_i = \partial_i \alpha / \partial x_i \) and the summation convention to compute

\[
A^2 V \nabla V \nabla \phi(\rho) = A \nabla V \nabla \phi \left[ A(A V) + e^{-2} A^3 V (1 - |V|^2) \right] \\
= A V \phi \left[ e^{-2} A^3 V (1 - |V|^2) + (A V)_V \right] \\
= \phi_i \left[ - (4 e^2)^{-1} A^4 (1 - |V|^2)^2 \right] + (4 e^2)^{-1} (1 - |V|^2)^2 (A^4)_V \\
- 2^{-1} (|V(AV)|^2) + [(A V)_V, (A V)_V] - A_i V (A V)_V.
\]

By virtue of this equality, integration by parts, and the fact that \( \nabla \phi(\rho) = 0 \) on \( \partial \Omega \) (see (3.8) and (3.9)), one gets

\[
\begin{align*}
\int_\Omega A^2 V \nabla V \nabla \phi(\rho) \, dx \\
= \int_\Omega [ A \phi (v) + (4 e^2)^{-1} A^4 (1 - |V|^2)^2 ] \nabla \ln a^2 \nabla \phi \\
- (A V)_V \phi_y A_i V (A V)_V \phi_y + A_i V (A V)_V \phi_y + \phi_i (A V)_V (A V)_V \phi_y \, dx.
\end{align*}
\]

(3.11)

But

\[
\phi_i (A V)_V (A V)_V = \phi_i \left[ (\ln A)_i A V \right] (A V)_V \\
= \frac{1}{2} \nabla \ln a \nabla \phi |A V|^2 + \frac{1}{2} \phi_i \left[ (\ln a)_i (A V)_V \right] A V.
\]

(3.12)

Combing (3.10), (3.11), and (3.12) yields

\[
\begin{align*}
\frac{d}{dt} \int_\Omega \phi(\rho(x, t)) e(V) \, dx \leq C(K, \sigma) - \frac{1}{2} \int_\Omega \phi(\rho) A^2 |V_t|^2 \\
+ \int_\Omega \left[ e(V) \phi_y (\ln a) - \nabla \ln a \nabla \phi \right] \\
+ I_3(V) \, dx + I_3(V),
\end{align*}
\]

(3.13)
where
\[ I_1(V) = (AV) \partial_y\phi_y - A\psi\phi(V) - (4\pi^2)^{-1}A^4(1-|V|^2)^2\nabla \ln a \nabla \phi \]
and
\[ I_2(V) = -\frac{1}{2} \int_D \phi_y (\ln a)_y (AV) \cdot AV - \int_D A_i V (AV) \phi_y \]
\[ \leq C(\sigma, a) \int_D \left( |V|^2 \frac{|
abla \phi|^2}{\phi} + \phi |\nabla (AV)|^2 \right) \]
\[ + \frac{1}{2} \int_D |V|^2 [A_i A_j \phi_y A_y + |A_i A_j \phi_y|] \]
\[ \leq C(\sigma, a) \left[ 1 + \int_D \phi |\nabla (AV)|^2 \right]. \tag{3.14} \]
If \( \rho(x, t) \geq \sigma \), by (3.8) and (3.9) one has
\[ e(V)|\phi(\rho)_y - \nabla \ln a \nabla \phi| + |I_1(V)| \leq C(\sigma, a) \phi(\rho) e(V). \tag{3.15} \]
If \( \rho(x, t) \leq \sigma \), on the other hand, then \( \phi(\rho(x, t)) = |x - y(t)|^2 \) for some \( l \).
Hence
\[ \phi_y = \delta_y \]
and
\[ I_1(V) = -\frac{A^4(1-|V|^2)^2}{2\pi^2} \left[ 1 + (x - y(t)) \cdot \nabla \ln a \right] \leq 0 \tag{3.16} \]
by (3.7). Moreover, using (1.2) we have
\[ \phi(\rho)_y - \nabla \ln a \phi = 2(x - y(t))(\nabla \ln a(y(t)) - \nabla \ln a(x)) \]
\[ \leq 2 |x - y(t)|^2 \||\ln a|_{C(\Omega)} \]
\[ = 2 \||\ln a|_{C(\Omega)} \phi(\rho(x, t)). \tag{3.17} \]
Combining (3.13)–(3.17), we obtain
\[ \frac{d}{dt} \int_{\Omega} \phi(\rho(x, t)) c_s(V_s) \, dx + \frac{1}{2} \int_{\Omega} \phi(\rho(x, t)) A^2 |V_s|^2 \]
\[ \leq C \left[ 1 + \int_{\Omega} \phi(\rho(x, t)) c_s(V_s) \, dx \right]. \]
for all $t \in [0, T]$. Hence, by Gronwall’s inequality and $(A_3)$, we deduce that

