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k-β and k-Nearly Uniformly Convex Banach Spaces

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Different uniform geometrical properties have been defined between the uniform convexity and the reflexivity of Banach spaces. In the present paper we introduce other properties of this type, namely k-β and k-nearly uniform convexity. The definitions, as well as some of the results presented here, are announced in [7].

Sullivan [24] has defined k-uniformly rotund Banach spaces which by a suggestion of Davis are all superreflexive. Fan and Glicksberg [1] have introduced fully k-convex Banach spaces.

Let $x_1, ..., x_{k+1}$ be vectors in a Banach space X. The k-dimensional volume enclosed by $x_1, ..., x_{k+1}$ is given by

$$V(x_1, x_2, ..., x_{k+1}) = \sup \left\{ \begin{vmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{vmatrix} : f_i \in X^*, \|f_i\| \leqslant 1, i = 1, 2, ..., k \right\}.$$

A Banach space X is said to be k-uniformly rotund (k-UR), $k \geqslant 1$, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $x_i \in X$, $\|x_i\| \leqslant 1$, $i = 1, 2, ..., k+1$, with $\|\sum_{i=1}^{k+1} x_i\|/(k+1) \geqslant 1 - \delta$, then $V(x_1, ..., x_{k+1}) < \varepsilon$. Clearly, 1-UR coincides with uniform convexity. For equivalent definition of k-UR see [11].

A Banach space X is said to be fully k-convex (kR), $k \geqslant 2$, if for every sequence $\{x_i\}$ in X such that $\lim_{n_1, ..., n_k \to \infty} \|\sum_{i=1}^{k} x_{n_i}\|/k = 1$, then $\{x_i\}$ is a Cauchy sequence in X.

Another uniform property is the nearly uniform convexity, introduced by Huff [3]. He has proved that the class of nearly uniformly convexifiable spaces is strictly between superreflexive and reflexive Banach spaces.

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A Banach space X is called nearly uniformly convex (NUC) if for each $\varepsilon > 0$ there is a $\delta, 0 < \delta < 1$ such that for any sequence $\{x_n\}$ in the closed unit ball B with

$$\operatorname{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon,$$

then $\operatorname{conv}(\{x_n\}) \cap (1-\delta)B \neq \emptyset$.

For equivalent definitions of NUC see [19].

The Kuratowski measure of non-compactness $\alpha(A)$ of a set A in X is the infimum of those $\varepsilon > 0$ for which there is a covering of A by a finite number of sets A_i such that $\operatorname{diam}(A_i) < \varepsilon$.

Let X be a Banach space with closed unit ball B. By the drop $D(x, B)$ defined by an element $x \in X \setminus B$, we mean $\operatorname{conv}(\{x\} \cup B)$ and let $R(x, B) = D(x, B) \setminus B$. Rolewicz [18] has proved that X is uniformly convex if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $1 < \|x\| < 1 + \delta$ implies $\operatorname{diam}(R(x, B)) < \varepsilon$. In connection with this he has also introduced [19] the property (β).

A Banach space X is called to have the property (β) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $1 < \|x\| < 1 + \delta$ implies $\alpha(R(x, B)) < \varepsilon$.

Rolewicz [19] has shown that $\mathrm{UC} \Rightarrow (\beta) \Rightarrow \mathrm{NUC}$. The class of Banach spaces with an equivalent norm with property (β) coincides neither with that of superreflexive spaces (independently done in [4] and [14]) nor with the class of nearly uniformly convexifiable spaces (see [5, 6]).

Yu Xin-Tai [25] has proved that k-UR implies NUC. In connection with this result we may ask if k-UR implies (β). The answer is negative. For this purpose we may present first the following.

PROPOSITION 1. *Let X be the l_1-direct sum of Banach spaces Y and Z, where Y is uniformly convex and Z is k-UR. Then X is $(k+1)$-UR.*

This is an addition to the results in [2, 13]. The author was informed by Prof. Bor-Luh Lin that the same result had been also obtained by Yu Xin-Tai.

EXAMPLE 2. There is a 2-UR Banach space which does not possess the property (β).

Proof. Let $Y = \mathbb{R}^1$ and $Z = l_2$. Denote by X the l_1-direct sum of Y and Z. According to Proposition 1, X is 2-UR. On the other hand, it is easy to observe that X fails to have the property (β) (cf. [19, 14]).

If in the above mentioned characterization of uniform convexity by drops [18] we replace $\operatorname{diam}(R(x, B)) < \varepsilon$ by $\sup\{\operatorname{diam}(C) : C \subset R(x, B), C \text{ convex}\} < \varepsilon$, obviously we have again UC. In contrast, if we do a similar substitution in the definition of (β), we obtain:

THEOREM 3. *A Banach space X is NUC if and only if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $1 < \|x\| < 1 + \delta$ implies*

$$\sup\{\alpha(C) : C \subset R(x, B),\ C \text{ convex}\} < \varepsilon,$$

where B is the closed unit ball of X.

Proof. Necessity. Assume the contrary; i.e., there exists an $\varepsilon > 0$, elements $x_n \in X$ with $1 < \|x_n\| < 1 + 1/n$ and convex sets C_n, $C_n \subset R(x_n B)$, so that $\alpha(C_n) \geq \varepsilon$, $n = 1, 2, \ldots$. For every integer n we have that the convex set C_n is disjoint with B and thus, there is a functional $f_n \in X^*$ with $\|f_n\| = 1$ which separates C_n and B, i.e.,

$$f(x) \geq 1 \qquad \text{for every} \quad x \in C_n.$$

Put $E_n = C_n/(1 + 1/n)$. Clearly, E_n is convex and $\alpha(E_n) \geq \varepsilon/2$ for every integer n. By the choice of x_n and C_n we obtain easily that $f(x) > 1 - 1/n$ for every $x \in E_n$, i.e.,

$$E_n \cap (1 - 1/n) B = \emptyset, \qquad n = 1, 2, \ldots.$$

By [19], this means that the norm is not NUC which is a contradiction.

