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The basic cohomology of the twisted $N = 16$, $D = 2$ super-Maxwell theory

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Abstract

We consider a recently proposed two-dimensional Abelian model for a Hodge theory, which is neither a Witten-type nor a Schwarz-type topological gauge theory. It is argumented that this model is not a good candidate for a Hodge theory because, on-shell, the BRST Laplacian vanishes. We show, that this model allows a natural extension such that the resulting topological theory is of Witten type and can be identified with the twisted $N = 16$, $D = 2$ super-Maxwell theory. Furthermore, the underlying basic cohomology preserves the Hodge-type structure and, on-shell, the BRST Laplacian does not vanish.

1. Introduction

It has been known for some time that two-dimensional Yang–Mills theory without matter is exactly soluble [1]. Moreover, it has been shown that Yang–Mills theory with matter on an arbitrary orientable two-dimensional manifold M of genus G and area A is equivalent to a closed, orientable, string theory with target space M [2]. Two-dimensional Yang–Mills theory was revisited using a non-Abelian version of the Duistermaat–Heckman formula in [3] and a simple mapping to topological Yang–Mills theory with an underlying $N_T = 1$ equivariantly nilpotent shift symmetry was given. For a previous work on quantum gauge theories in two dimensions see, e.g., [4]. In this Letter we consider a Hodge type cohomological gauge theory with an underlying $N_T = 8$ strictly nilpotent shift and co-shift symmetry (i.e., even prior to the introduction of the gauge ghost and antighost fields), which leads to other topological observables.

The study of such theory was motivated by a recently series of papers [5] where a class of topological gauge theories in two dimensions was presented which is neither of Witten nor of Schwarz type. Rather, they show some of the characteristic features of both types of topological quantum field theories (TQFT), namely, the form of the action turns out to be of Witten type whereas the underlying supersymmetries are reminiscent of Schwarz type (for a review of TQFT, see, e.g., [6]).

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The aim of this Letter is to reveal these rather unusual properties of a TQFT. Thereby, we focus on the simplest of the models considered in [5], namely the Euclidean Maxwell theory in two dimensions in the Feynman gauge. Their action is given by

$$S = \int\limits_E d^2x \left\{ \frac{1}{4} F^{ab}(A) F_{ab}(A) + \frac{1}{2} \partial^a A_a \partial^b A_b + \partial^a \bar{C} \partial_a C \right\}, \tag{1}$$

with $F_{ab}(A) = \partial_a A_b - \partial_b A_a$, where A_a is the Abelian gauge field and C, $\bar{C}$ are the gauge (anti)ghost fields, respectively.

In [4] it has been shown that the action (1) is not only invariant under the BRST symmetry, generated by Ω, but also under a co-BRST symmetry, generated by ${}^\star\Omega$, which, together with the BRST Laplacian W, obey the following BRST-complex:

$$\Omega^2 = 0, \qquad {}^\star\Omega^2 = 0, \qquad W = \{\Omega, {}^\star\Omega\} \neq 0, \qquad [\Omega, W] = 0, \qquad [{}^\star\Omega, W] = 0. \tag{2}$$

Representations of this superalgebra for the first time have been considered in [7]. Since Ω and ${}^\star\Omega$ are nilpotent hermitian operators they are realized in a Krein space $\mathcal{K}$ [8] whose non-degenerate indefinite scalar product $\langle \chi | \psi \rangle := (\chi, J\psi)$ is defined by the help of a self-adjoint metric operator $J \neq 1$, $J^2 = 1$. With respect to the inner product $(\,,\,)$ the operators Ω and ${}^\star\Omega = \pm J\Omega J$ are adjoint to each other, $(\chi, {}^\star\Omega\psi) = (\Omega\chi, \psi)$, but with respect to the scalar product $\langle\,|\,\rangle$ they are self-adjoint. From these definitions one obtains a remarkable correspondence between the BRST cohomology and the de Rham cohomology [9]:

BRST operator	Ω,	differential	d,
co-BRST operator	${}^\star\Omega = \pm J\Omega J$,	co-differential	$\delta = \pm \star d \star$,
duality operation	J,	Hodge star	$\star$,
BRST Laplacian	$W = \{\Omega, {}^\star\Omega\}$,	Laplacian	$\Delta = \{d, \delta\}$.