$$
\int_\Omega \phi(p(x, t)) e_{\epsilon_t} (V_t) \, dx + \frac{1}{2} \int_0^T \int_\Omega e^{C(t-t')} \phi(p(x, t')) A^2 |\partial_t V_t|^2 \, dx \, ds \\
\leq C(\sigma, T, K).
$$

This result, together with the fact that $\phi(p(x, t)) \geq \delta^2$ for all $t \in [0, T]$, all $x \in \Omega \setminus \bigcup_{j=1}^m B_{\delta_0}(y_j(t))$ and any $\delta \in (0, \sigma(T))$, immediately implies the conclusion of Lemma 3.2.

Now we are in the position to prove Theorem 1.2. Recall the $G_j$ in $(A_4)$ and let

$$
\delta_0 = \min_{1 \leq j \leq m} \mathrm{dist}(G_j, \partial \Omega). \quad (3.18)
$$

Then by Theorem 1.1, we have

$$
\min_{1 \leq j \leq m} \mathrm{dist}(y_j(t), \partial \Omega) \geq \delta_0, \quad \forall t \in [0, \infty). \quad (3.19)
$$

Set

$$
\Omega(t) = \Omega \bigcup_{j=1}^m B_{\delta_0}(y_j(t)). \quad (3.20)
$$

For any $T > 0$ and any $\delta \in (0, \delta_0)$, it follows from Lemma 3.2 and $(A_2)$ that

$$
\int_0^T \int_{\Omega(t)} \left| \frac{\partial V_t}{\partial t} \right|^2 \, dx \, dt + \sup_{0 \leq t \leq T} \int_{\Omega(t)} \left[ |\nabla V_t|^2 + \frac{1}{\epsilon^2} (1 - |V_t|^2)^2 \right] \, dx \leq C(\delta, T) \quad (3.21)
$$

and

$$
\int_0^T \int_{\Omega(t)} \left| \frac{\partial u_t}{\partial t} \right|^2 \, dx \, dt + \sup_{0 \leq t \leq T} \int_{\Omega(t)} \left[ |\nabla u_t|^2 + \frac{1}{\epsilon^2} (a(x) - |u_t|^2)^2 \right] \, dx \leq C(\delta, T) \quad (3.22)
$$

for all $\epsilon > 0$. This shows that the set $\{u_\epsilon : \epsilon > 0\}$ is bounded in $H^1_{\text{loc}}(\Omega(a))$.

Using (3.22) and applying a diagonal method for $\delta \downarrow 0$ and $T \uparrow \infty$, we see that, for any sequence $\epsilon_n \to 0$, there is a subsequence $u_{\epsilon_n}$ (denoted still by itself) such that $u_{\epsilon_n} \rightharpoonup u$ weakly in $H^1_{\text{loc}}(\Omega(a))$. Moreover, (3.22), (3.1), and Lebesgue’s dominated convergence theorem imply that

$$
|u(x, t)| = A(x) \quad \text{a.e. in } \Omega(a). \quad (3.23)
$$
By taking the wedge product of $u_n$ with the equation in (1.1), we have

$$u_n \wedge \frac{\partial u_n}{\partial t} = \text{div}(u_n \wedge \nabla u_n).$$

Passing to the limit, we conclude that

$$u \wedge \frac{\partial u}{\partial t} = \text{div}(u \wedge \nabla u) \quad \text{in} \quad D'(\Omega(a)).$$

But (3.23) yields

$$\frac{\partial u}{\partial t} = 0, \quad \text{div}(u \nabla u) = \frac{Aa}{2} \quad \text{in} \quad D'(\Omega(a)).$$

Combining the last three equations with (3.23), one gets

$$\begin{cases}
u_t - Au = \frac{2|\nabla u|^2 - Aa}{2a} u & \text{in} \quad D'(\Omega(a)) \\
|u| = A(x) & \text{in} \quad \Omega(a) \\
u = g_1 & \text{on} \quad \partial \Omega.
\end{cases}$$

This completes the proof of Theorem 1.2.