Sufficiency. Assume that the norm is not NUC. Then, following [19], we may find an $\varepsilon > 0$ and a sequence $\{f_n\} \subset X^*$ with $\|f_n\| = 1$ so that

$$\alpha(S(f_n, 1/n)) \geq \varepsilon, \qquad n = 1, 2, \ldots,$$

where $S(f, \delta) = \{x \in B : f(x) \geq 1 - \delta\}$ for $f \in X^*$ and $0 < \delta < 1$. Take x_n with $1 + 2/n \leq \|x_n\| \leq 1 + 3/n$ and $f_n(x_n) = 1 + 2/n$, $n = 1, 2, \ldots$. Then, if we denote by C_n the set $x_{n/2} + S(f_n, 1/n)/2$, we have that C_n is convex and $C_n \subset R(x_n, B)$. Moreover, $\alpha(C_n) \geq \varepsilon/2$ for every n, which is a contradiction.

We are now in a position to introduce the notions of k-β and k-nearly uniform convexity.

DEFINITION 4. Let $k \geq 1$ be an integer. A Banach space X with closed unit ball B is called to be k-β, provided for each $\varepsilon > 0$ there exists a $\delta > 0$ so that $1 < \|x\| < 1 + \delta$ implies

$$\sup\{\alpha(C)\} < \varepsilon,$$

where the supremum is taken over all subsets C of $R(x, B)$ such that for every choice of elements $\{x_i\}_{i=1}^k \subset C$ and scalars $\gamma_i \geq 0$, $i = 1, \ldots, k$ with $\sum_{i=1}^k \gamma_i = 1$; then $\sum_{i=1}^k \gamma_i x_i \in R(x, B)$.

Clearly, 1-β coincides with the property (β).

DEFINITION 5. Let $k \geqslant 2$ be an integer. A Banach X is called to be k-nearly uniformly convex (k-NUC), provided for each $\varepsilon > 0$ there exists a δ, $0 < \delta < 1$, such that for every sequence $\{x_n\} \subset X$, with $\|x_n\| \leqslant 1$ and $\operatorname{sep}(x_n) > \varepsilon$, there are indices $\{n_i\}_{i=1}^{k}$ and scalars $\gamma_i \geqslant 0$, $i = 1, ..., k$, with $\sum_{i=1}^{k} \gamma_i = 1$ so that

$$\left\| \sum_{i=1}^{k} \gamma_i x_{n_i} \right\| \leqslant 1 - \delta.$$

Evidently, k-NUC implies NUC.

THEOREM 6. *Let X be a Banach space and $k \geqslant 2$ be an integer. Then the following are equivalent*:

(i) *X is k-NUC;*

(ii) *for each $\varepsilon > 0$ there is a δ, $0 < \delta < 1$ such that for every set E contained is the closed unit ball B with $\alpha(E) > \varepsilon$, there exist elements $\{x_i\}_{i=1}^{k} \subset E$ so that* $\operatorname{conv}(\{x_i\}_{i=1}^{k}) \cap (1-\delta) B \neq \emptyset$;

(iii) *for each $\varepsilon > 0$ there is a δ, $0 < \delta < 1$, such that for every sequence $\{x_n\} \subset B$ with* $\operatorname{sep}(x_n) > \varepsilon$, *there are indices $\{n_i\}_{i=1}^{k}$ so that*

$$\left\| \sum_{i=1}^{k} x_{n_i} \right\| \Big/ k \leqslant 1 - \delta.$$

Proof. The equivalence between (i) and (ii) follows immediately from the properties of the Kuratowski measure of non-compactness. It remains to show that (i) $\Rightarrow$ (iii). Let $\varepsilon > 0$ and choose $\delta > 0$ according to Definition 5. Put $\eta = \delta/k$. Let $\{x_n\} \subset B$ be an arbitrary sequence with $\operatorname{sep}(x_n) > \varepsilon$. Then there exists a point $y = \sum_{i=1}^{k} \gamma_i x_{n_i}$ with $\gamma_i \geqslant 0$, $\sum_{i=1}^{k} \gamma_i = 1$ so that $\|y\| \leqslant 1 - \delta$. Without loss of generality we may assume that

$$\gamma_1 = \max\{\gamma_i : 1 \leqslant i \leqslant k\}.$$

We have the obvious representation

$$\frac{1}{k} \sum_{i=1}^{k} x_{n_i} = \frac{1}{k} \left(\frac{1}{\gamma_1} y + \sum_{i=2}^{k} \left(1 - \frac{\gamma_i}{\gamma_1}\right) x_{n_i} \right).$$

By the choice of γ_1, $1 - \gamma_i/\gamma_1 \geqslant 0$, $i = 2, ..., k$, and thus

$$\frac{1}{k} \left\| \sum_{i=1}^{k} x_{n_i} \right\| \leqslant \frac{1}{k} \left(k - \frac{\delta}{\gamma_1} \right) \leqslant 1 - \eta,$$

which concludes the proof.