Hence, the action (1) provides a Hodge-type field theoretical model in two dimensions. However, owing to the absence of a shift and co-shift symmetry, the action is not of Witten type and, because the Maxwell action in a gravitational background is not metric independent, it is also not of Schwarz type. Differently, the topological nature of that model is a consequence of the fact that in two dimensions there are no propagating degrees of freedom associated with the gauge field. On the other hand, as there is no topological supersymmetry, on-shell, the BRST Laplacian vanishes. This is, in fact, an unsatisfactory property of a Hodge-type theory because their physical states should lie entirely in the set of harmonic states, i.e., the set of the zero modes of the BRST Laplacian. Therefore, in order to incorporate a topological supersymmetry into that model, it will be shown that it can be regarded as part of a more complex topological model of Witten type, namely the twisted $N = 16$, $D = 2$ super-Maxwell theory with global symmetry group $SU(4)$, whose *basic cohomology* [10] possesses actually a Hodge-type structure.

The Letter is organized as follows: in Section 2, as a first step, we substitute in (1) the Maxwell action by the cohomological action of twisted $N = 16$, $D = 2$ super-Maxwell theory with global symmetry group $SU(4)$. And we show that the BRST complex of the 8 twisted scalar supercharges, i.e., the generators of the shift and co-shift symmetries, is really of Hodge type. In Section 3, as a second step, we complete the cohomological action by introducing the ordinary gauge fixing terms and verify that the basic cohomology, i.e., the BRST complex including also the ordinary gauge symmetry, preserves the underlying Hodge-type structure. In Section 4 we construct topological observables for that theory.

2. The BRST complex of the twisted $N = 16$, $D = 2$ super-Maxwell theory with global symmetry group $SU(4)$

As mentioned above, in order to avoid the vanishing of the BRST Laplacian of the topological model proposed in [5] we view the Maxwell action in (1) as the classical part of the twisted action of $N = 16$, $D = 2$ super-Maxwell theory with global symmetry group $SU(4)$ (see, e.g., [11] where the non-Abelian extension of that action has been constructed),

$$S_T^{(N_T=8)} = \int\limits_E d^2x \left\{ \frac{1}{4} F^{ab}(A + iV) F_{ab}(A - iV) + \frac{1}{2} \partial^a V_a \partial^b V_b + \frac{1}{8} \partial^a M_{\alpha\beta} \partial_a M^{\alpha\beta} \right.$$
$$\left. - \epsilon^{ab} \zeta_\alpha \partial_a \psi_b^\alpha - \eta_\alpha \partial^a \psi_b^\alpha \right\}. \tag{3}$$

Here, V_a is a co-vector field which is combined with A_a to forms the complexified gauge fields $A_a \pm iV_a$, i.e., the action (3) localizes onto the moduli space of complexified flat connections. Furthermore, we have introduced a $SU(4)$-quartet of Grassmann-odd vector fields ψ_a^α, two $SU(4)$-quartets of Grassmann-odd scalar fields, η_α and ζ_α, which transform as the fundamental and its complex conjugate representation of $SU(4)$, respectively, and a $SU(4)$-sextet of Grassmann-even complex scalar fields $M_{\alpha\beta} = (1/2)\epsilon_{\alpha\beta\gamma\delta} M^{\gamma\delta}$, which transform as the second-rank complex selfdual representation of $SU(4)$, where $\alpha = 1, 2, 3, 4$ denotes the internal group index of $SU(4)$. ϵ_{ab} is the antisymmetric Levi-Civita tensor in $D = 2$.

Let us notice that the action (3) can be obtained from the Euclidean $N = 16$, $D = 2$ super-Maxwell theory with R-symmetry group $SU(4) \otimes U(1)$ by twisting the Euclidean rotation group $SO_E(2) \sim U_E(1)$ in $D = 2$ by the $U(1)$ of the R-symmetry group (by simply putting together both $U(1)$ charges), thereby leaving the group $SU(4)$ intact [12].

The action (3) is invariant under the following discrete Hodge type $\star$-symmetry, defined by the replacements

$$\varphi \equiv \begin{bmatrix} \partial_a & A_a & V_a & \\ \psi_a^\alpha & \eta_\alpha & \zeta_\alpha & M^{\alpha\beta} \end{bmatrix} \quad \Rightarrow \quad \star\varphi = \begin{bmatrix} \epsilon_{ab}\partial^b & \epsilon_{ab}A^b & -\epsilon_{ab}V^b & \\ -i\psi_a^\alpha & -i\zeta_\alpha & i\eta_\alpha & -M^{\alpha\beta} \end{bmatrix}, \tag{4}$$

with the property $\star(\star\varphi) = \pm\varphi$.