4. PROOF OF THEOREM 1.3

As in the last section, we always assume $(A_1), (A_2), (A_3)$, and $(A_4)$ in this section. Let $u_\varepsilon$ be classical solution to (1.1) and define $V_\varepsilon$ as in (3.4). Then all conclusions in the last section hold true.

**Lemma 4.1.** For any $T_0 > 0$, there exist constants $C(T_0) > 0$ and $\sigma_1 = \sigma_1(T_0)$, $\sigma_1 \in (0, \sigma(T_0)/4)$ with the same $\sigma(T_0)$ as in Lemma 3.2 such that for all $t \in [0, T_0]$, all $x_0 \in \Omega \setminus \bigcup_{j=1}^m B_{r_\varepsilon}(y_j(t))$, and all $r \in (0, \sigma_1)$, one has

$$\begin{aligned}
\int_{\partial B_r(x_0)} |x - x_0| \frac{\partial V_\varepsilon}{\partial N} dx + \int_{\Omega(x_0)} \frac{1}{\varepsilon} (1 - |V_\varepsilon|^2)^2 dx \\
\leq C(T_0) \left( \int_{\partial B_r(x_0)} |x - x_0| \left[ 1 + \frac{|\nabla V_\varepsilon|^2}{|T|} + \frac{1}{4\varepsilon^2} (1 - |V_\varepsilon|^2)^2 \right] dx \\
+ \int_{\Omega(x_0)} |x - x_0| \left[ 1 + \frac{1}{2} \left( |\nabla V_\varepsilon|^2 + |\nabla^2 V_\varepsilon| \left[ \frac{1}{|T|} \right] \right) \right] dx \right),
\end{aligned}$$

(4.1)
where \( \Omega_\varepsilon(x_0) = B_\varepsilon(x_0) \cap \Omega \), \( N \) and \( T \) are, respectively, the exterior unit normal vector and tangent vector of \( \partial \Omega \), such that \((N, T)\) is direct.

**Proof.** By Lemma 3.2 and the fact that \( \Omega \) is smooth, we can find positive constants \( \varepsilon = \varepsilon(T_0), \sigma_1 = \sigma_1(T_0) \in (0, \sigma(T_0)/4) \) such that for all \( t \in [0, T_0] \), all \( x_0 = x_\varepsilon(t) \in \Omega\setminus \bigcup_{t=1}^{n} B_{\sigma_1}(y(t)) \) and all \( r \in (0, \sigma_1) \),

\[
(x - x_0) \cdot N \geq \varepsilon|x - x_0|, \quad \forall \in \partial \Omega_s(x_0).
\tag{4.2}
\]

Multiply the equation in (3.5) by \( \nabla V \cdot (x - x_0) \) and integrate it over \( \Omega_\varepsilon = \Omega_s(x_0) \). Neglecting the subscript \( \varepsilon \), we obtain that

\[
\frac{1}{4\varepsilon^2} \int_{\partial \Omega} (1 - |\nabla|^2)^2 \text{div}(A^3 \cdot (x - x_0))
= \frac{1}{4\varepsilon^2} \int_{\partial \Omega} (1 - |\nabla|^2)^2 A^3 N \cdot (x - x_0) + I_3 + I_4 + I_5, \tag{4.3}
\]

where

\[
I_3 = \int_{\partial \Omega} \frac{\partial( - AV)}{\partial N} \cdot (\nabla V \cdot (x - x_0))
,
I_4 = \int_{\Omega} \nabla(AV) \cdot \nabla V \cdot (x - x_0)),
\]
and

\[
I_5 = \int_{\partial \Omega} A \partial_t V \cdot (\nabla V \cdot (x - x_0)).
\]