THEOREM 7. *Let X be a Banach space with closed unit ball B and* $k \geqslant 1$ *be an integer. Then the following are equivalent*:

(i) *X is* k-β;

(ii) *for each* $\varepsilon > 0$ *there is a* $\delta > 0$ *so that* $1 < \|x\| < 1 + \delta$ *implies*

$$\sup\{\operatorname{sep}(x_n)\} < \varepsilon,$$

where the supremum is taken over all sequences $\{x_n\} \subset R(x, B)$ *so that the convex hull of every k elements* $\{x_{n_i}\}_{i=1}^k$ *also belongs to* $R(x, B)$;

(iii) *for each* $\varepsilon > 0$ *there exists a* δ, $0 < \delta < 1$, *so that for every element* $x \in B$ *and every sequence* $\{x_n\} \subset B$ *with* $\operatorname{sep}(x_n) > \varepsilon$, *there are indices* $\{n_i\}_{i=1}^k$ *so that* $\operatorname{conv}(\{x\} \cup \{x_{n_i}\}_{i=1}^k) \cap (1-\delta) B \neq \emptyset$;

(iv) *for each* $\varepsilon > 0$ *there exists a* δ, $0 < \delta < 1$, *so that for every element* $x \in B$ *and every sequence* $\{x_n\} \subset B$ *with* $\operatorname{sep}(x_n) > \varepsilon$, *there are indices* $\{n_i\}_{i=1}^k$ *so that*

$$\left\| x + \sum_{i=1}^{k} x_{n_i} \right\| \Big/ (k+1) \leqslant 1 - \delta.$$

Proof. The equivalence between (i) and (ii) follows from the properties of the measure of non-compactness.

(ii) $\Rightarrow$ (iii). Let $\varepsilon > 0$. According to (ii), fix for $\varepsilon/2$ a corresponding $0 < \delta < 1$. Put $\eta = \delta/4$. Take an element x and a sequence $\{x_n\}$ in B.

Assume that for every choice of indices $\{n_i\}_{i=1}^k$, we have

$$\operatorname{conv}\left(\{x\} \cup \{x_{n_i}\}_{i=1}^k\right) \cap (1-\eta) B = \emptyset. \tag{1}$$

Put $y = (1+\delta)x$ and let $y_n = (y + x_n)/2$. Clearly, $y_n \in D(y, B)$. Moreover,

$$y_n = \left(1 + \frac{\delta}{2}\right)\left(\frac{1+\delta}{2+\delta} x + \frac{1}{2+\delta} x_n\right)$$

and hence for arbitrary choice $\{n_i\}_{i=1}^k$ and convex combination with coefficients $\gamma_i \geqslant 0$, $i = 1, ..., k$, $\sum_{i=1}^k \gamma_i = 1$, we get

$$\sum_{i=1}^{k} \gamma_i y_{n_i} = \left(1 + \frac{\delta}{2}\right)\left(\frac{1+\delta}{2+\delta} x + \frac{1}{2+\delta} \sum_{i=1}^{k} \gamma_i x_{n_i}\right).$$

Thus, by (1)

$$\left\| \sum_{i=1}^{k} \gamma_i y_{n_i} \right\| > \left(1 + \frac{\delta}{2}\right)(1 - \eta) > 1.$$

Therefore, $\text{conv}(\{y_{n_i}\}_{i=1}^k) \subset R(y, B)$ and $1 < \|y\| < 1+\delta$. Then we get from (ii) that $\text{sep}(y_n) < \varepsilon/2$, whence $\text{sep}(x_n) < \varepsilon$.

(iii) $\Rightarrow$ (ii). Let $\varepsilon > 0$. According to (iii), choose for $\varepsilon/2$ a corresponding $0 < \delta < \frac{1}{2}$. Take an $x \in X$ with $1 < \|x\| < 1+\delta$ and a sequence $\{x_n\} \subset R(x, B)$ as in (ii). By convexity we also get for every choice $\{n_i\}_{i=1}^k$ and scalars $\gamma_i \geqslant 0$, $i = 0, 1, ..., k$, $\sum_{i=0}^k \gamma_i = 1$; then

$$\gamma_0 x + \sum_{i=1}^k \gamma_i x_{n_i} \in R(x, B).$$

Denote $y = (1-\delta)x$ and $y_n = (1-\delta)x_n$. Evidently, $y, y_n \in B$. Moreover,

$$\left\| \gamma_0 y + \sum_{i=1}^k \gamma_i y_{n_i} \right\| > 1 - \delta.$$

Therefore, $\text{sep}(y_n) < \varepsilon/2$ and thus, $\text{sep}(x_n) < \varepsilon$.

(iv) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv). Let $\varepsilon > 0$. Find a δ according to (iii). Put $\eta = \delta/(k+1)$. Take an $x \in B$ and $\{x_n\} \subset B$ with $\text{sep}(x_n) > \varepsilon$. Then there are indices $\{n_i\}_{i=1}^k$ and scalars $\gamma_i \geqslant 0$, $i = 0, 1, ..., k$, $\sum_{i=0}^k \gamma_i = 1$, so that

$$\left\| \gamma_0 x + \sum_{i=1}^k \gamma_i x_{n_i} \right\| \leqslant 1 - \delta.$$

As in the proof of Proposition 6, we get

$$\left\| x + \sum_{i=1}^k x_{n_i} \right\| \Big/ (k+1) \leqslant 1 - \eta.$$

THEOREM 8. *Let X be a Banach space. Then,*

(i) *k-$\beta \Rightarrow (k+1)$-NUC for every $k \geqslant 1$;*

(ii) *k-NUC $\Rightarrow k$-β for every $k \geqslant 2$.*

Proof. (i) Let $\varepsilon > 0$. Select a corresponding $\delta > 0$ according to Theorem 7(iv). Take $\{x_n\} \subset B$ with $\text{sep}(x_n) > \varepsilon$. Then for $\{x_n\}$ and $x = x_1$ there are indices $\{n_i\}_{i=2}^{k+1}$ so that the norm of the arithmetic mean of x_1 and $\{x_{n_i}\}_{i=2}^{k+1}$ is less than or equal to $1-\delta$; i.e., denoting $n_1 = 1$ we get

$$\left\| \sum_{i=1}^{k+1} x_{n_i} \right\| \Big/ (k+1) \leqslant 1 - \delta.$$

(ii) Let $\varepsilon > 0$. Choose $\delta > 0$ according to Theorem 6(iii). Then the condition (iv) of Theorem 7 is fulfilled for $\delta_1 = k\delta/(k+1)$.