Let us now describe the full set of twisted supersymmetry transformations which leave the action (3) invariant. The transformation rules for the *on-shell* shift symmetries Q^α are

$$Q^\alpha A_a = \psi_a^\alpha, \qquad Q^\alpha V_a = -i\psi_a^\alpha, \qquad Q^\alpha M_{\beta\gamma} = 2i\delta^\alpha{}_{[\beta}\zeta_{\gamma]}, \qquad Q^\alpha \psi_a^\beta = -i\epsilon_{ab}\partial^b M^{\alpha\beta},$$
$$Q^\alpha \eta_\beta = -i\delta^\alpha{}_\beta \partial^a V_a, \qquad Q^\alpha \zeta_\beta = \frac{1}{2}\delta^\alpha{}_\beta \epsilon^{ab} F_{ab}(A - iV). \tag{5}$$

From combining Q^α with the above displayed Hodge-type $\star$-symmetry one gets the corresponding transformation rules for the *on-shell* co-shift symmetries ${}^\star Q^\alpha$, i.e., ${}^\star Q^\alpha \varphi = \pm \star Q^\alpha \star \varphi$, where the signs are the same as in the relation $\star(\star\varphi) = \pm\varphi$.

Furthermore, by making use of the identity $\epsilon_{ab}\partial_c + \epsilon_{bc}\partial_a + \epsilon_{ca}\partial_b = 0$, one simply verifies that (3) is also invariant under the following *on-shell* vector supersymmetries $Q_{a\alpha}$,

$$Q_{a\alpha} A_b = \delta_{ab}\eta_\alpha - \epsilon_{ab}\zeta_\alpha, \qquad Q_{a\alpha} V_b = -i\delta_{ab}\eta_\alpha - i\epsilon_{ab}\zeta_\alpha, \qquad Q_{a\alpha} M^{\beta\gamma} = 2i\epsilon_{ab}\delta_\alpha{}^{[\beta}\psi^{b\gamma]},$$
$$Q_{a\alpha}\psi_b^\beta = i\delta_\alpha{}^\beta \delta_{ab}\partial^c V_c - \delta_\alpha{}^\beta \partial_a(A_b + iV_b) + \delta_\alpha{}^\beta \partial_b(A_a - iV_a),$$
$$Q_{a\alpha}\eta_\beta = i\epsilon_{ab}\partial^b M_{\alpha\beta}, \qquad Q_{a\alpha}\zeta_\beta = i\partial_a M_{\alpha\beta}. \tag{6}$$

In principle, owing to the Hodge type $\star$-symmetry, one can also introduce *on-shell* co-vector supersymmetries ${}^\star Q_{a\alpha}$, namely, similar as before, according to ${}^\star Q_{a\alpha}\varphi = \pm \star Q_{a\alpha} \star \varphi$. However, they become i times the *on-shell* vector supersymmetries $Q_{a\alpha}$, i.e., ${}^\star Q_{a\alpha} = i Q_{a\alpha}$. Hence, it holds $(Q^\alpha, {}^\star Q^\alpha, Q_{a\alpha}) S_T^{(N_T=8)} = 0$, and the total number of (real) supercharges is actually $N = 16$.

By an explicit calculation one establishes that the 8 scalar supercharges Q^α and ${}^\star Q^\alpha$, being interrelated by the $\star$-operation, together with the 8 vector supercharges $Q_{\mu\alpha}$, obey the following topological superalgebra *on-shell*,

$$\begin{aligned}
&\{Q^\alpha, Q^\beta\} \doteq 0, \qquad \{Q^\alpha, {}^\star Q^\beta\} \doteq -2\delta_G(M^{\alpha\beta}), \qquad \{{}^\star Q^\alpha, {}^\star Q^\beta\} \doteq 0,\\
&\{Q^\alpha, Q_{a\beta}\} \doteq -2\delta^\alpha{}_\beta\big(\partial_a - \delta_G(A_a - iV_a)\big), \qquad \{{}^\star Q^\alpha, Q_{a\beta}\} \doteq -2\delta^\alpha{}_\beta\big(\partial_a - \delta_G(A_a + iV_a)\big),
\end{aligned}$$

where the field-dependent gauge transformations $\delta_G(\varphi)$, with $\varphi = (M^{\alpha\beta}, A_a \pm iV_a)$, are defined by $\delta_G(\varphi)A_a = \partial_a\varphi$, and zero otherwise. (The symbol $\doteq$ means that the corresponding relation is satisfied only *on-shell*.)

Finally, let us show that the twisted action (3) is actually of Witten type. For that purpose we break down the global symmetry group $SU(4)$ in such way that the resulting action splits into a topological term (Q-cocycle) and a Q-exact term, where Q is a certain linear combination of Q^α and ${}^\star Q^\alpha$, and that it possesses a discrete Hodge type $\star$-symmetry. This is precisely what we need in order to put the theory on a two-dimensional compact Riemannian manifold.

