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Note

An Extension of the Erdös, Ko, Rado Theorem to t-Designs

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The following theorem is proved:

THEOREM. *Let $\mathscr{B}$ represent the set of blocks of a $t-(v, k, \lambda)$ design. Given $0 < s < t \leqslant k$, then there exists a function $f(k, t, s)$ with the following property: suppose there is a set $\mathscr{C} \subseteq \mathscr{B}$ of blocks such that for all $A, B \in \mathscr{C}$, $|A \cap B| \geqslant s$; then if $v \geqslant f(k, t, s)$,*

$$|\mathscr{C}| \leqslant b_s = \text{the number of blocks through } s \text{ points.}$$

Furthermore, the only families of blocks reaching this bound are those consisting of all blocks through some s points.

If $s < t - 1$, then $f(k, t, s) \leqslant s + \binom{k}{s}(k - s + 1)(k - s)$.

If $s = t - 1$, then $f(k, t, s) \leqslant s + (k - s)\binom{k}{s}^2$.

Terminology and Notes

A $t-(v, k, \lambda)$ design is an ordered pair $(X, \mathscr{B})$, where X is a set of size v, and $\mathscr{B}$ is a family of k-subsets (blocks) of X with the property that any t-subset of X is contained in exactly λ of the blocks of $\mathscr{B}$. To avoid degenerate cases, it is assumed that $0 < t \leqslant k \leqslant v$.

It is well known that for any $s \leqslant t$, the number of blocks, b_s, through any collection of s points of X, is independent of the points, and

$$b_s = \lambda \binom{v-s}{t-s} \bigg/ \binom{k-s}{t-s}.$$

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Extensive use is made of the fact that

$$\binom{n}{r}\Big/\binom{n-1}{r-1}=n/r.$$

Let $\mathscr{P}_k(v)$ denote the set of all k-subsets of a v-set. Then it may be regarded as a $k-(v, k, 1)$ design. So the theorem has the following theorem due to Erdös *et al.* [1] as an immediate corollary:

THEOREM. *Given $0 < s \leqslant k \leqslant v$, then there exists a function $g(k, s)$ with the following property: suppose there is a set $\mathscr{A}$ of k-subsets of a v-set such that for all $A, B \in \mathscr{A}$, $|A \cap B| \geqslant s$; then if $v \geqslant g(k, s)$,*

$$|\mathscr{A}| \leqslant \binom{v-s}{k-s}.$$

Frankl [2] has shown that if $s \geqslant 15$, then,

$$g(k, s) = (k - s + 1)(s + 1).$$

Proof of the theorem. Let $\mathscr{A}$ be a family of blocks, satisfying the conditions of the theorem. Let $\mathscr{C}$ be the set of s-subsets, which are at the intersection of at least two blocks of $\mathscr{A}$. Let n_p be the number of blocks containing the s-subset p of the family $\mathscr{C}$. Let $|\mathscr{A}| = w$.

Count (p, B) such that $p \in \mathscr{C}$, $p \subseteq B \in \mathscr{A}$, to obtain,

$$\sum_{p\in\mathscr{C}} n_p \leqslant w\binom{k}{s}.$$

Count (p, B, A) such that $p \in \mathscr{C}$, $p \subseteq B \cap A$, with $A, B \in \mathscr{A}$;

$$\sum_{p\in\mathscr{C}} n_p(n_p-1) = \sum_{\substack{A\neq B\\ A,B\in\mathscr{A}}} \binom{|A\cap B|}{s} \geqslant w(w-1).$$

Now if $\mathscr{A}$ is not the set of all blocks through an s-subset, then, for each $p \in \mathscr{C}$, there is some block $B \in \mathscr{A}$ with $p \notin B$. Any other block $A \in \mathscr{A}$, which contains p, contains at least s points of B. So if d is the maximum number of blocks of $\mathscr{B}$ which contain p and at least s points of B, then $n_p \leqslant d$. Hence,

$$w(w-1) \leqslant (d-1)\sum_{p\in\mathscr{C}} n_p \leqslant (d-1)\, w\binom{k}{s},$$

$$w - 1 \leqslant (d-1)\binom{k}{s} \qquad \text{and so} \qquad w < d\binom{k}{s}.$$

If $d\binom{k}{s} \leqslant b_s$ we are done.

The following lemma gives an upper bound for d.

LEMMA. *Let p be an s-subset, and B a block not containing p. Let d be the number of blocks containing p and at least s points of B. Then*

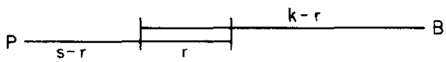
(i) *if $s \leqslant t/2$ and $v \geqslant k^2 + 2t$, or*

(ii) *if $t/2 < s < t-1$ and $v \geqslant s + \binom{k}{s}(k-s)$,*

then $d \leqslant (k-s-1)[\binom{v-s-1}{t-s-1}/\binom{k-s-1}{t-s-1}]\lambda$;

(iii) *if $s = t-1$ then $d \leqslant \binom{k}{s}\lambda$.*

Proof of (i). Take an s-subset, p, and a block B containing r points of p, for some $r < s$.