We shall prove in the sequel that the converse implications are not true; i.e., all the notions of k-β and k-NUC are isometrically different. For this purpose we investigate first their relation to the other uniform geometrical properties.

A Banach space X is said to be uniformly Kadec–Klee (UKK) if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every sequence $\{x_n\}$, $\|x_n\| \leqslant 1$ which converges weakly to x and $\text{sep}(x_n) > \varepsilon$, we have $\|x\| \leqslant 1 - \delta$. X is NUC if and only if it is UKK and reflexive [3]. X is said to be weakly uniformly Kadec–Klee (wUKK or UKKε) if there exists an $0 < \varepsilon < 1$ so that there is a $\delta > 0$ for which $\|x\| \leqslant 1 - \delta$ if $x_n \to x$ weakly, $\|x_n\| \leqslant 1$, sep $(x_n) > \varepsilon$. X is NUCε for some $0 < \varepsilon < 1$; i.e., there is a $0 < \delta < 1$ such that for every sequence $\{x_n\} \subset B$ with $\text{sep}(x_n) > \varepsilon$ we have $\text{conv}(\{x_n\}) \cap (1 - \delta)B$, if and only if X is wUKK and reflexive (cf., e.g., [10]).

Similarly, we may define k-NUCε for $0 < \varepsilon < 1$, $k \geqslant 2$. Clearly, k-NUC$\varepsilon \Rightarrow$ NUCε.

A Banach space X has the Banach–Saks property (BS) whenever every bounded sequence in X has a subsequence whose arithmetic means converge in norm.

In [6] it is proved that every space with property (β) (i.e., 1-β) has (BS). Partington [16] has proved that Baernstein's example of a reflexive space without (BS) is NUC. Since (BS) is invariant under isomorphisms, (β) and NUC are isomorphically different [6].

THEOREM 9. *Let $k \geqslant 2$ be an integer. If X is k-NUCε for some $0 < \varepsilon < 1$ then X possesses the Banach–Saks property.*

Since X is reflexive, it suffices to show that X does not have a spreading model isomorphic to l_1. The last fact can be easily proved as in [6]. We prefer to give here an alternative argument based on a result of Partington [15] because it clarifies also the relation of k-NUC to the properties A_k defined in [15].

Let $k \geqslant 2$. A Banach space X has property A_k if it is reflexive and there exists a number η, $0 < \eta < 1$, such that, whenever $x_n \to 0$ weakly, $\|x_n\| \leqslant 1$, then there exist $n_1, ..., n_k$ with $\|\sum_{i=1} x_{n_i}\|/k \leqslant 1 - \eta$. Partington has proved that $A_k \Rightarrow$ (BS) and every superreflexive space has A_k for some $k \geqslant 2$.

PROPOSITION 10. *If X is k-NUCε for some $0 < \varepsilon < 1$ then X has the property A_k.*

Proof. Let $\delta > 0$ be chosen according to the definition of k-NUCε. Fix ν, $\varepsilon < \nu < 1$. Put $\eta = \min\{1 - \nu, \delta\}$. Take now an arbitrary sequence $\{x_n\}$, $\|x_n\| \leqslant 1$ with $x_n \to 0$ weakly. The interesting case is when $\|x_n\| > \nu$ for infinitely many n. For brevity let $\|x_n\| > \nu$ for all n. Put $n_1 = 1$. Having chosen $x_{n_1}, ..., x_{n_m}$ with $\|x_{n_i} - x_{n_j}\| > \nu$ whenever $i \neq j$, $1 \leqslant i, j \leqslant m$, select supporting functionals f_i at x_{n_i}, $i = 1, ..., m$, i.e., $\|f_i\| = 1$, $f_i(x_{n_i}) = \|x_{n_i}\|$. Since $x_n \to 0$ weakly, find $x_{n_{m+1}}$ so that $|f_i(x_{n_{m+1}})| < \|x_i\| - \nu$ for every $i = 1, ..., m$. Thus for every $1 \leqslant i \leqslant m$,

$$\|x_{n_i} - x_{n_{m+1}}\| \geqslant f_i(x_{n_i}) - |f_i(x_{n_{m+1}})| > \nu.$$

Hence, for the above constructed subsequence $\{x_{n_m}\}_{m=1}^{\infty}$ we get $\operatorname{sep}(x_{n_m}) > \varepsilon$. Therefore, there are idices $j_i = n_{m_i}$, $i = 1, ..., k$ so that

$$\left\| \sum_{i=1}^{k} x_{j_i} \right\| \Big/ k \leqslant 1 - \delta,$$

which completes the proof.

The converse implication of Proposition 10 is not fulfilled.

EXAMPLE 11. There exists a Banach space which has the property A_n for some $n \geqslant 2$ and fails to be k-NUCε for any integer $k \geqslant 2$ and $0 < \varepsilon < 1$.