By virtue of (4.2) and the smallness of \( r \) we may assume

\[
\text{div}(A^3 \cdot (x - x_0)) \geq \lambda > 0 \tag{4.4}
\]

for all \( x \in \Omega_\varepsilon \) and some constant \( \lambda \) depending only on \( A \). On the other hand, the integrand in \( I_3 \) can be written as

\[
\left(- A \frac{\partial V}{\partial N} \frac{\partial A}{\partial N} V \right) \cdot \left( \left( \frac{\partial V}{\partial N} N + \frac{\partial V}{\partial T} \right)(x - x_0) \right)
\leq -\frac{4}{5} A \left| \frac{\partial V}{\partial N} \right|^2 N \cdot (x - x_0) + C \left| \frac{\partial V}{\partial T} \right|^2 + |V|^2 |x - x_0|. \tag{4.5}
\]
The integrand in $I_4$ is nothing but

$$
\text{div}\left(\frac{1}{2} \left| \nabla V (x-x_0) \right|^2 (x-x_0) \cdot \nabla A + \text{div} \{ \nabla \cdot (x-x_0) \cdot V \} \right)
$$

$$
- AA [\nabla V \cdot (x-x_0)] \cdot V - \sum_{i,j,l=1}^2 \frac{\partial A}{\partial x^i} (x^i-x_0^i) \frac{\partial V^j}{\partial x^l} \frac{\partial V^l}{\partial x^j}.
$$

Hence, we have

$$
I_4 \leq \int_{\partial \Omega} \left[ \frac{A}{2} \left| \nabla V \right|^2 N \cdot (x-x_0) + \frac{\partial A}{\partial N} \left( \nabla V \cdot (x-x_0) \cdot V \right) \right] ds
$$

$$
+ C \int_{\Omega} \left[ \left| \nabla V \right|^2 + \left| V \right|^2 \right] |x-x_0| dx
$$

$$
\leq \frac{3}{4} \int_{\partial \Omega} A \left| \frac{\partial V}{\partial N} \right|^2 N \cdot (x-x_0) ds + C \int_{\partial \Omega} \left[ \left| \nabla V \right|^2 + \left| V \right|^2 \right] |x-x_0| ds
$$

$$
+ C \int_{\Omega} \left( \left| \nabla V \right|^2 + \left| V \right|^2 \right) |x-x_0| dx. \quad (4.6)
$$

Combining (4.2)-(4.6) and using (3.1), we have deduced the desired (4.1).

We will prove Theorem 1.3 by the coming two lemmas which are motivated by a method in [14].

**Lemma 4.2.** For any $\delta > 0$ and any measurable set $I \subset (0, \infty)$ with $|I| > 0$, there exist $t \in I$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$,

$$
\int_{\partial \Omega(t)} \left| \frac{\partial V}{\partial N} \right|^2 dx + \int_{\partial \Omega(t)} \left[ \left| \nabla V \right|^2 + \frac{1}{\epsilon} (1 - \left| V \right|^2)^2 \right] dx \leq C(I, \delta) \quad (4.7)
$$

and

$$
\left\{ x \in \Omega : |V(x, t)| < \frac{1}{2} \right\} \subseteq \bigcup_{j=1}^m B_{\delta_1}(y_j(t)). \quad (4.8)
$$

**Proof.** Inequality (4.7) follows easily from (3.21) and (3.22).

Obviously, it is enough to prove (4.8) for sufficiently small $\delta > 0$ and the same $t$ as in (4.7). If the conclusion were not true, we could find $\delta_1 \in (0, \sigma(T_0))$ (with the same $\sigma(T_0)$ as in Lemma 3.2), a sequence $\epsilon_k \searrow 0$, $\epsilon_k \in (0, \delta_1)$, and $\{ x_k \} \subset \Omega \setminus \bigcup_{j=1}^m B_{\delta_1}(y_j(t))$ such that $|V(x_k, t)| < \frac{1}{2}$ for all $k$. Hence, by virtue of (3.2) and the fact $|V| = |g| = 1$ on $\partial \Omega$, we see that there is $B_{C \epsilon_k}(x_k) \subset \Omega \setminus \bigcup_{j=1}^m B_{\delta_1}(y_j(t))$ for some constant $C > 0$ with
\[ |V_{n}(x, t)| \leq \frac{k}{4} \text{ for all } x \in B_{C_{0}}(x_{k}) \text{ and all sufficiently large } k \geq k_{0}. \]

Let \( r_{k} = R_{k} \), \( B_{k} = B_{r_{k}}(x_{k}) \) and \( V_{k} = V_{r_{k}} \). Then we have
\[
\varepsilon_{k}^{-2} \int_{B_{k}} (1 - |V_{k}(x, t)|^{2})^{2} \, dx \geq C_{2} > 0 \tag{4.9}
\]
for all \( k \geq k_{0} \) and some positive constant \( C_{2} \) depending only on \( C_{1} \).