To begin with, let us consider the following $N_T = 1$ topological action in five dimensions,

$$\begin{aligned}
S_T^{(N_T=1)} = \int_E d^5x \Big\{ &\frac{1}{4}F^{AB}(A+iV)F_{AB}(A-iV) + \frac{1}{2}\partial^A V_A \partial^B V_B - \frac{1}{8}i\epsilon_{ABCDE}\chi^{AB}\partial^C\chi^{DE}\\
&- i\chi^{AB}\partial_A\psi_B - i\eta\partial^A\psi_A\Big\},
\end{aligned} \tag{7}$$

which is build up from the Abelian gauge field A_A, the co-vector field V_A and the Grassmann-odd scalar, vector and antisymmetric tensor fields η, ψ_A and χ_{AB}, respectively. Here, the space index A runs from 1 to 5 and ϵ_{ABCDE} is the complete antisymmetric Levi-Civita tensor in $D = 5$. This action can be obtained from the Euclidean $N = 2$, $D = 5$ super-Maxwell theory with R-symmetry group $SO(5)$ by twisting the Euclidean rotation group $SO_E(5)$ in $D = 5$ by the R-symmetry group (for details, we refer to [11], where the non-Abelian extension of that action was given).

The full set of twisted *on-shell* supersymmetry transformations, generated by the scalar, vector and antisymmetric tensor supercharges Q, Q_A and Q_{AB}, respectively, are given by

$$\begin{aligned}
&QA_A = \psi_A, \qquad QV_A = -i\psi_A, \qquad Q\eta = -\partial^A V_A, \qquad Q\psi_A = 0,\\
&Q\chi_{AB} = -iF_{AB}(A - iV),
\end{aligned} \tag{8}$$

$$\begin{aligned}
&Q_A A_B = \delta_{AB}\eta - \chi_{AB}, \qquad Q_A V_B = -i\delta_{AB}\eta - i\chi_{AB}, \qquad Q_A\eta = 0,\\
&Q_A\psi_B = \delta_{AB}\partial^C V_C + i\partial_A(A_B + iV_B) - i\partial_B(A_A - iV_A), \qquad Q_A\chi_{BC} = -\frac{1}{2}\epsilon_{ABCDE}F^{DE}(A+iV)
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
&Q_{AB}A_C = -\delta_{C[A}\psi_{B]} - \frac{1}{2}\epsilon_{ABCDE}\chi^{DE}, \qquad Q_{AB}V_C = -i\delta_{C[A}\psi_{B]} + \frac{1}{2}i\epsilon_{ABCDE}\chi^{DE},\\
&Q_{AB}\eta = -iF_{AB}(A+iV), \qquad Q_{AB}\psi_C = -\frac{1}{2}i\epsilon_{ABCDE}F^{DE}(A-iV),\\
&Q_{AB}\chi_{CD} = \delta_{C[A}\delta_{B]D}\partial^E V_E - i\delta_{[C[A}\big(\partial_{B]}(A_{D]} - iV_{D]}) - \partial_{D]}(A_{B]} + iV_{B]})\big).
\end{aligned} \tag{10}$$

By the help of the identity $(1/2)\epsilon_{ABCGH}\epsilon^{DEFGH} = \delta^D_{[A}\delta^E_B\delta^F_{C]}$ it is straightforward, but tedious to prove that the action (7) is really left invariant under the above transformations.

Let us now group the components of the 5-dimensional fields A_A, V_A, χ^{AB}, ψ_A, η into those of the 2-dimensional fields A_a, V_a, $M_{\alpha\beta}$, η_α, ζ_α, ψ_a^α according to

$$
\begin{aligned}
&M_{12} = A_3 + iV_3, && M_{31} = A_4 + iV_4, && M_{14} = A_5 + iV_5, && \\
&M_{34} = A_3 - iV_3, && M_{42} = A_4 - iV_4, && M_{23} = A_5 - iV_5, && \\
&\eta_1 = i\eta, && \eta_2 = i\chi^{45}, && \eta_3 = -i\chi^{53}, && \eta_4 = i\chi^{34}, \\
&\zeta_1 = \frac{1}{2}i\epsilon_{ab}\chi^{ab}, && \zeta_2 = -i\psi_3, && \zeta_3 = i\psi_4, && \zeta_4 = -i\psi_5, \\
&\psi_a^1 = \psi_a, && \psi_a^2 = \epsilon_{ab}\chi^{b3}, && \psi_a^3 = -\epsilon_{ab}\chi^{b4}, && \psi_a^4 = \epsilon_{ab}\chi^{b5},
\end{aligned}
$$

where (A_A, V_A) agrees with (A_a, V_b) for $A = 1, 2$. Then, it is easily seen that by dimensional reducing the action (7) and the supersymmetry transformations (8)–(10) onto two dimensions, one arrives precisely at the $N_T = 8$ topological action (3) and the supersymmetry transformations (5) and (6).