Proof. Let X be the space l_2 supplied with the emquivalent norm $\|\cdot\|_a$ considered in [26]. It has been shown there that $(l_2, \|\cdot\|_a)$ is non-wUKK and hence X fails to be even NUCε for any $0 < \varepsilon < 1$. On the other hand, X is superreflexive and by the result of Partington, it has A_n for some $n \geqslant 2$.

Actually, the properties k-NUC are isomorphically stronger than A_k, as we can see in the next example.

EXAMPLE 12. There exists a Banach space which has the property A_2 but fails to have an equivalent NUC norm.

Proof. Let X be the l_2-direct sum of the spaces $X_n = l_n$, $n \geqslant 2$. It is not difficult to show that X has the property A_2. On the other hand, X fails to have an equivalent NUC norm (cf. [3, 19]).

As a consequence of Proposition 10 we obtain the following.

COROLLARY 13. *The properties k-β and k-NUC are isomorphically different from NUC.*

Bor-Luh Lin and Yu Xin-Tai [13] have proved that every strictly convex k-UR space is $(k+1)R$. We improve this result in the following way.

THEOREM 14. *Let X be a Banach space. Then,*

(i) *if X is k-UR for some integer $k \geqslant 1$, then X is k-β;*

(ii) *if X is strictly convex and k-NUC for some integer $k \geqslant 2$, then X is kR.*

Proof. (i) Assume the contrary; i.e., there exists an $\varepsilon > 0$ such that for every integer n there is an element $x_0^{(n)}$ and a sequence $\{x_m^{(n)}\}_{m=1}^{\infty}$ with $\|x_m^{(n)}\| \leqslant 1$, $m = 0, 1, 2, ..., n = 1, 2, ...$ and $\text{sep}(x_m^{(n)}) > \varepsilon$, $n = 1, 2, ...$, so that for arbitrary choice $\{m_i\}_{i=1}^{k}$ of indices we get

$$\left\| x_0^{(n)} + \sum_{i=1}^{k} x_{m_i}^{(n)} \right\| \Big/ (k+1) > 1 - 1/n. \tag{2}$$

It is easily seen that for every bounded set E with $\alpha(E) > \varepsilon$,

$$\sup_{x \in E} \{ \inf_{y \in L} \|x - y\| \} > \varepsilon/2, \tag{3}$$

for any finite-dimensional subspace L.

Put $m_0 = 0$. By (3), we may find successively elements $\{x_{m_i}^{(n)}\}_{i=1}^{k}$, so that for every $n = 1, 2, ...$,

$$\text{dist}(x_{m_{i+1}}^{(n)}, [x_{m_0, ...}^{(n)} x_{m_i}^{(n)}]) > \varepsilon/2, \qquad i = 1, 2, ..., K, \tag{4}$$

where $[x_{m_0, ...}^{(n)} x_{m_i}^{(n)}]$ means the affine hull of $\{x_{m_j}^{(n)}\}_{j=1}^{i}$. It follows from (4) and [2] that

$$V(x_{m_0}^{(n)}, ..., x_{m_k}^{(n)}) > (\varepsilon/2)^k, \qquad n = 1, 2, ... \tag{5}$$

On the other hand, (2) implies that

$$\left\| \sum_{i=0}^{k} x_{m_i}^{(n)} \right\| \Big/ (k+1) \to 1, \qquad \text{as} \quad n \to \infty,$$

which according to k-UR contradicts (5).

(ii) Let $\{x_n\}$ be a sequence for which

$$\lim_{n_1, ..., n_k \to \infty} \left\| \sum_{i=1}^{k} x_{n_i} \right\| \Big/ k = 1. \tag{6}$$

Without affecting the generality we may suppose that $\|x_n\| = 1$ for every n. We prove first that every subsequence of $\{x_n\}$ has Cauchy subsequence. Take an arbitrary subsequence of $\{x_n\}$, for brevity denote it again by $\{x_n\}$, and assume that it does not have a Cauchy subsequence. Then, $\alpha(\{x_n\}) > 0$. Hence, one can find a subsequence $\{x_m\}$ with $\text{sep}(x_m) > \alpha(\{x_n\})/3 = \varepsilon$. For this $\varepsilon > 0$ there exists by k-NUC a $\delta > 0$ so that for every sequence $\{y_n\}$ in the closed unit ball with $\operatorname{sep}(y_n)>\varepsilon$, there are indices $\{n_i\}_{i=1}^k$

$$\left\|\sum_{i=1}^{k} y_{n_i}\right\|\Big/k\leqslant 1-\delta. \tag{7}$$

It follows from (6) that there is a number N so that for $m_i\geqslant N$,

$$\left\|\sum_{i=1}^{k} x_{m_i}\right\|\Big/k>1-\delta. \tag{8}$$

Since the separation of $\{x_m\}_{m\geqslant N}$ is also greater than ε, then (8) contradicts (7), which proves our claim.

It remains to show that $\{x_n\}$ has unique cluster point. Assume that there are subsequences $\{x_{n_i}\}\to x$, $\{x_{m_i}\}\to y$. By (6) and the triangle inequality, $\lim_{i\to\infty}\|x_{n_i}+x_{m_i}\|/2=1$, whence $\|x+y\|/2=1$. Since X is strictly convex, this implies $x=y$, which completes the proof.

Remark 15. In view of Corollary 12, Theorem 14(i) isomorphically improves the implication k-UR $\Rightarrow$ NUC of Yu Xin-Tai [25]. As we mentioned in the beginning, k-UR spaces are superreflexive [24] and not every space with property (β) is superreflexive [4, 14]. Bor-Luh Lin and Pei-Kee Lin [12] have shown that the Baerstein's space which fails to have (BS), admits an equivalent $2R$ norm. Since k-β and k-NUC spaces have (BS), this shows that Theorem 14 isomorphically improves the result of Bor-Luh Lin and Yu Xin-Tai [13].