On the other hand, since \( r_{k} \to 0 \), (4.7) and (3.21) imply that for all \( k \geq k_{0} \),
\[
\begin{align*}
\frac{1}{2} \int_{\Omega} \frac{\partial V_{k}}{\partial t} \, dx + \int_{\Omega} |\nabla V_{k}|^{2} \, dx &\leq C(I, \delta) \tag{4.10}.
\end{align*}
\]

Thus, letting
\[
f_{k}(r) = \int_{r \leq \sqrt{r_{k}}} \left[ |\nabla V_{k}|^{2} + \varepsilon_{k}^{-2}(1 - |V_{k}|^{2})^{2} \right] \, ds
\]
we have
\[
C(T, \delta) \geq \int_{\Omega} \left[ |\nabla V_{k}|^{2} + \varepsilon_{k}^{-2}(1 - |V_{k}|^{2})^{2} \right] \, dx
= \int_{r_{k}}^{\sqrt{r_{k}}} \frac{f_{k}(r)}{dr} \, dr
\geq \frac{1}{2} |\ln r_{k}| \min_{r \leq r_{k} < \sqrt{r_{k}}} \{ f_{k}(r) \}
\]
for all \( k \geq k_{0} \). Therefore, for each \( k \geq k_{0} \), we can find a \( \lambda_{k} \in (r_{k}, \sqrt{r_{k}}) \) such that
\[
\lambda_{k} f_{k}(\lambda_{k}) \leq 2 |\ln r_{k}|^{-1} C(I, \delta). \tag{4.11}
\]

Using Lemma 4.1 for \( x_{0} = x_{k} \) and \( r = \lambda_{k} \), we obtain
\[
\begin{align*}
\varepsilon_{k}^{-2} \int_{B_{k}} (1 - |V_{k}|^{2})^{2} &\leq \varepsilon_{k}^{-2} \int_{\Omega} (1 - |V_{k}|^{2})^{2} \\
&\leq C(T) \lambda_{k} \int_{\Omega} (1 + |\nabla V_{k}|^{2} + |\nabla V_{k}| |\nabla V_{k}|) \, dx \\
&\quad + \int_{\Omega} \frac{|\nabla^{2} V_{k}|^{2}}{\partial t} \, ds + \int_{B_{R_{k}}} \left( \frac{|\nabla V_{k}|^{2}}{\partial t} + \frac{(1 - |V_{k}|^{2})^{2}}{4\varepsilon_{k}^{2}} \right) \, ds \\
&\leq C(\delta, T, I) \left[ \lambda_{k} + 2 |\ln \lambda_{k}|^{-1} \right] \tag{by(4.10) and (4.11)}.
\end{align*}
\]
This contradicts (4.9) because of the fact $\lambda_k \to 0$. In this way, we have proved Lemma 4.2.

**Lemma 4.3.** With the same $\delta, I, t \in I$ and $e_0$ as in Lemma 4.2, we have that for all $x \in \Omega \setminus \bigcup_{j=1}^{m} B_d(y_j(t))$,\n\[
A(x) - C(\delta, I) e \leq |u_p(x, t)| \leq A(x) + C(\delta, I) e, \quad \forall e \in (0, e_0). \tag{4.12}
\]