Let d_r be the number of blocks containing p and at least s points of B. Then $d_r \leqslant \binom{k-r}{s-r} b_{2s-r} = \lambda\binom{k-r}{s-r}\binom{v-2s+r}{t-2s+r}/\binom{k-2s+r}{t-2s+r} = e_r$, say, since there are $\binom{k-r}{s-r}$ s-subsets of B which contain all points of $B \cap p$, and these, together with the remaining $s-r$ points of p, each determine a family of b_{2s-r} blocks with the required property. These families have in their union all such blocks. Clearly,

$$e_{r+1}/e_r = (s-r)(v-2s+r+1)/(k-r)(k-2s+r+1)$$
$$> (v-2t)/k^2$$

so if $v \geqslant k^2 + 2t$, then $e_{r+1} > e_r$ for all $r < s-1$, and so

$$d_r \leqslant e_{s-1} = \lambda(k-s+1)\binom{v-s-1}{t-s-1}\Big/\binom{k-s-1}{t-s-1}.$$

Hence d is bounded as required.

Proof of (ii) *and* (iii). Let c_j = maximum number of blocks containing j points of X. So if $j \leqslant t$, then $c_j \leqslant b_j$, but, if $j > t$, then $c_j \leqslant \lambda$.

Then, with d_r as in the proof of case (i),

$$d_r \leqslant \binom{k-r}{s-r} c_{2s-r}.$$

So,

if $r \leqslant 2s-t$, then $d_r \leqslant \lambda\binom{k-r}{s-r} \leqslant \binom{k}{s}\lambda$;

if $r = 2s-t$, then $d_r \leqslant \lambda\binom{k-2s+t}{2t-2s}$;

if $r > 2s-t$, then $d_r \leqslant \lambda\binom{k-r}{s-r}\binom{v-2s+r}{t-2s+r}/\binom{k-2s+r}{t-2s+r}$.

It is clear, using the same argument as in (i), that,

$$d = \max d_r \leqslant \max(d_0, d_{s-1})$$
$$\leqslant \max\left(\binom{k}{s}\lambda, \lambda(k-s-1)\binom{v-s-1}{t-s-1}\Big/\binom{k-s-1}{t-s-1}\right),$$

as long as $v \geqslant k^2 + 2t$.

But if $s < t-1$, then d $\max((\binom{k}{s})\lambda, \lambda(v-t)/k)$. So if $v \geqslant t + \binom{k}{s}k$, then $d \leqslant \lambda(k-s-1)b_{s+1}$. If $s = t-1$, then $d \leqslant \max(\binom{k}{s}\lambda, \lambda(k-t)) = \binom{k}{s}\lambda$. So ends the proof of the lemma.

We want $d\binom{k}{s} \leqslant b_s$.

In cases (i) and (ii)

$$\lambda\binom{k}{s}(k-s+1)\binom{v-s-1}{t-s-1}\Big/\binom{k-s-1}{t-s-1} \leqslant \lambda\binom{v-s}{t-s}\Big/\binom{k-s}{t-s},$$

i.e.,

$$\binom{k}{s}(k-s+1)(k-s)/(t-s) \leqslant (v-s)/(t-s),$$

or

$$v \geqslant s + \binom{k}{s}(k-s+1)(k-s).$$

In case (iii)

$$\lambda\binom{k}{s}^2 \leqslant \lambda(v-t+1)/(k-t+1)$$

or

$$v \geqslant s + (k-s)\binom{k}{s}^2.$$

Since we have strict inequality, in $|\mathcal{C}| < b_s$, when v satisfies these conditions, and $\mathcal{C}$ does not consist of all blocks containing an s-subset, then the conclusion of the theorem holds.

Conclusion

The bound $f(k, t, s)$ is not the best possible, but is sometimes surprisingly good, especially as it is independent of λ. For example, consider the case $t = 2$, $s = 1$; then the theorem gives

$$f(k, 2, 1) \leqslant 1 + k^2(k-1) = k^3 - k^2 + 1.$$

It is not difficult to show that if $\lambda = 1$, then for $v > ((k-1)^4 - 1)/(k-2) = k^3 - 2k^2 + 2k$ the conclusions of the theorem hold. However, to prove this it seems necessary to make use of the fact that a $2-(v, k, 1)$ design is a partial geometry. For $\lambda > 1$, no such alternative method is available. In the case $\lambda = 1$, and $v = k^3 - 2k^2 + 2k$, take the 2-design $PG(3, k-1)$, consisting of the points and lines of projective 3-space (so $k - 1$ must be a prime power). The family of $k^2 - k + 1 = b_1$ lines lying in a plane have the property that any two intersect, but do not form the set of blocks through a point.

Acknowledgments

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References

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2. P. Frankl, The Erdös, Ko, Rado Theorem is true for $n = ckt$, *in* "Colloquia Mathematica Societatis Janos Bolyai," Vol. 18, "Combinatorics" (A. Hajoal and V. Sós, Eds.), Keszthely, Hungary, 1976.