We are ready now to distinguish isometrically k-β and k-NUC.

THEOREM 16. *For the properties k-β and k-NUC the following hold*:

(i) *for every $k\geqslant 2$, there exists a strictly convex Banach space X_k, isomorphic to l_2, which is k-β but is not k-NUC*;

(ii) *for every $k\geqslant 1$, there exists a Banach space Y_k, isomorphic to l_2, which is $(k+1)$-NUC but is not k-β.*

Proof. (i) Let X_k be the example of a strictly convex Banach space which is k-UR but is not kR, given in [13] (it is a modification of an example in [23]). From Theorem 14 (i) and (ii) it follows immediately that X_k is k-β but is not k-NUC.

(ii) Put $X_1=l_2$ endowed with its usual norm and let for $k\geqslant 2 X_k$ be as in (ii). Denote by Y_k, $k=1,2,\ldots$ the l_1-direct sum of X_k and $\mathbb{R}^1$. By Theorem 8(i), X_k is $(k+1)$-NUC and thus, by [8], Y_k is also $(k+1)$-NUC. The space Y_1 is not 1-β (see Example 2). Let $k\geqslant 2$. Since X_k is not

k-NUC, there exists an $\varepsilon > 0$ so that for each $\delta > 0$ there is a sequence $\{x_n(\delta)\}$, $\|x_n(\delta)\| \leqslant 1$ with $\text{sep}(x_n(\delta)) > \varepsilon$ and for every choice of indices $\{n_i\}_{i=1}^k$,

$$\left\| \sum_{i=1}^{k} x_{n_i}(\delta) \right\| \Big/ k > 1 - \delta.$$

Let x be the unit of $\mathbb{R}^1$ considered as element of Y_k. Then, for every choice $\{n_i\}_{i=1}^k$ we get

$$\left\| x + \sum_{i=1}^{k} x_{n_i}(\delta) \right\| \Big/ (k+1) = \left\| \sum_{i=1}^{k} x_{n_i}(\delta) \right\| \Big/ (k+1) + 1/(k+1)$$
$$> 1 - k\delta/(k+1),$$

which means that Y_k is not k-β.

In connection with Theorems 8 and 16, Corollary 13, and Remark 15 one can ask about the isomorphic relationship between the properties k-β, $k \geqslant 1$ and k-NUC, $k \geqslant 2$. In this direction we show a representative of k-NUC spaces which fails to have an equivalent 1-β norm. For this purpose we use an example of Schachermayer [20, 21]. In the above sense Schachermayer's space is a typical example of a k-NUC space.

The definition of Schachermayer's space E [21] is as follows.

Let $\gamma = \{n_1, n_2, ..., n_m\}$ be an increasing finite sequence of natural numbers. Write $n_i = 2^{u_i} + v_i$ where this expression is unique, if we require that $0 \leqslant v_i < 2^{u_i}$. Associate to every n_i the real number $t(n_i) = v_i/2^{u_i} \in [0, 1)$ and call γ admissible if

(i) $m \leqslant n_1$,

(ii) for every $0 \leqslant j < 2^{u_i+1}$ there is at most one i such that $t(n_i) \in [j/2^{u_1+1}, (j+1)/2^{u_1+1})$.

For an admissible $\gamma = \{n_1, ..., n_m\}$ and $x \in \mathbb{R}^{(\mathbb{N})}$, the space of finite sequences, define

$$\|x\|_\gamma = \sum_{i=1}^{m} |x_{n_i}|.$$

Put

$$\|x\| = \sup \left\{ \left(\sum_{j=1}^{\infty} \|x\|_{\gamma_j}^2 \right)^{1/2} \right\},$$

where the supremum is taken over all increasing sequences $\{\gamma_j\}_{j=1}^\infty$ of admissible sets (i.e., the last member of γ_j is smaller than the first member of γ_{j+1}).

By $(E, \|\cdot\|)$ denote the completion of $\mathbb{R}^{(\mathbb{N})}$ with respect to this norm. Let $\{e_n\}_{n=1}^\infty$ be its natural basis.

THEOREM 17. *Schachermayer's space E is 8-NUC and it does not admit an equivalent 1-β norm.*

Proof. In [5] we have shown that the example from [9] of a reflexive Banach space which does not admit an equivalent norm, uniformly differentiable in every direction, is NUC but it fails to have an equivalent norm with property (β), i.e., 1-β. The proof of the last fact was based on Day's technique. After showing that $(\beta) \Rightarrow (\mathrm{BS})$, we managed to give a simpler proof of the fact that (β) and NUC are isomorphically different [6], using Baernsfein's space. In view of Theorem 9 such a method is not applicable to distinguish isomorphically k-NUC and 1-β. Still, one can repeat almost literally the proof in [5] to show that Schachermayer's space E fails to have an equivalent norm with property 1-β.

Let us mention that the original norm $\|\cdot\|$ is not 2-NUC. Indeed, one can easily construct a sequence of integers $\{n_i\}_{i=1}^{\infty}$ so that all the two-point sets $\{n_i, n_j\}$ are admissible, and then consider in E the sequence $\{e_{n_i}\}_{i=1}^{\infty}$.