On the other hand, on-shell, upon using the equation of motion for η, the action (7) can be written as a sum of a topological term (Q-cocycle) and a Q-exact term,

$$
S_T^{(N_T=1)} \doteq -\int_E d^5x \left\{ \frac{1}{8} i\epsilon_{ABCDE}\chi^{AB}\partial^C\chi^{DE} \right\} + Q\Psi, \qquad Q^2 \doteq 0, \tag{11}
$$

with the gauge fermion

$$
\Psi = \int_E d^5x \left\{ \frac{1}{4} i\chi^{AB} F_{AB}(A + iV) - \frac{1}{2}\eta\partial^A V_A \right\}.
$$

Hence, by reducing also (11) in the same way as above into two dimensions we get exactly the decomposition of the action (3) we are looking for, consisting of a topological term (Q-cocycle) and of a Q-exact term.

More precisely, if we group the components A_M, V_M, ψ_M, χ_{aM}, χ_{MN} $(M, N = 3, 4, 5)$ into $SU(2)$ triplets M^{ij}, N^{ij}, ρ^{ij}, χ_a^{ij}, λ^{ij} $(i, j = 1, 2)$ and identify $(1/2)\epsilon^{ab}\chi_{ab}$ with the $SU(2)$ singlet ζ, we obtain the following $N_T = 8$ topological action with a residual global symmetry group $SU(2) \otimes U(1)$,

$$
\begin{aligned}
S_T^{(N_T=8)} = \int_E d^2x \Big\{ & \frac{1}{4} F^{ab}(A + iV) F_{ab}(A - iV) + \frac{1}{2}\partial^a V_a \partial^b V_b + \frac{1}{2}\partial^a (M_{ij} + iN_{ij})\partial_a \big(M^{ij} - iN^{ij}\big) \\
& - i\epsilon^{ab}\lambda_{ij}\partial_a\chi_b{}^{ij} - i\rho_{ij}\partial^a\chi_a{}^{ij} - i\epsilon^{ab}\zeta\partial_a\psi_b - i\eta\partial^a\psi_a \Big\}.
\end{aligned} \tag{12}
$$

The transformation rules for the *on-shell* shift symmetry Q are given by

$$
\begin{aligned}
&QA_a = \psi_a, \qquad QV_a = -i\psi_a, \qquad QM^{ij} = \rho^{ij}, \qquad QN^{ij} = -i\rho^{ij}, \qquad Q\lambda^{ij} = 0, \qquad Q\rho^{ij} = 0, \\
&Q\eta = -\partial^a V_a, \qquad Q\zeta = -\frac{1}{2} i\epsilon^{ab} F_{ab}(A - iV), \qquad Q\psi_a = 0, \qquad Q\chi_a^{ij} = -i\partial_a\big(M^{ij} - iN^{ij}\big).
\end{aligned}
$$

Moreover, the action (12) is also invariant under the following duality $\star$-operation, which maps Q to ${}^\star Q$,

$$
\varphi \equiv \begin{bmatrix} \partial_a & A_a & V_a \\ M^{ij} & N^{ij} & \\ \psi_a & \eta & \zeta \\ \chi_a^{ij} & \rho^{ij} & \lambda^{ij} \end{bmatrix} \quad \Rightarrow \quad \star\varphi = \begin{bmatrix} \epsilon_{ab}\partial^b & \epsilon_{ab}A^b & -\epsilon_{ab}V^b \\ -M^{ij} & -N^{ij} & \\ -i\psi_a & -i\zeta & i\eta \\ -i\chi_a^{ij} & -i\lambda^{ij} & i\rho^{ij} \end{bmatrix}.
$$

Hence, as anticipated, after breaking down the $SU(4)$-symmetry to $SU(2) \otimes U(1)$ the Hodge-type structure of the theory is still preserved. Obviously, because the maximal number of scalar supercharges of twisted $N = 16$, $D = 2$ super-Maxwell theory is $N_T = 8$, the generator Q must be a certain linear combination of Q^α and ${}^\star Q^\alpha$.

Up to now we have shown that the BRST complex of the twisted $N = 16$, $D = 2$ super-Maxwell theory possesses actually a Hodge-type structure. Moreover, we have verified that the cohomological action is of Witten type and that it corresponds to the case with the maximal number $N_T = 8$ of scalar supercharges and the largest possible global symmetry group $SU(4)$. On the other hand, we could also perform a topological twist of $N = 8$, $D = 2$ super-Maxwell theory, whose underlying BRST complex possesses a Hodge-type structure, too. In that case we would get the minimal number of $N_T = 4$ scalar supercharges and the lowest possible global symmetry group $SU(2)$ [13].