**Proof.** Fix $x_0 \in \partial \Omega$ and let $\sigma_2 = \frac{1}{4} \min \{1, \|\sigma(\sup I)\|^2\}$. Using (4.7) and the arguments from (4.10) to (4.11) one can easily see that\n\[
\int_{B_2 R_0 + r_0} |\nabla u_p|^2 ds + \int_{\partial \Omega \times (t - C_\varepsilon, t + C_\varepsilon)} \left( |\nabla u_p|^2 + \frac{\partial (u_p)^{1/2}}{\partial t} + \frac{a(x) - |u_p|^2}{\varepsilon^2} \right) dx \leq C(\delta, I) \tag{4.13}
\]
for some $\lambda_2 \in [\sigma_2, \sqrt{\sigma_2}]$ and all $e \in (0, e_0)$. Moreover, (4.13), (4.7), and Lemma 4.1 imply that \[\int_{\Omega_0} |\partial V / \partial N|^2 \leq C\] independent of $e$. Therefore, we deduce that\n\[
\int_{\partial \Omega} |\nabla u_p|^2 ds + \int_{\partial \Omega \times (t - C_\varepsilon, t + C_\varepsilon)} \left( |\nabla u_p|^2 + \frac{\partial u_p}{\partial t} + \frac{(a(x) - |u_p|^2)}{\varepsilon^2} \right) dx \leq C(\delta, I) \tag{4.14}
\]
and\n\[
\alpha_1 \geq |u_p(x, t)|^2 \geq \frac{1}{2} a(x) \geq \alpha_2 > 0, \quad \forall x \in \Omega_0(t) \text{ (by (3.1) and (4.8))} \tag{4.15}
\]
for all $e \in (0, e_0)$ and some $\varepsilon_0 = \varepsilon_0(\delta) \in (0, 1)$. Let $R_0$ be a positive constant to be determined later. As we will see, it depends only on $\alpha_1, \alpha_2$ and $C(\delta, I)$ in (4.14). Fix a constant $r_0 \in (0, \min \{\frac{2}{3}, R_0\})$ which will be suitably small at last. For an arbitrary $y \in \Omega_0(t)$, choose a number $R \in (0, \min \{\frac{2}{3}, \frac{R_0 - r_0}{2}\})$ satisfying $B_{2R + r_0}(y) \subset \Omega_0(t)$. As the arguments in [6, pp. 345–346], we write\n\[
u_p(x, t) = p_p(x, t) e^{i\phi_p(x, t)}
\]
on $B_{2R + r_0}(y) \times (t - C_\varepsilon, t + C_\varepsilon)$ for some $C_\varepsilon > 0$ (see (3.2)) so that the equation in (1.1) turns to be\n\[
div(p_p^2 \nabla \psi_p) = p_p^2 \partial_t \psi_p 
\text{ in } B_{2R + r_0}(y) \tag{4.16}
\]
and
\[
\Delta \rho_\varepsilon + \frac{(a(x) - p_\varepsilon^2)}{\varepsilon^2} \rho_\varepsilon = |\nabla \psi_\varepsilon|^2 \rho_\varepsilon + \bar{\rho}_\varepsilon \quad \text{in} \quad B_{2R + r_0}(y). \tag{4.17}
\]

Moreover, using (4.14) and Fubini’s theorem (see the arguments from (4.10) to the (4.11)), we obtain \(R \in (R, R + r_0/2)\) such that
\[
\int_{\partial B_R(y)} |\nabla u_\varepsilon|^2 + \frac{(a(x) - |u_\varepsilon|^2)^2}{\varepsilon^2} \leq 2r_0^{-1} C(\delta, I). \tag{4.18}
\]

It easily follows from (4.18) and Lemma 3.1 that
\[
\max_{x \in \partial B_R(y)} |a(x) - |u_\varepsilon(x, t)||^2 \leq C(r_0^{-1}, \delta, \|a\|_{C^1(I)}) \varepsilon
\]

which, together with (3.1), implies that
\[
|A(x) - |u_\varepsilon(x, t)|| \leq C_4 \varepsilon \tag{4.19}
\]

for all \(x \in \partial B_R(y)\), all \(\varepsilon \in (0, \varepsilon_0)\) and some constant \(C_4\) depending only on \(r_0^{-1}, \delta, I\), and \(||a||_{C^1(I)}\).