Thus, it remains to see that E is 8-NUC. Let $\varepsilon > 0$. Consider an arbitrary sequence $\{y_n\}$, $\|y_n\| \leqslant 1$ with $\mathrm{sep}(y_n) > \varepsilon$. Passing to a subsequence (denote again by $\{y_n\}$) we may assume that y_n is weakly convergent because of reflexivity of E. Put $y_n = x_0 + x_n$, where $\{x_n\}$ is weakly null sequence. Observe by a standard perturbation argument that there is no loss of generality to suppose that $\{x_n\}_{n=0}^{\infty}$ are supported by finite increasing sets $\{I_n\}_{n=0}^{\infty}$ (i.e., the last member of I_n is less than the first member of I_{n+1}). Thus,

$$x_n = \sum_{i \in I_n} \lambda_i^{(n)} e_i, \qquad n = 0, 1, 2, \ldots. \tag{9}$$

Passing to a subsequence assume that $\{\|x_n\|\}_{n=1}^{\infty}$ is convergent, moreover without affecting the generality we may suppose that $\|x_n\| = b$, $n = 1, 2, \ldots$. By (9) and the definition of the norm in E, we obtain

$$\|x_0 + x_n\|^2 \geqslant \|x_0\|^2 + b^2. \tag{10}$$

It follows from $\mathrm{sep}(x_n) = \mathrm{sep}(y_n) > \varepsilon$ that

$$b > \varepsilon/2. \tag{11}$$

Schachermayer [21] has proved that every normalized weakly null sequence $\{z_n\}_{n=1}^{\infty}$ has a subsequence $\{z_{n_i}\}_{i=1}^{\infty}$ so that for every finite sequence of scalars $\{a_i\}_{i=1}^{k}$

$$\left\| \sum_{i=1}^{k} a_i z_{n_i} \right\| \leqslant \sqrt{6 + 1/2} \left(\sum_{i=1}^{k} a_i^2 \right)^{1/2}$$

holds. Hence, we may choose a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ so that for every integer k,

$$\left\| \sum_{i=1}^{k} \frac{1}{k} x_{n_i} \right\| \leqslant \sqrt{6+1/2}\, b/k^{1/2}. \tag{12}$$

For the sake of brevity we denote the above subsequence by $\{x_m\}$. Write $i \in I_0$ in the form $i = 2^u + v$, $0 \leqslant v < 2^u$. Put

$$r = \max\{u+1 : i \in I_0\}.$$

Let $\Delta_j = [j/2^r, (j+1)/2^r)$, $0 \leqslant j < 2^r$. Consider for $0 \leqslant j < 2^r$ and $m = 1, 2, ...$

$$\mu_j^{(m)} = \max\{|\lambda_i^{(m)}| : t(i) \in \Delta_j\}$$

(write 0 if $\{i \in I_m : t(i) \in \Delta_j\} = \varnothing$).

Since $\{x_m\}$ is bounded, the sequences $\{\mu_j^{(m)}\}_{m=1}^{\infty}$, $j = 0, ..., 2^r$ are also bounded. Thus, passing to a subsequence, we may suppose that for every $0 \leqslant j < 2^r$,

$$\lim_{m \to \infty} \mu_j^{(m)} = \mu_j.$$

Fix k. Then for m large enough, say $m > s$, we get

$$|\mu_j^{(m)} - \mu_j| < \varepsilon^2/64k, \qquad 0 \leqslant j < 2^r. \tag{13}$$

Consider

$$y = \sum_{m=s+1}^{s+k} \frac{1}{k} y_m = x_0 + \sum_{m=s+1}^{s+k} \frac{1}{k} x_m.$$

Let $\{\gamma_j\}_{j=1}^{q}$ be an increasing sequence of admissible sets such that

$$\|y\| = \left(\sum_{j=1}^{q} \|y\|_{\gamma_j}^2 \right)^{1/2}.$$

Note that if there is no γ_j which intersects simultaneously I_0 and $I = \bigcup_{m=s+1}^{s+k} I_m$ then it is easy to see that

$$\|y\|^2 = \|x_0\|^2 + \left\| \sum_{m=s+1}^{s+k} \frac{1}{k} x_m \right\|^2$$

and thus by (12),

$$\|y\|^2 \leqslant \|x_0\|^2 + (6+1/2) b^2/k.$$

It follows from (10) that $\|x_0\|^2 \leqslant 1 - b^2$ and therefore by (11),

$$\|y\|^2 \leqslant 1 - (k - 6 - 1/2)\varepsilon^2/4k.$$

Let there exist a $\gamma_p = \{i_1 < i_2 < \cdots < i_p\}$ which intersects both I_0 and I. Then $i_1 = 2^{u_1} + v_1 \in I_0$ whence $r_1 = u_1 + 1 \leqslant r$. Put $\Delta'_j = [j/2^{r_1}, (j+1)/2^{r_1})$, $0 \leqslant j < 2^{r_1}$. Clearly, for $v_j^{(m)} = \max\{|\lambda_i^{(m)}| : t(i) \in \Delta'_j\}$ we have that

$$v_j^{(m)} = \max\{\mu_l^{(m)} : 2^{r - r_1} j \leqslant l < 2^{r - r_1}(j+1)\}, \qquad 0 \leqslant j < 2^{r_1}.$$

Evidently, (13) implies that for every $m > s$,

$$|v_j^{(m)} - v_j| < \varepsilon^2/64k, \qquad 0 \leqslant j < 2^{r_1}, \tag{14}$$

where $v_j = \lim_{m \to \infty} v_j^{(m)}$.