3. The Hodge-type structure of the basic cohomology

The twisted action (3) has still an ordinary gauge symmetry, which we have not considered yet. Now, we want to show that by adding to (3) the usual gauge fixing and (anti)ghost dependent terms the Hodge-type structure of the underlying *basic cohomology*, i.e., the BRST complex including also the ordinary gauge symmetry, is preserved. To this end we introduce two $SU(4)$-quartets of Grassmann-odd scalar ghost and antighost fields, C^α and $\bar{C}_\alpha$, which transform as the fundamental and its complex conjugate representation of $SU(4)$, respectively.

Then, the complete gauge-fixed action reads

$$S^{(N_T=8)} = \int_E d^2x \Big\{ \frac{1}{4} F^{ab}(A + iV) F_{ab}(A - iV) + \frac{1}{2} \partial^a (A_a + iV_a) \partial^b (A_b - iV_b) + \frac{1}{8} \partial^a M_{\alpha\beta} \partial_a M^{\alpha\beta} - \epsilon^{ab} \zeta_\alpha \partial_a \psi_b^\alpha - \eta_\alpha \partial^a \psi_b^\alpha + \partial^a \bar{C}_\alpha \partial_a C^\alpha \Big\}. \tag{13}$$

This action, in spite of the fact that the gauge symmetry is fixed, exhibits still an invariance under the following bosonic symmetry,

$$W^{\alpha\beta} A_a = -2\partial_a M^{\alpha\beta}, \qquad W^{\alpha\beta} M_{\gamma\delta} = -4\delta^\alpha{}_{[\gamma} \delta^\beta{}_{\delta]} \partial^a A_a, \qquad W^{\alpha\beta} \psi_a^\gamma = 2i\epsilon^{\alpha\beta\gamma\delta} \big(\partial_a \zeta_\delta + \epsilon_{ab} \partial^b \eta_\delta \big), \tag{14}$$

where we have written down only the non-vanishing transformation rules. However, below it will be shown that the generator $W^{\alpha\beta}$ of that symmetry is just the BRST Laplacian.

Furthermore, in order to ensure the nilpotency of the BRST and co-BRST operators, we introduce a set of auxiliary fields, namely, the $SU(4)$-singlets of Grassmann-even scalar fields B, $\bar{B}$ and G, $\bar{G}$. By the help of these additional fields the action (13) can be rewritten as

$$S^{(N_T=8)} = \int_E d^2x \Big\{ \frac{1}{4} i\epsilon^{ab} B F_{ab}(A + iV) - \frac{1}{4} i\epsilon^{ab} \bar{B} F_{ab}(A - iV) - \frac{1}{2} \bar{B} B + \frac{1}{2} iG \partial^a (A_a + iV_a) - \frac{1}{2} i\bar{G} \partial^a (A_a - iV_a) - \frac{1}{2} \bar{G} G + \frac{1}{8} \partial^a M_{\alpha\beta} \partial_a M^{\alpha\beta} - \epsilon^{ab} \zeta_\alpha \partial_a \psi_b^\alpha - \eta_\alpha \partial^a \psi_b^\alpha + \partial^a \bar{C}_\alpha \partial_a C^\alpha \Big\}, \tag{15}$$

and the Hodge-type $\star$-symmetry (4) must be supplemented by the following replacements,

$$\varphi \equiv \begin{bmatrix} B & C^\alpha & G \\ \bar{B} & \bar{C}_\alpha & \bar{G} \end{bmatrix} \quad \Rightarrow \quad \star\varphi = \begin{bmatrix} -\bar{B} & C^\alpha & -\bar{G} \\ -B & \bar{C}_\alpha & -G \end{bmatrix}, \tag{16}$$

where, again, two successive $\star$-operations on φ yield $\star(\star\varphi) = \pm\varphi$.

Let us now give the transformations rules for the generators of the basic cohomology, which, just as in (2), will be denoted by Ω^α (BRST operator), ${}^\star\Omega^\alpha = \pm \star \Omega^\alpha \star$ (co-BRST operator) and $W^{\alpha\beta} = \{\Omega^\alpha, {}^\star\Omega^\beta\}$ (BRST

Laplacian). Thereby, Ω^α and ${}^\star\Omega^\alpha$ include besides the shift and co-shift symmetries, Q^α and ${}^\star Q^\alpha$, the ghost-dependent ordinary gauge symmetries $\delta_G(C^\alpha)$ as well.