On one hand, applying Theorem 2.2 of Chapter V in [15] to Eq. (4.16) for \(\psi_\varepsilon\) with the coefficient \(\rho_\varepsilon\) satisfying (4.15) and using the notation \(\frac{1}{p} = \frac{1}{p} E\), we obtain that
\[
\left( \int_{\partial B_R(y)} |\nabla \psi_\varepsilon|^p \, dx \right)^{1/p} \leq C_5 \left( \left( \int_{\partial B_R(y)} |\nabla \psi_\varepsilon|^2 \, dx \right)^{1/2} + R \left( \int_{\partial B_R(y)} |\nabla \psi_\varepsilon|^{2p/3} \, dx \right)^{3/2p} \right)
\]

for some \(p \in (2, 3]\) depending only on \(\varepsilon_1, \varepsilon_2\) in (4.15), some \(R_0 > 0\) depending only on \(\varepsilon_1, \varepsilon_2\) and \(C\) in (4.14), some \(C_5 > 0\) depending only on \(p, \varepsilon_1\) and \(\varepsilon_2\), all \(R < (R_0 - r_0)/2\), and all \(\varepsilon \in (0, \varepsilon_0)\).

On the other hand, Eq. (4.17) implies that the function
\[
\bar{U}_\varepsilon(x, t) \equiv A(x) - \rho_\varepsilon(x, t)
\]
satisfies
\[
-A \bar{U}_\varepsilon + \frac{C_3(x)}{\varepsilon} \bar{U}_\varepsilon = f_\varepsilon
\]
in $B_R(y)$ with $0 < C(x_1) \leq C(x) \equiv (A + p_r) \rho_r \leq C(x_2)$ and
\[
f_r \equiv |\nabla \psi_r|^2 \rho_r - AA + \partial_i \rho_r \in L^{p^*}(B_R(y)).
\]
By (4.20) and (4.14), we see that
\[
f_r = \mathcal{L}_p(B_B(y)) \subseteq C(\lambda_1, \lambda_2, \delta), \quad \forall \varepsilon \in (0, \varepsilon_0).
\]
Moreover, (4.19) yields $-C_4 \varepsilon \leq u_{r, \varepsilon} \leq C_4 \varepsilon$ on $\partial B_r(y)$. Therefore, a
standard elliptic estimate [16, Theorem 8.16] gives us
\[
|\bar{U}_{r, \varepsilon}| \leq C(C_4, \varepsilon_1, \varepsilon_2) \varepsilon \quad \text{in} \quad B_R(y) \times \{t\}.
\]
Particularly, we have
\[
A(x) - C\varepsilon \leq \rho_s(x) = |u_{r, \varepsilon}(x, t)| \leq A(x) + C\varepsilon, \quad \forall x \in B_R(y), \quad \forall \varepsilon \in (0, \varepsilon_0).
\]
(4.21)
Now for any $G_0 \subset \subset \Omega$, choose $r_0 = \frac{1}{4} \min \{R_0, \delta, \dist(G_0, \Omega)\}$ and fix
\[
R = \min \left\{ \frac{\delta}{8}, \frac{R_0 - r_0}{2}, \frac{1}{4} \dist(G_0, \partial \Omega) \right\}.
\]
Then, by the arbitrariness of $y \in G_0 \setminus \bigcup_{j=1}^m B_d(y_j(t))$, we can find finite balls,
$B_d(y_j), i = 1, 2, \ldots, N$, such that $\bigcup_{j=1}^m B_d(y_j) \supset G_0 \setminus \bigcup_{j=1}^m B_d(y_j(t))$ and
(4.21) holds true for all $x \in B_d(y_j)$ and all $i = 1, 2, \ldots, N$. In this way, we conclude
\[
A(x) - C\varepsilon \leq |u_{r, \varepsilon}(x, t)| \leq A(x) + C\varepsilon, \quad \forall x \in G_0 \setminus \bigcup_{j=1}^m B_d(y_j(t)), \forall \varepsilon \in (0, \varepsilon_0)
\]
(4.22)
for some constant $C$ and $\varepsilon_0$ both independent of $\varepsilon$. Moreover, using the fact
\[
\int_{\partial \Omega} |\nabla u|^2 \leq C(\delta) \quad \text{(see (4.14))}
\]
and repeating the argument above for $\Omega_R = B_d(y) \cap \Omega$ with $y \in \partial \Omega$, we can find a domain $G' \subset \subset \Omega$ such that
(4.22) holds true for all $x \in \Omega \setminus G'$. Combining this result and (4.22), we have proved Lemma 4.3.

Finally, combining Lemmas 4.2 and 4.3, we have completed the proof of Theorem 1.3.
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