Denote

$$A = \{0 \leqslant j < 2^{r_1} : \text{there is an } i(j) \in \gamma_p \cap I_0 \text{ with } t(i(j)) \in \Delta'_j\},$$

$$B = \{0 \leqslant j < 2^{r_1} : \text{there is an } i(j) \in \gamma_p \cap I \text{ with } t(i(j)) \in \Delta'_j\}.$$

Note that for every $j \in A \cup B$ the element $i(j)$ is unique. We write

$$c = \sum_{j=1}^{p-1} \left(\sum_{i \in \gamma_j} |\lambda_i^{(0)}| \right)^2 \qquad \text{and} \qquad a = \sum_{j \in A} |\lambda_{i(j)}^{(0)}|.$$

By the definition of he norm, we have for every m

$$c + \left(a + \sum_{j \in B} v_j^{(m)} \right)^2 \leqslant \|y_m\|^2,$$

whence

$$c + \left(a + \sum_{j \in B} v_j \right)^2 \leqslant 1. \tag{15}$$

It follows from (14) and the definition of the norm that

$$\|y\|^2 \leqslant c + \left(a + \sum_{j \in B} v_j/k \right)^2 + \left\| \sum_{m=s+1}^{s+k} x_m/k \right\|^2 + \varepsilon^2/16k.$$

Clearly, (15) yields $2a \sum_{j \in B} v_j \leqslant 1 - c - a^2 - (\sum_{j \in B} v_j)^2$ and thus,

$$c + \left(a + \sum_{j \in B} v_j/k \right)^2 \leqslant \frac{1}{k} + \frac{k-1}{k}(c + a^2).$$

Taking into account (12), $\|x_0\| + b^2 \leqslant 1$ and $c + a^2 \leqslant \|x_0\|^2$, we obtain

$$\|y\|^2 \leqslant \frac{1}{k} + \frac{k-1}{k}(1-b^2) + \left(6 + \frac{1}{2}\right)\frac{b^2}{k} + \frac{\varepsilon^2}{16k}$$
$$= 1 - \frac{k-7-1/2}{k} b^2 + \frac{\varepsilon^2}{16k}.$$

Let now $k = 8$. Because of (11), we have

$$\|y\|^2 \leqslant 1 - \varepsilon^2/16k.$$

Evidently, this implies that the norm is 8-NUC.

Moreover, the following holds true.

PROPOSITION 18. *The dual E^* of Schachermayer's space does not admit an equivalent NUC norm.*

Proof. Huff [3] has introduced a property (*), suggested by J. Bourgain. Let X be a Banach space. Given a set $A \subset X$ and $\varepsilon > 0$, define the ε-derived set of A to be the set

$$\eta_\varepsilon(A) = \{x : \text{there exists } \{x_n\} \subset A \text{ with } \operatorname{sep}(x_n) > \varepsilon \text{ and } x_n \to x \text{ weakly}\}.$$

Denote by B the closed unit ball in X. In [3] it is proved that if X has an equivalent norm which is UKK, then

$$(*) \begin{cases} \text{for every } \varepsilon > 0 \text{ there exists } n \text{ such that} \\ \eta_\varepsilon^{(n)}(B) = \varnothing. \end{cases}$$

Let B^* be the closed unit ball of E^*. Consider $\varepsilon = \frac{1}{2}$. We show that for every n, $\eta_{1/2}^{(n)}(B^*) \neq \varnothing$, which implies that E^* fails to have an equivalent NUC norm. Clearly, it is enough to take $n = 2^j$, $j = 1, 2, \ldots$. For every admissible set γ consider its characteristic function χ_γ as an element of E^*, evidently $\|\chi_\gamma\| = 1$ (here $\|\cdot\|$ stands for the dual norm). Fix $n = 2^j$. Consider admissible sets γ whose first element n_1 is greater or equal to n. Let $\gamma = \{n_1 < n_2 < \cdots < n_{2^k}\}$. Fix the first $2^k - 1$ elements and vary $n_{2^k} = m$. Obviously, there are infinitely many permissible m. For the characteristic functions χ' and χ'' of the admissible sets $\gamma' = \{n_1, \ldots, n_{2^k-1}, m'\}$ and $\gamma'' = \{n_1, \ldots, n_{2^k-1}, m''\}$ with $m' \neq m''$, we get

$$\|\chi' - \chi''\| \geqslant 1.$$

So, if $\chi^{(m)}$ is the characteristic function of $\{n_1, ..., n_{2^k-1}, m\}$, then $\text{sep}(\chi^{(m)}) \geqslant 1$. Since the biorthogonal basis $\{e_n^*\}$ tends weakly to zero, we obtain that

$$\chi^{(m)} \xrightarrow[m]{} \chi_{\{n_1, ..., n_{2^k-1}\}} \text{ weakly,}$$

whence $\chi_{\{n_1, ..., n_{2^k-1}\}}$ belongs to $\eta_{1,2}^{(1)}(B^*)$.

Repeating the argument, we get

$$w\text{-}\lim_{n_1 \to \infty} (\cdots (w\text{-}\lim_{n_{2^k} \to \infty} (\chi_{\{n_1, ..., n_{2^k}\}})) \cdots) = 0.$$

Thus, $0 \in \eta_{1,2}^{(n)}(B^*)$, which ends the proof.

Remark 19. Sekowski and Stachura [22] and Prus [17] have defined the notion of nearly uniform smoothness (NUS). They have proved that a Banach space X (resp. X^*) is NUS if and only if X^* (resp. X) is NUC. Prus has characterized the existence of an equivalent NUS or NUC norm in Banach spaces with countable basis. Using his results, we may give an alternative proof of Proposition 18, Theorem 17 and Proposition 18 show that Schachermayer's space is an example of a k-NUC space which does not admit an equivalent NUS norm.

To conclude we mention the following simple statement.

PROPOSITION 20. *Let X be a Banach space and Y be its subspace. If X is k-β (resp. k-NUC), then*

(i) *Y is k-β (resp. k-NUC);*

(ii) *the quotient space X/Y is k-β (resp. k-NUC).*

The proof is evident in view of Theorem 7.

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