The transformation rules for the *off-shell* BRST symmetries Ω^α are

$$\Omega^\alpha A_a = \psi_a^\alpha + \partial_a C^\alpha, \qquad \Omega^\alpha \psi_a^\beta = -i\epsilon_{ab}\partial^b M^{\alpha\beta}, \qquad \Omega^\alpha V_a = -i\psi_a^\alpha, \qquad \Omega^\alpha M_{\beta\gamma} = 2i\delta^\alpha{}_{[\beta}\zeta_{\gamma]},$$
$$\Omega^\alpha \zeta_\beta = i\delta^\alpha{}_\beta B, \qquad \Omega^\alpha C^\beta = 0, \qquad \Omega^\alpha \eta_\beta = i\delta^\alpha{}_\beta G, \qquad \Omega^\alpha \bar{C}_\beta = \frac{1}{2} i\delta^\alpha{}_\beta (G - \bar{G}), \qquad \Omega^\alpha B = 0,$$
$$\Omega^\alpha \bar{B} = 0, \qquad \Omega^\alpha G = 0, \qquad \Omega^\alpha \bar{G} = 0. \tag{17}$$

From combining Ω^α with the Hodge-type $\star$-symmetry (4) and (15) one gets the corresponding transformation rules for the *off-shell* co-BRST symmetries ${}^\star\Omega^\alpha$.

Then, for the non-vanishing transformations generated by the BRST Laplacian $W^{\alpha\beta}$ one obtains

$$W^{\alpha\beta} A_a = -2\partial_a M^{\alpha\beta}, \qquad W^{\alpha\beta} M_{\gamma\delta} = -2i\delta^\alpha{}_{[\gamma}\delta^\beta{}_{\delta]}(G - \bar{G}), \qquad W^{\alpha\beta}\psi_a^\gamma = 2i\epsilon^{\alpha\beta\gamma\delta}\big(\partial_a \zeta_\delta + \epsilon_{ab}\partial^b \eta_\delta\big),$$

which, after elimination of G and $\bar{G}$ through their equations of motion, agree, as promised, with (14). Let us emphasize, that the BRST Laplacian $W^{\alpha\beta}$, in contrast to [1], does *not* vanish on-shell, due to the presence of scalar fields $M^{\alpha\beta}$. This is a consequence of the fact that the topological nature of our model is actually encoded in the shift and co-shift symmetries Q^α and ${}^\star Q^\alpha$, and not in the vanishing of the BRST Laplacian as in [5].

Furthermore, by a straightforward calculation it can be verified that Ω^α and ${}^\star\Omega^\alpha$ actually leave the action (15) invariant, i.e., it holds $(\Omega^\alpha, {}^\star\Omega^\alpha)S^{(N_T=8)} = 0$. Both operators, together with $W^{\alpha\beta}$, satisfy the following BRST complex,

$$\big\{\Omega^\alpha, \Omega^\beta\big\} = 0, \qquad W^{\alpha\beta} = \big\{\Omega^\alpha, {}^\star\Omega^\beta\big\} \neq 0, \qquad \big\{{}^\star\Omega^\alpha, {}^\star\Omega^\beta\big\} = 0,$$
$$\big[W^{\alpha\beta}, \Omega^\gamma\big] \doteq 0, \qquad \big[W^{\alpha\beta}, {}^\star\Omega^\gamma\big] \doteq 0.$$

Obviously, this basic cohomology is analogous to the de Rham cohomology: the both nilpotent BRST and co-BRST operators, Ω^α and ${}^\star\Omega^\alpha = \pm\star\Omega^\alpha\star$, being interrelated by the duality $\star$-operation, correspond to the exterior and the co-exterior derivatives, d and $\delta = \pm\star d\star$, respectively, and the BRST Laplacian $W^{\alpha\beta} = \{\Omega^\alpha, {}^\star\Omega^\alpha\}$ is the analogue of $\Delta = \{d, \delta\}$, so that we have indeed a perfect example of a Hodge-type cohomological theory in two dimensions.

4. Topological observables

As already pointed out earlier, when the gauge-fixed action (15) is formulated on a compact two-dimensional Riemannian manifold we break down the global symmetry group $SU(4)$ in such a way that the resulting action splits into a topological term (Ω-cocycle) and a Ω-exact term, and that the discrete Hodge-type $\star$-symmetry is preserved, mapping Ω to ${}^\star\Omega$. Alternatively, the same action can be also obtained from the cohomological action (12) by adding the usual gauge-fixing and (anti)ghost dependent terms, see Eq. (3), and by introducing an appropriate set of auxiliary fields.

With regard to this, let us note two unusual features of the action (12) which are relevant for the construction of two-dimensional observables of that topological model. Its most striking property is that the both, shift and co-shift symmetry Q and ${}^\star Q$, are not equivariantly nilpotent (due to the absence of the usual ghost for ghost field ϕ) but, on-shell, rather they are strictly nilpotent even prior to the introduction of the ghost and antighost fields C and $\bar{C}$.

Another remarkable property is that $A_a - iV_a$ and $A_a + iV_a$ are invariant under one of the supercharges, namely Q in the former and ${}^\star Q$ in the latter case. Thus, for the both BRST and co-BRST operators Ω and ${}^\star\Omega$ one should expect the existence of two different sets of observables, depending either on $A_a - iV_a$ in the former case or on $A_a + iV_a$ in the latter case. In fact, these observables can be constructed in a similar way as in the case of the topological sigma models [14]. Therefore, we shall omit any details and simply quote the results.

To begin with, we first associate the zero-forms W_0 and ${}^\star W_0$ to the BRST and co-BRST transforms $\Omega\bar{C}$ and ${}^\star\Omega\bar{C}$ (which, on-shell, correspond to the gauge-fixing function $\partial^a A_a$), respectively, via the BRST and co-BRST invariant ghost field C. These zero-forms can be used as building blocks for constructing the following both sets of k-forms, W_k and ${}^\star W_k$, being interrelated by the Hodge-type $\star$-operation,

$$W_0 = (\Omega\bar{C})C, \qquad W_1 = dx^\mu\big((\Omega\bar{C})(A_\mu - iV_\mu) - C\partial_\mu\bar{C}\big),$$
$$W_2 = dx^\mu \wedge dx^\nu\big(\bar{C}\partial_\mu(A_\nu - iV_\nu) - (A_\mu - iV_\mu)\partial_\nu\bar{C}\big),$$

and

$${}^\star W_0 = ({}^\star\Omega\bar{C})C, \qquad {}^\star W_1 = \delta x^\mu\big(({}^\star\Omega\bar{C})(A_\mu + iV_\mu) - C\partial_\mu\bar{C}\big),$$
$${}^\star W_2 = \delta x^\mu \wedge \delta x^\nu\big(\bar{C}\partial_\mu(A_\nu + iV_\nu) - (A_\mu + iV_\mu)\partial_\nu\bar{C}\big),$$

$dx^\mu = e^\mu{}_a\, dx^a$ and $\delta x^\mu = \epsilon^{\mu\nu}e_{\nu a}\, dx^a$, with $e^\mu{}_a$ being the two-bein on a smooth connected, oriented Riemannian manifold M endowed with metric $g^{\mu\nu}$. Here, $d = dx^\mu\,\partial_\mu$ and $\delta = dx^\mu\,\epsilon_{\mu\nu}\partial^\nu$ are the exterior and co-exterior derivative, respectively.

These k-forms obey the following recursion relations, which are typical for any topological gauge theory,

$$0 = \Omega W_0, \qquad dW_0 = \Omega W_1, \qquad dW_1 = \Omega W_2, \qquad dW_2 = 0,$$
$$0 = {}^\star\Omega{}^\star W_0, \qquad \delta{}^\star W_0 = {}^\star\Omega{}^\star W_1, \qquad \delta{}^\star W_1 = {}^\star\Omega{}^\star W_2, \qquad \delta{}^\star W_2 = 0. \tag{18}$$

Now, if γ is a k-dimensional homology cycle, $\partial\gamma = 0$, on M then the integrated k-forms

$$I_k(\gamma) = \int\limits_\gamma W_k,$$

by virtue of (18), are Ω-invariant,

$$\Omega I_k(\gamma) = \int\limits_\gamma \Omega W_k = \int\limits_\gamma dW_{k-1} = 0, \quad k > 0.$$

Moreover, if $\beta = \partial\alpha$ is the boundary of a $(k+1)$-dimensional surface, $k < 2$, so that β is trivial in homology, then $I_k(\gamma)$ depends only upon the homology class of γ up to a Ω-exact term,

$$I_k(\gamma + \partial\alpha) = \int\limits_{\gamma+\partial\alpha} W_k = I_k(\gamma) + \int\limits_\alpha dW_k = I_k(\gamma) + \int\limits_\alpha \Omega W_{k+1} = I_k(\gamma).$$

Finally, following [14], one can introduce gauge invariant correlation functions of arbitrary products of the $I_k(\gamma)$,

$$Z(\gamma_1, \ldots, \gamma_r) = \int D\varphi \exp\big(-S(\varphi)\big) \prod_{i=1}^{r} \int\limits_{\gamma_i} W_{k_i}(\varphi),$$

which, by construction, both are Ω-invariant and invariant under metric deformations which preserve the holonomy structure. The same constructions hold for the k-forms ${}^\star W_k$.

Summarizing, we have shown that, on-shell, the vanishing of the BRST Laplacian of the Hodge theory proposed in [5] can be avoided, if we view the Maxwell action as the classical part of a more involved cohomological action, which is obtained by a $N_T = 8$ topological twist of $N = 16$, $D = 2$ super-Maxwell theory with global symmetry group $SU(4)$. Then, the complete gauge-fixed cohomological action is of Witten type and the underlying basic cohomology is really of Hodge type. The non-Abelian case will be presented elsewhere.

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